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QUADRATURE METHOD FOR SOLVING LINEAR MIXED VOLTERRA-FREDHOLM INTEGRAL EQUATION

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ABSTRACT

In this paper, new algorithms for finding numerical solution of Linear mixed Volterra-Fredholm integral equations of the second kind (LMVFIE's) are introduced to solve (LMVFIE's) based on some method. These methods namely are Simpson's 3/8 method, Durand's method and Weddle method. Two examples are given and their results shown in tables and figures to illustrate the efficiency and accuracy of this methods to find the result by using modify program (MATLAB) version 7.11.0, product 2010.

1. INTRODUCTION

A numerical quadrature (numerical integration) rules is the approximate computation of an integral using numerical techniques so is a primary tool used by engineers and scientists to obtain approximate answers for definite integrals that cannot be solved analytically [1]. Numerical integration has always been useful in biostatistics to evaluate distribution functions and other quantities.

Integral equations have received considerable interest in the mathematical literatures, because of their many fields of application in different areas of sciences. Integral equations are encountered in various fields of science and numerous applications in elasticity, plasticity, heat and mass transfer, approximation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, electrical engineering, economics and medicine [2,3].

Let us consider the following (LMVFIE's):

$$u(x) = f(x) + \int_a^x K(x,t)\,u(t)dt + \int_a^b L(x,t)u(t)dt \qquad (1.1)$$

Where $a \le x \le b$, $f(x)$, $K(x,t)$ and $L(x,t)$, are given continuous functions and $u(x)$ is the unknown function to be determined. Some kinds of Volterra-Fredholm integral equations had been solved numerically, by different methods that are indicated below.

Many researchers studied and discussed (LMVFIE's), Muna M. and Iman N. in [4] using Lagrange polynomials for solving the linear Volterra-Fredholm integral equation. Hendi F. and Bakodah H. in [5] employed discrete adomain decomposition method to solve Fredholm-Volterra integral equation in two dimensional space. Majeed S. and Omran H. in[6] applied the repeated Trapezoidal method and the repeated Simpson's 1/3 method for solving linear Fredholm-Volterra integral equation, Omran H. in[7] applied the repeated Trapezoidal method and the repeated Simpson's method for solving the first order linear Fredholm-Volterra integro-differential equations. Maleknejad K. and Mahdiani K. in [8] using Piecewise Constant block-pulse functions for solving linear two Dimensional Fredholm-Volterra Integral Equations. Hendi F. and Albugami A. in [9] adopt collocation and Galerkin methods for solving Fredholm–Volterra integral equation of the second kind.

In this paper, we show how the numerical methods which are based on the Simpson's 3/8 quadrature formula, Weddle quadrature formula and Durand's quadrature formula can be used to solve (LMVFIE's).

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This paper is organized as follows:

In section 2 we will present quadrature method for solving (LMVFIE's). In section 3 we solve equation (1.1) by Simpson's 3/8 method. In section 4 we solve equation (1.1) by Durand's method. So in section5 we solve equation (1.1) by Weddle method. In section6 we apply the proposed method in some examples, showing the accuracy and efficiency of the method. Finally, the report ends with a brief conclusion.

2. NEW ALGORITHMS (QUADRATURE METHOD FOR SOLVING LINEAR MIXED VOLTERRA-FREDHOLM INTEGRAL EQUATION):

The numerical integration methods (also called quadrature rule) is an approximation of the definite integral of a function, usually stated as a weighted sum of function values at specified points within the domain of integration. the term quadrature means the process of finding square with the same area as the area enclosed by the arbitrary closed curve. Integration by quadrature either means solving an integral analytically (i.e., symbolically in terms of known functions), or solving of an integral numerically (Gaussian quadrature, Newton-Cotes formulas).

