International Journal of Mathematical Archive-6(11), 2015, 65-70
IJMA Available online through www.ijma.info ISSN 2229 – 5046

CHARACTERIZATIONS OF I_{gg} -CLOSED SETS

K. BHAVANI*[1], T. BABITHA[2]

[1]Department of Mathematics,
SRM University Ramapuram, Chennai, Tamil Nadu, India.

[2]Department of Mathematics,
MVM Government Arts College for women, Dindugal, Tamilnadu, India.

(Received On: 08-09-15; Revised & Accepted On: 17-10-15)

ABSTRACT

We discuss various properties of I_{gg} -closed sets in terms of g - closed sets, g⋆ -closed sets and I_g -closed sets. Also, the applications of I_{gg} -closed sets in $T^*_{1/2}$ -spaces and $^*T_{1/2}$ -spaces are discussed.

Keywords and Phrases: *I_g -closed set, g -closed set, g⋆ -closed set, g -local function, $T^*_{1/2}$ -space and relative topology.*

AMS 2010 Subject classification: *54 A 05, 54 A 10.*

1. INTRODUCTION AND PRELIMINARIES

An *ideal* I [7] on a nonempty set X is a nonempty collection of subsets of X satisfying the following: (i) If $A\in\mathrm{I}$ and $B\subset A$; then $B\in\mathrm{I}$; and (ii) If $A\in\mathrm{I}$ and $B\in\mathrm{I}$; then $A\cup B\in\mathrm{I}$. A topological space (X,τ) together with an ideal I is called an *ideal topological space* and is denoted by (X, τ, I). For each subset A of X, $A^*(\mathrm{I},\tau) = \{X\in X \mid U\cap A\notin \mathrm{I}\}$ for every open set U containing $X\}$ is called the *local function* of A [7] with respect to I and τ. We simply write A^* instead of $A^*(\mathrm{I}, \tau)$ in case there is no chance for confusion. We often use the properties of the local function stated in Theorem 2.3 of [6] without mentioning it. Moreover, $cl^*(A) = A\cup A^*$ [9] defines a Kuratowski closure operator for a topology τ^* on X which is finer than τ. A subset A of an ideal space (X,τ) is said to be g -closed [8], if $cl(A)\subset U$ whenever $A\subset U$ and U is open. The complement of a g -closed set is called a g -open set [8]. A subset A of an ideal space (X, τ, I) is said to be I_g-closed [5], if $cl^*(A)\subset U$ whenever $A\subset U$ and U is open. The complement of an I_g-closed set is called an I_g -open set [5]. The collection of all g -open sets in a topological space (X, τ, I) is denoted by τ_g. The g -closure of A denoted by $cl_g(A)$ [1] is the intersection of all g -closed sets containing A and the g -interior of A denoted by $int_g(A)$ defined as the union of all g -open sets contained in A. For every $A\in P(X)$; $A^*(\mathrm{I},\tau_g) = \{X\in X \mid U\cap A\notin \mathrm{I}$ for every g - open set U containing $X\}$ is called the g -local function of A[1] with respect to I and τ_g is denoted by A^*_τ. Also, $cl^*_g(A) = A\cup A^*_g$ [1] is a Kurotowski closure operator for a topology $\tau^*_g = \{X - A \mid cl^*_g(A) = A\}$ [1] on X which is finer than τ_g. A subset A of an ideal space (X, τ, I) is said to be τ^*_g [1], if $cl^*_g(A) = A$ or $A^*_g\subset A$. A subset A of a topological space (X,τ) is said to be a g^* *-closed set* [10], if $cl(A)\subset U$ whenever $A\subset U$ and U is g -open in X. It is clear that every g^*-closed set is a g -closed set [10].

Corresponding Author: K. Bhavani*[1], [1]Department of Mathematics, SRM University Ramapuram, Chennai, Tamil Nadu, India.