An obvious numerical procedure is to approximate the integral term (1.1) via a quadrature rule which integrates over the variable t for a fixed value x. it is natural to choose a regular mesh in x and t ; thus setting $x = x_i = a + ih$, where $h = (b - a)/n$ is the fixed step length. We approximate in an obvious notation the integral term in linear equation (1.1) by

$$\int_a^{x_i} K(x_i,t)u(t)dt + \int_a^b L(x_i,t)u(t)dt \approx h\left[\sum_{j=0}^{i} w_{ij}\, K(x_i,t_j)u(t_j) + \sum_{j=0}^{n} w_{ij}\, L(x_i,t_j)u(t_j)\right]$$
$$= h\left[\sum_{j=0}^{i}\left(w_{ij}\, K_{ij} + w_{ij}\, L_{ij}\right)u_j + \sum_{j=i+1}^{n} w_{ij}\, L_{ij}\, u_j\right]$$

Where $x_i = t_i,\ i = 0,1,..,n$. This quadrature rule leads to the following set of equations:

$$u(x_0) = f(x_0) + h\sum_{j=0}^{n} w_{0j}\, L_{0j}\, u_0$$
$$u(x_1) = f(x_1) + h\left[(w_{10}K_{10}u_0 + w_{11}K_{11}u_1) + \sum_{j=0}^{n} w_{1j}\, L_{1j}\, u_j\right] + E(K(x,t),(L(x,t)u(t))$$
$$u(x_i) = f(x_i) + h\left[\sum_{j=0}^{i}\left(w_{ij}\, K_{ij} + w_{ij}\, L_{ij}\right)u_j + \sum_{j=i+1}^{n} w_{ij}\, L_{ij}\, u_j\right] + E((K(x,t),L(x,t))u(t)), \qquad i = 2,\dots,n$$

where $E((K(x,t),L(x,t))u(t))$ represents the error term in the quadrature rule. The set w_{ij} represents the weight function for an point quadrature rule of Newton-cotes type for the interval [0, ih].

Now let us start by (Simpsons 3/8 rule).

3. ALGORITHM1 (SIMPSON'S 3/8 RULE)

Simpson's 3/8 rule is numerical method use to solve numerical integration proposed by Thomas Simpson. This approach approximates the function *u(x)* by cubic curve and the area contained in three strips under a curve can be evaluated from x_0 to x_3.

Consider the (LMVFIE's) given by equation (1.1). To solve this equation we divide the finite interval [a, b] into 3n smaller interval of width h, where h=(b − a)/3n. The i-th point of subdivision is denoted by x_i, such that $i = 0, 1,\dots, n$. The approximate solution will be defined at the mesh point x_{3i} is denoted by $u(x_{3i})$ and is given by:

$$u(x_{3i}) = f(x_{3i}) + \int_a^{x_{3i}} K(x_{3i},t)\, u(t)dt + \int_a^b L(x_{3i},t)u(t)dt \qquad (3.1)$$
$$i = 0,1,..,n.$$

And in the odd nods

$$u(x_m) = f(x_m) + \int_a^{x_m} K(x_m,t)u(t)dt + \int_a^b L(x_m,t)u(t)dt, \qquad (3.2)$$
$$m = 5,7,11,..,3n-1 \quad if\ n\ is\ even\ (n = 2,4,6,\dots),$$
$$m = 5,7,11,..,3n-2 \quad if\ n\ is\ odd\ \ (n = 3,5,7,\dots).$$

And in the even nods

$$u(x_r) = f(x_r) + \int_a^{x_r} K(x_r,t)u(t)dt + \int_a^b L(x_r,t)u(t)dt, \tag{3.3}$$

$r = 2,4,8,\ldots,3n-2 \quad if\ n\ is\ even\ (n = 2,4,6,..),$
$r = 2,4,8,\ldots,3n-1 \quad if\ n\ is\ odd\ (n = 3,5,7,\ldots).$

If we approximation the integrals that appeared in equations (3.1) - (3.3) by the Simpson's 3/8 formula which will yield the following system of equations: (remark we divide the Volterra integral equation see [10])

$$u_0 = f_0 + \frac{3h}{8}\left[L_{0,0}u_0 + 3\sum_{j=0}^{n-1} L_{0,3j+1}u_{3j+1} + 2\sum_{j=1}^{n-1} L_{0,3j}u_{3j} + 3\sum_{j=1}^{n} L_{0,3j-1}u_{3j-1} + L_{0,3n}u_{3n}\right].$$

$$\begin{aligned} u_{3i} = f_{3i} + \frac{3h}{8}\Bigg[&(L_{3i,0}+K_{3i,0})u_0 + 3\sum_{j=0}^{i-1}(L_{3i,3j+1}+K_{3i,3j+1})u_{3j+1} + 3\sum_{j=1}^{i}(L_{3i,3j-1}+K_{3i,3j-1})u_{3j-1} + \\ &+ 2\sum_{j=1}^{i-1}(L_{3i,3j}+K_{3i,3j})u_{3j} + (2L_{3i,3i}+K_{3i,3i})u_{3i} + 2\sum_{j=i+1}^{n-1} L_{3i,3j}u_{3j} \\ &+ 3\sum_{j=i}^{n-1} L_{3i,3j+1}u_{3j+1} + 3\sum_{j=i+1}^{n} L_{3i,3j-1}u_{3j-1} + L_{3i,3n}u_{3n}\Bigg], i = 1,2,\ldots,n-1. \end{aligned}$$

$$\begin{aligned} u_m = f_m + \frac{3h}{8}\Bigg[&(L_{m,0}+K_{m,0})u_0 \\ &+ 3\sum_{j=1}^{2}(L_{m,j}+K_{m,j})u_j + \left(2L_{m,3} + \frac{136}{72}K_{m,3}\right)u_3 \\ &+ 3\left(\sum_{j=1}^{n-1} L_{m,3j+1}u_{3j+1} + \sum_{j=2}^{n} L_{m,3j-1}u_{3j-1}\right) + 2\sum_{j=2}^{n-1} L_{m,3j}u_{3j} + \frac{32}{9}\sum_{j=4,6,8,10,..}^{m-1} K_{m,j}u_j \\ &+ \frac{16}{9}\sum_{j=7,9,11,..}^{m} K_{m,j-2}u_{j-2} + \frac{8}{9}K_{m,m}u_m + L_{m,3n}u_{3n}\Bigg], \end{aligned}$$

$m = 5,7,11,..,3n-1 \quad if\ n\ is\ even\ (n = 2,4,6,\ldots),$

$m = 5,7,11,..,3n-2 \quad if\ n\ is\ odd\ (n = 3,5,7,\ldots).$

$$\begin{aligned} u_r = f_r + \frac{3h}{8}\Bigg[&\left(L_{r,0} + \frac{8}{9}K_{r,0}\right)u_0 \\ &+ 3\left(\sum_{j=0}^{n-1} L_{r,3j+1}u_{3j+1} + \sum_{j=1}^{n} L_{r,3j-1}u_{3j-1}\right) + 2\sum_{j=1}^{n-1} L_{r,3j}u_{3j} \\ &+ \frac{32}{9}\sum_{j=1,3,5,..}^{r-1} K_{r,j}u_j + \frac{16}{9}\sum_{j=4,6,8,..}^{r} K_{r,j-2}u_{j-2} + \frac{8}{9}K_{r,r}u_r + L_{r,3n}u_{3n}\Bigg], \end{aligned}$$

$r = 2,4,8,\ldots,3n-2 \quad if\ n\ is\ even\ (n = 2,4,6,..),$
$r = 2,4,8,\ldots,3n-1 \quad if\ n\ is\ odd\ (n = 3,5,7,\ldots).$

$$\begin{aligned} u_{3n} = f_{3n} + \frac{3h}{8}\Bigg[&(L_{3n,0}+K_{3n,0})u_0 + 3\sum_{j=0}^{n-1}(L_{3n,3j+1}+K_{3n,3j+1})u_{3j+1} + \\ &+ 3\sum_{j=1}^{n}(L_{3n,3j-1}+K_{3n,3j-1})u_{3j-1} + 2\sum_{j=1}^{n-1}(L_{3n,3j}+K_{3n,3j})u_{3j} + (L_{3n,3n}+K_{3n,3n})u_{3n}\Bigg]. \end{aligned} \tag{3.4}$$

where:
$K_{ij} = K(x_i, x_j) \quad (j = 0,1, \dots, i)$,
$L_{ik} = L(x_i, x_k) \quad (k = 0,1, \dots, n)$,
$g_i = g(x_i)$, and u_i is the approximate value of the unknown function u at the node x_i, $i = 0,1,..,$ n.

By solving the system given by equation (3.2) which consists of (n+1) equations and (n+1) unknowns, the approximate solution of (1.1), is obtained.