A topological space (X, τ) is said to be a $T^*_{1/2}$- *space* [10], if every g^*- closed set is closed. Equivalently, a topological space (X,τ) is called a $T^*_{1/2}$- *space* [10] if and only if every singleton set is either *g*-closed or open. A topological space (X,τ) is said to be a ${}^*T_{1/2}$ - *space* [10] if every g –closed set is a g^* -closed set. Equivalently, a topological space (X,τ) is called a ${}^*T_{1/2}$- space if and only if every singleton set is either closed or g^*-open [10]. A topological space (X,τ) is a $T_{1/2}$ -space if and only if it is ${}^*T_{1/2}$ and $T^*_{1/2}$ [10].

A subset A of X is said to be an I_{gg} - closed set [2] if $A^*_g \subset$ U whenever $A \subset$ U where U is a g -open set in X, equivalently, $cl^*_g(A) \subset$ U whenever $A \subset$ U where U is a g -open set in X. A is said to be an I_{gg} -open set if $X - A$ is an I_{gg} -closed set.

Lemma 1.1: [2] Let (X, τ , I) be an ideal topological space and A $\subset X$. Then the following are equivalent.
(a) A is I_{gg} -closed.
(b) For all $x \in cl^*_g(A)$, $cl_g(\{x\}) \cap A \neq 0$.
(c) $cl^*_g(A) - A$ contains no nonempty g -closed set.
(d) $A^*_g - A$ contains no nonempty g -closed set.

Lemma 1.2: [1] Let (X, τ , I) be an ideal topological space and A$\subset X$. If $A \subset A^*_g$, then the following holds for every subset $A \subset X$:
(a) $A^* = A^*_g = cl^*(A) = cl(A) = cl^*_g(A)$.
(b) A^*_g is a g -closed set.
(c) $A^*_g = cl^*_g(A^*_g)$.
(d) $A^*_g = cl_g(A)$.

2. CHARACTERIZATIONS OF I_{gg} - CLOSED SETS

Theorem 2.1: Let (X, τ , I) be an ideal topological space and A be an I_{gg} -closed subset X. Then the following are equivalent.
(a) A is a τ^*_g -closed set.
(b) $A^*_g -$ A is a g -closed set.

Proof:
(a) ⇒ (b): If A is a τ^*_g - closed set, then $A^*_g - A = \phi$ and so $A^*_g - A$ is a g -closed set.
(b) ⇒ (a): Suppose that $A^*_g - A$ is a g -closed set. Since A is I_{gg} -closed, by Lemma 1.1(d), $A^*_g - A = \phi$ and so $A^*_g \subset A$. Therefore, A is τ^*_g -closed.

Corollary 2.2: Let (X, τ , I) be an ideal topological space and A be an I_{gg} -closed subset of X. If $A \subset A^*_g$; then the following are equivalent.
(a) A is a g -closed set.
(b) $A^*_g - A$ is a g -closed set.

Proof:
(a) ⇒ (b): If A is a g - closed set, then A is a τ^*_g -closed set. Then by Theorem 2.1, $A^*_g - A$ is a g -closed set.
(b) ⇒ (a): If $A^*_g - A$ is g -closed and since A is I_{gg} - closed, by Lemma1.1(d), $A^*_g - A = \phi$ and hence $A^*_g \subset A$. Since $A \subset A^*_g$; $cl_g(A) = A^*_g = A$ by Lemma 1.2(d) and so A is a g -closed set.

Theorem 2.3: Let (X, τ, I) be an ideal topological space and $A \subset X$. Then the following are equivalent.
(a) A is an I_{gg} -closed set.
(b) $cl_g^*(A) \cap F = \phi$ whenever $A \cap F = \phi$ and F is g -closed.

Proof:
(a) ⇒ (b): Suppose that $A \cap F = \phi$ and F is g -closed. Then $A \subset X-F$ and $X-F$ is g -open. Since A is an I_{gg} - closed set, $cl_g^*(A) \subset X-F$ which implies that $cl_g^*(A) \cap F = \phi$
(b) ⇒ (a): Let U be a g -open set containing A. Then $A \cap (X-U) = \phi$ and $X - U$ is g - closed. By (b), $cl_g^*(A) \cap (X - U) = \phi$ and so $cl_g^*(A) \subset (X - U)$. Therefore, A is an I_{gg} -closed set.

Theorem 2.4: Let (X, τ, I) be an ideal topological space and $A \subset X$. If A is a closed set, then A is an I_{gg} -closed set.