4. ALGORITHM2 (DURAND'S RULE)

Durand's rule is one of a family of formulas for numerical integration called Newton–Cotes formulas. the formula approximates the function on (x_0, x_3) by a curve that passes through four points $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3))$ which results in Durand's rule.

Consider the (LMVFIE's) given by equation (1.1). Here we use Durand's method to find the solution of equation (1.1).To do this, we divide the finite interval [a,b] into 3n smaller interval of width h, where h = (b − a)/3n. The approximate solution of (1.1) will be defined at the mesh point x_i is denoted by $u(x_i)$ and is given by

$$u(x_i) = f(x_i) + \int_a^{x_i} K(x_i, t)u(t)dt + \int_a^b L(x_i, t)u(t)dt, \quad i = 0,1,2,..,n. \tag{4.1}$$

By using the Durand's formula to approximate the integrals that appeared in equations (4.1) can get the following system of equations:

$$u_0 = f_0 + h\left[\frac{2}{5}L_{0,0}u_0 + \frac{11}{10}L_{0,1}u_1 + \sum_{j=2}^{3n-2} L_{0,j}u_j + \frac{11}{10}L_{0,3n-1}u_{3n-1} + \frac{2}{5}L_{0,3n}u_{3n}\right].$$

$$u_1 = f_1 + h\left[\left(\frac{1}{2}K_{1,0} + \frac{2}{5}L_{1,0}\right)u_0 + \left(\frac{1}{2}K_{1,1} + \frac{11}{10}L_{1,1}\right)u_1 + \sum_{j=2}^{3n-2} L_{1,j}u_j + + \frac{11}{10}L_{1,3n-1}u_{3n-1} + \frac{2}{5}L_{1,3n}u_{3n}\right].$$

$$u_2 = f_2 + h\left[\left(\frac{1}{3}K_{2,0} + \frac{2}{5}L_{2,0}\right)u_0 + \left(\frac{4}{3}K_{2,1} + \frac{11}{10}L_{2,1}\right)u_1 + \left(\frac{1}{3}K_{2,2} + L_{2,2}\right)u_2 \right.$$
$$\left. + \sum_{j=3}^{3n-2} L_{2,3n-2}u_{3n-2} + \frac{11}{10}L_{2,3n-1}u_{3n-1} + \frac{2}{5}L_{2,3n}u_{3n}\right].$$

$$u_i = f_i + h\left[\frac{2}{5}(K_{i,0} + L_{i,0})u_0 + \frac{11}{10}(K_{i,1} + L_{i,1})u_1 + (\sum_{j=2}^{i-2} K_{i,j} + \sum_{j=2}^{3n-2} L_{i,j})u_j + \frac{11}{10}(K_{i,i-1}u_{i-1} + L_{i,3n-1}u_{3n-1}) \right.$$
$$\left. + \frac{2}{5}(K_{i,i}u_i + L_{i,3n})u_{3n}\right],$$

$i = 3,4,5,6, \dots, 3n - 1.$

$$u_{3n} = f_{3n} + h\left[\frac{2}{5}(K_{3n,0} + L_{3n,0})u_0 + \frac{11}{10}(K_{3n,1} + L_{3n,1})u_1 \right.$$
$$\left. + \sum_{j=2}^{3n-2}(K_{3n,j} + L_{3n,j})u_j + \frac{11}{10}(K_{3n,3n-1} + L_{3n,3n-1})u_{3n-1} + \frac{2}{5}(K_{3n,3n} + L_{3n,3n})u_{3n}\right]. \tag{4.2}$$

5. ALGORITHM3 (WEDDLE RULE)

Is a numerical method that approximates the value of a definite integral and so is one of the Newton cotes formulas. The formula approximates the function on (x_0, x_6) by a curve that passes through seven points $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2)),$ $(x_3, f(x_3)), (x_4, f(x_4)), (x_5, f(x_5))$ *and* $(x_6, f(x_6))$, which results in Weddle's rule.

Consider the (LMVFIE's) given by equation (1.1). Here we use Weddle method to find the solution of equation (1.1).To do this, we divide the finite interval [a, b] into 6n smaller interval of width h, where h = (b − a)/6n. The approximate solution of (1.1) will be defined at the mesh point x_{6i} is denoted by $u(x_{6i})$ and is given by

$$u(x_{6i}) = f(x_{6i}) + \int_a^{x_{6i}} K(x_{6i},t)u(t)dt + \int_a^b L(x_{6i},t)u(t)dt,\ \ i = 0,1,..,n. \tag{5.1}$$

$$u(x_m) = f_m + \int_a^{x_m} K(x_m,t)u(t)dt + \int_a^b L(x_m,t)u(t)dt, \qquad m = 1,..,5. \tag{5.2}$$

$$u(x_r)\ = f_r + \int_a^{x_r} K(x_r,t)u(t)dt + \int_a^b L(x_r,t)u(t)dt,\ \ r = 7,8,9,..,6n-1. \tag{5.3}$$

$$u(x_0) = f_0 + \frac{3h}{10}\Bigg[L_{0,0}u_0 + 5\sum_{j=1}^{n} L_{0,6j-1}u_{6j-1} + 5\sum_{j=0}^{n-1} L_{0,6j+1}u_{6j+1} + \sum_{j=1}^{n} L_{0,6j-2}u_{6j-2} + \sum_{j=0}^{n-1} L_{0,6j+2}u_{6j+2} + 6\sum_{j=0}^{n-1} L_{0,6j+3}u_{6j+3} + 2\sum_{j=2}^{n-2} L_{0,6j}u_{6j} + L_{0,6n}u_{6n}\Bigg].$$

$$\begin{aligned} u(x_{6i}) = f_{6i} + \frac{3h}{10}\Bigg[&(K_{6i,0} + L_{6i,0})u_0 + 5\sum_{j=1}^{i}(K_{6i,6j-1} + L_{6i,6j-1})u_{6j-1} \\ &+ 5\sum_{j=0}^{i-1}(K_{6i,6j+1} + L_{6i,6j+1})u_{6j+1} + \sum_{j=1}^{i}(K_{6i,6j-2} + L_{6i,6j-2})u_{6j-2} \\ &+ \sum_{j=0}^{i-1}(K_{6i,6j+2} + L_{6i,6j+2})u_{6j+2} + 6\sum_{j=0}^{i-1}(K_{6i,6j+3} + L_{6i,6j+3})u_{6j+3} \\ &+ 2\sum_{j=2}^{i}(K_{6i,6j-6} + L_{6i,6j-6})u_{6j-6} + (K_{6i,6i} + 2L_{6i,6i})u_{6i} \\ &+ 5\sum_{j=i+1}^{n} L_{6i,6j-1}u_{6j-1} + 5\sum_{j=i}^{n-1} L_{6i,6j+1}u_{6j+1} + \sum_{j=i+1}^{n} L_{6i,6j-2}u_{6j-2} \\ &+ \sum_{j=i}^{n-1} L_{6i,6j+2}u_{6j+2} + 6\sum_{j=i}^{n-1} L_{6i,6j+3}u_{6j+3} + 2\sum_{j=i}^{n-2} L_{6i,6j}u_{6j} + L_{6i,6n}u_{6n}\Bigg]. \end{aligned}$$

Now, we will show table to illustrates divided the equation (5.2) m=1,…5. coefficient of $K_{m,j}$

m	*J*					
1	*5/3*	*5/3*	*Trapezoidal*			
2	*10/9*	*40/9*	*10/9*	*Simpsoin1/3*		
3	*5/4*	*15/4*	*15/4*	*5/4*	*Simpsoin3/8*	
4	*10/9*	*40/9*	*20/9*	*40/9*	*10/9*	*Composite Simpsoin1/3*
5	*5/3*	*10/3*	*10/3*	*10/3*	*10/3*	*5/3* *Comp. Trapezoidal*