Proof: Let $A \subset U$ where $A \subset X$ and $U \in \tau_g$. Since U is g -open, $cl_g^*(A) \subset cl(A) = A \subset U$ by hypothesis and so A is an I_{gg} -closed set.

The following Example 2.5 shows that the converse of Theorem 2.4 is not true.

Example 2.5: Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $I = \{\phi, \{a\}\}$. If $A = \{b\}$ and $A \subset U$ where $U = \{b\}$ is a g -open set, then $cl(A) = \{b, c\} \neq A$, but $A_g^* = \{b\} \subset U$, which implies that A is an I_{gg} -closed set but not a closed set.

Theorem 2.6: Let (X, τ, I) be an ideal topological space and $A \subset X$. Then A_g^* is always an I_{gg} -closed set.

Proof: Let $A_g^* \subset U$ where U is a g -open set. Since $(A_g^*)_g^* \subset A_g^* \subset U$ [1, Theorem 3.7(e)] whenever $A_g^* \subset U$ which implies that A_g^* is an I_{gg} –closed set.

The following theorem 2.7 shows that every g*- closed set is an I_{gg} -closed set and Example 2.8 below shows that the converse is not true. Also, if $\tau_g = \tau_g^*$ then g^* -closed sets coincide with I_{gg} -closed sets.

Theorem 2.7: Let (X, τ, I) be an ideal topological space and $A \subset X$. If A is a g^* -closed set, then A is an I_{gg} -closed set.

Proof: Let $A \subset U$ where $A \subset X$ and $U \in \tau_g$. Since U is g -open, $cl_g^*(A) \subset cl(A) \subset U$ by hypothesis and so A is an I_{gg} -closed set.

Example 2.8: Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $I = \{\phi, \{a\}\}$. If $A = \{b\}$ and $A \subset U$ where $U = \{b\}$ is a g -open set, then $cl(A) = \{b, c\} \not\subset U$ but $A_g^* = \{b\} \subset U$, which implies that A is an I_{gg} -closed set but not a g^* -closed set.

Theorem 2.9: If (X, τ, I) is an ideal topological space, A is an I_{gg} –closed subset of X and $A \subset A_g^*$, then A is a g^* -closed set.

Proof: Let $A \subset U$ where U is g -open. Since A is I_{gg} -closed, $cl_g^*(A) \subset U$. Since $A \subset A_g^*$, by [1, Theorem 3.10(a)], $cl(A) = cl_g^*(A) \subset U$ which implies that A is a g^* -closed set.

Corollary 2.10: If (X, τ , I) is an ideal topological space where I = { ϕ }, then A is I_{gg} - closed if and only if A is g^* - closed.

Proof: If I = { ϕ }, then by [1, Theorem 3.7(f)], A_g^* = $cl_g(A) \supset A$ and so $A \subset A_g^*$. If A is I_{gg} - closed, then by Theorem 2.9, A is g^* - closed. The converse is clear from Theorem 2.7.

Theorem 2.11: Let (X, τ , I) be an ideal topological space where I = { ϕ }. If (X, τ , I) is a $T_{1/2}^* -$space, then every I_{gg} - closed set is a closed set.

Proof: By Corollary 2.10, the family of all g^*- closed sets coincide with the family of all I_{gg} - closed sets, since I = { ϕ }. Since every g^*- closed set is a closed set in every $T_{1/2}^*$ -space, every g^*- closed set is a closed set and so every I_{gg} – closed set is a closed set.

Theorem 2.12: Let (X, τ , I) be an ideal topological space. If X is a ${}^*T_{1/2}$ -space, then for each $x \in X$; {x} is either closed or I_{gg} - open.

Proof: Suppose that (X, τ , I) is a ${}^*T_{1/2}$ -space and $x \in X$. If {x} is not a closed set, then X − {x} is not an open set. This implies that X − {x} is a g -closed set, since X is the only open set which contains X − {x}. Since X is a ${}^*T_{1/2}$ -space, X − {x} is a g^*-closed set. Therefore, X − {x} is an I_{gg} -closed set by Theorem 2.7 or equivalently {x} is an I_{gg} -open set.

Theorem 2.13: Let (X, τ , I) be an ideal topological space. Then either {x} is g- closed or $\{x\}^c$ is I_{gg} - closed.