Coefficient of $L_{m,6n}$*(Weddle rule)*

1	*5*	*1*	*6*	*1*	*5*	*2*	*5*	*1*	*6*	*1*	*5*	*1*	…	*6n(point)*

u_m = *3h/10*(*$K_{m,j}u_j + L_{m,l}u_l$*)*, $\quad j = 0,1,2,3,4,5$ *and* $l = 0,1,..,6n$

m	$L_{m,0} + K_{m,0}$	$L_{m,1} + K_{m,1}$	$L_{m,2} + K_{m,2}$	$L_{m,3} + K_{m,3}$	$L_{m,4} + K_{m,4}$	$L_{m,5} + K_{m,5}$	$L_{m,6}$	$L_{m,7}$	$L_{m,8}$	$L_{m,9}$	…			$L_{m,l}$
1	*1+5/3*	*5+5/3*	*1*	*6*	*1*	*5*	*2*	*5*	*1*	*6*	*1*	*5*	*1*	*6n(point*
2	*1+10/9*	*5+40/9*	*1+10/9*	*6*	*1*	*5*	*2*	*5*	*1*	*6*	*1*	*5*	*1*	…
3	*1+5/4*	*5+15/4*	*1+15/4*	*6+5/4*	*1*	*5*	*2*	*5*	*1*	*6*	*1*	*5*	*1*	…
4	*1+10/9*	*5+40/9*	*1+20/9*	*6+40/9*	*1+10/9*	*5*	*2*	*5*	*1*	*6*	*1*	*5*	*1*	…
5	*1+5/3*	*5+10/3*	*1+10/3*	*6+10/3*	*1+10/3*	*5+5/3*	*2*	*5*	*1*	*6*	*1*	*5*	*1*	…

$$u(x_r) = f_r + \frac{3h}{10}\Bigg[(K_{r,0} + L_{r,0})u_0 + 5\sum_{j=0}^{1}(K_{r,4j+1} + L_{r,4j+1})u_{4j+1} + \sum_{j=0}^{1}(K_{r,2j+2} + L_{r,2j+2})u_{2j+2} + 6(K_{r,3} + L_{r,3})u_3 + \frac{8}{3}K_{r,6}u_6 + \frac{5}{3}K_{r,r}u_r + \frac{10}{3}\sum_{j=7}^{r-1}K_{r,j}u_j + 5\sum_{j=7,11,13,..}^{6n-1}L_{r,j}u_j + 2\sum_{j=3*(even\ num.)}^{6n-6}L_{r,j}u_j + \sum_{j=8,10,14,..}^{6n-2}L_{r,j}u_j + 6\sum_{j=3*(odd\ num.)}^{6n-3}L_{r,j}u_j + L_{r,6n}u_{6n}\Bigg],$$
$$r = 7,8,9,..,6n-1. \tag{5.4}$$

6. NUMERICAL EXAMPLE

In this section we give some of the numerical examples to illustrate the above methods for solving the (LMVFIE's). In all case we chose *f(x)* in such a way that we know the exact solution. This exact solution is used only to show that the numerical solution obtained with our method is correct. Then, in such example, we calculate the errors at some points. We solve these examples by using MATLAB version 7.11.0**.**

Example 1: Consider the (LMVFIE) of the second kind:

$$u(x) = (cosx - 1)x^2 + (2cos1 - cosx - sin1 - 1)x + 2sinx + \int_0^x (x^2 - t)u(t)\,dt + \int_0^1 (xt + x)u(t)\,dt$$

for which the exact solution is *u(x) = sin(x)*. Tables 1.1, 1.2 and Figures 1.1, 1.2, 1.3 and 2.1, 2.2, 2.3 show that the approximation and exact solution by using Simpson's 3/8 method, Durand's method and Weddle method respectively for n= 5, 7.

Table 1.1: *The Error of Example 1 by using Simpson's 3/8 rule, Durand's rule and Weddle rule respectively with n=5,7*

x	Error simpson's3/8 $=\lvert u - u_n \rvert$	Error Durand's rule	Error Weddle rule
0	0.000e+000	0.000e+000	0.000e+000
0.067	2.098e-005	5.385e-005	5.211e-005
0.133	5.659e-005	8.958e-006	1.039e-004
0.2	8.438e-005	4.270e-005	1.542e-004
0.267	1.117e-004	3.695e-005	2.169e-004
0.333	9.014e-005	3.144e-005	2.788e-004
0.4	1.669e-004	2.623e-005	3.404e-004
0.467	2.036e-004	2.140e-005	4.024e-004
0.533	2.215e-004	1.697e-005	4.655e-004
0.6	2.510e-004	1.292e-005	5.308e-004
0.667	2.811e-004	9.213e-006	5.998e-004
0.733	5.006e-004	5.785e-006	6.739e-004
0.8	3.500e-004	2.544e-006	7.550e-004
0.867	6.957e-004	6.238e-007	8.455e-004
0.933	4.388e-004	3.856e-006	9.479e-004
1	4.971e-004	7.312e-006	1.066e-003