Proof: Suppose that {x} is not g -closed. Then $\{x\}^c$ is not a g -open set and the only g -open set containing $\{x\}^c$ is X. Therefore, $(\{x\})_g^* \subset X$ and so $\{x\}^c$ is I_{gg} -closed.

Theorem 2.14: Let (X, τ , I) be an ideal topological space. Then the following are equivalent.
(a) Every I_{gg} -closed set is τ_g^* -closed.
(b) Every singleton subset of X is either g -closed or τ_g^* -open.

Proof:
(a) ⇒ (b): Let $x \in X$. If {x} is not g -closed, then by Theorem 2.13, $\{x\}^c$ is I_{gg} -closed and so τ_g^* -closed by hypothesis. Therefore, $\{x\}$ is τ_g^* -open.
(b) ⇒ (a): Let A be an I_{gg} -closed set and $x \in cl_g^*(A)$. Then we have the following two cases.

Case-1: Suppose that {x} is g -closed. By Lemma 1.1(c), $cl_g^*(A)$ −A contains no nonempty g -closed set. Therefore, X $\notin$ $cl_g^*(A)$ − A which implies that $x \in A$.

Case-2: Suppose that {x} is τ_g^* -open. Since x $\in cl_g^*(A)$, {x} $\cap$ $A \neq \phi$ Therefore, $x \in A$. Thus in both cases, $x \in A$ and so $cl_g^*(A)$ = A which shows that A is τ_g^* -closed.

Theorem 2.15: Let (X, τ, I) be an ideal topological space and $A \subset X$. Then $cl_g^*(A)$ is a closed set.

Proof: If $x \in cl(cl_g^*(A))$ and U be an open set containing x such that $U \cap cl_g^*(A) \neq \phi$. Let $y \in U \cap cl_g^*(A)$ for some $y \in X$. Since $U \in \tau_g(y)$ and $y \in cl_g^*(A)$, there exists a τ_g^* -open set V such that $V \cap A \neq \phi$ Since $U \cap V$ is a τ_g^* -open neighbourhood of y, $(U \cap V) \cap A \neq \phi$ which implies that $x \in cl_g^*(A)$. Therefore, $cl_g^*(A)$ is a closed set.

Theorem 2.16: Let (X, τ, I) be an ideal topological space and let U and A be subsets of X such that $A \subset U \subset A_g^*$ Then $U \subset U_g^*$, $(U_g^*)_g^* = U_g^*$ and $(A_g^*)_g^* = A_g^*$.

Proof: Since $A \subset U \subset A_g^*$, $A_g^* = U_g^*$ and so $U \subset U_g^*$. Since $(A_g^*)_g^* \subset A_g^*$, $A \subset A_g^*$, implies that $(A_g^*)_g^* = A_g^*$ and so $(U_g^*)_g^* = U_g^*$.

3. APPLICATIONS OF I_{gg} -CLOSED SETS

If Y is a nonempty subset of an ideal topological space (X, τ, I), then (Y, τ_Y, I_Y) is an ideal topological subspace of X where $I_Y = \{ I \cap Y \mid I \in I \}$ [4] is an ideal on Y, the restriction of I to Y and $\tau_Y = \{U \cap Y \mid U \in \tau\}$ is the relative topology on Y.

Lemma 3.1: [6, Lemma 6.6] Let (X, τ, I) be an ideal topological space and $A \subset X$. Then $(\tau_A)^*(I_A) = \tau^*(I)_A$.

Lemma 3.2: [3] Let (X, τ, I) be an ideal topological space and $A \subset Y \subset X$. If Y is g -open in X, then $A_g^*(I_Y, \tau_{gY}) = A_g^*(I, \tau_g) \cap Y$ where $\tau_{gY} = \tau_g \mid Y$.

Proof: Let $x \notin A_g^*(I, \tau_g) \cap Y$. Then either $X \in Y$ or $x \notin Y$.

Case-1: Suppose that $x \notin Y$. Since $A_g^*(I_Y, \tau_{gY}) \cap Y \subset Y$, $x \notin A_g^*(I_Y, \tau_{gY})$.

Case-2: Suppose that $x \in Y$. Since $x \notin A_g^*(I, \tau_g)$, there exists a g -open set V in X containing X, such that $V \cap A \in I$. Since $x \in Y$ and Y is g –open in X, $Y \cap V \in \tau_g$ such that $(Y \cap V) \cap A \in I$ and so $(Y \cap V) \cap A \in I_Y$. Consequently, $x \notin A_g^*(I_Y, \tau_{gY})$. Hence $A_g^*(I_Y, \tau_{gY}) \subset A_g^*(I, \tau_g) \cap Y$. To prove the converse, consider $x \notin A_g^*(I_Y, \tau_{gY})$. Then for some g -open set V in (Y, τ_Y) containing X, there exists $U \in \tau_g$ such that $V = U \cap Y$ and so $(U \cap Y) \cap A \in I_Y$. Since $A \subset Y$, $U \cap A \in I_Y \subset I$ gives $U \cap A \in I$ for some g -open set U containing X. Therefore, $x \notin A_g^*(I, \tau_g)$. Therefore, $A_g^*(I_Y, \tau_{gY}) = A_g^*(I, \tau_g) \cap Y$.

Theorem 3.3: Let (X, τ, I) be an ideal topological space and $A \subset Y \subset X$. If A is an I_{gg} -closed set in (Y, τ_Y, I_Y) and Y is g -open and τ_g^*-closed in X, then A is an I_{gg} - closed set in X.

Proof: Let $A \subset U$ where U is g -open in X. Then $U \cap Y$ is g -open in Y. This implies that $A_g^*(I_Y, \tau_{gY}) = A_g^*(I, \tau_g) \cap Y$ by Lemma 3.2. Since Y is τ_g^* -closed in X, $A_g^*(I, \tau_g) \subset Y_g^*(I, \tau_g) \subset Y$ and so $A_g^*(I_Y, \tau_{gY}) = A_g^*(I, \tau_g) \cap Y$. Since A is I_{gg} -closed in Y and $A \subset U \cap Y$, $A_g^*(I, \tau_g) = A_g^*(I_Y, \tau_{gY}) \subset U \cap Y \subset U \cup (X - A_g^*(I, \tau_g))$. Therefore, $A_g^*(I, \tau_g) \subset U \cup (X - A_g^*(I, \tau_g))$ which implies that $A_g^*(I, \tau_g) \subset U$ and so A is an I_{gg} -closed set in X.

Theorem 3.4: Let (X, τ , I) be an ideal topological space and $A \subset Y \subset X$. If A is I_{gg} -closed in X and Y is g -open in X, then A is I_{gg} -closed in (Y, τ_Y , I_Y).

Proof: Let U be a g -open subset of (Y, τ_Y , I_Y) such that $A \subset$U. Since Y is g -open in X, U is g -open in X and so $A_g^*(I, \tau_g) \subset$U. By Lemma 3.2, $A_g^*(I_Y, \tau_{gY}) = A_g^*(I, \tau_g) \cap Y \subset$ U $\cap$ Y = U and hence $A_g^*(I_Y, \tau_{gY})$ Therefore, A is an I_{gg} -closed set in (Y, τ_Y , I_Y).

Corollary 3.5: Let (X, τ , I) be an ideal topological space where Y is a g -open and τ_g^* -closed subset of X. Then A is I_{gg} -closed in (Y, τ_Y , I_Y) if and only if A is I_{gg} -closed in X.

Theorem 3.6: Let (Y, τ_Y , I_Y) be a g -closed subspace of an ideal topological space (X, τ , I) and U be I_{gg} - open in X. Then U $\cap$ Y is I_{gg} -open in Y.

Proof: Let F be a g -closed subset of (Y, τ_Y , I_Y) such that $F \subset$U$\cap Y$. Since U is I_{gg} -open in X, by [2, Theorem 2.4], $F \subset \mathrm{int}_g^*(U)$ and $F = F \cap Y \subset \mathrm{int}_g^*(U) \cap Y \subset \mathrm{int}_{gY}^*(U \cap Y)$. Therefore, U$\cap Y$ is I_{gg} -open in Y.

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Source of support: Nil, Conflict of interest: None Declared