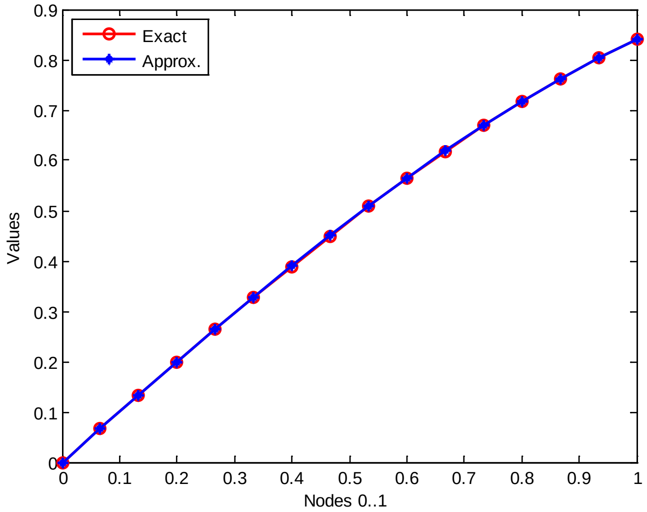

Figure 1.1: shows both the exact and the approximate Simpson's 3/8 rule with n =5

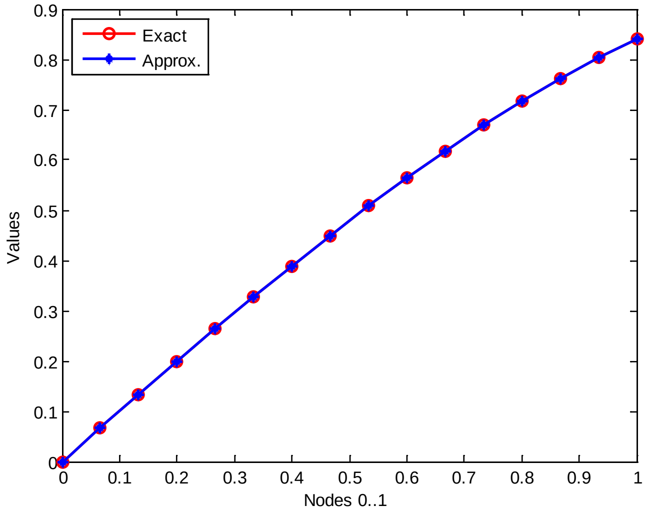

Figure 1.2: compares the exact solution $u(x) = sinx$ with the approximate Durand's rule with n=5

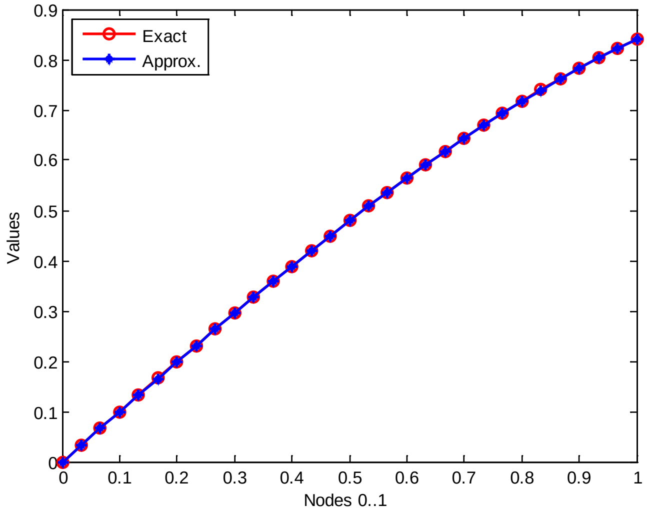

***Figure 1.3:** compares the exact solution with the approximate Weddle solution.*

Table 1.2: With n=7

x	Error simpson's3/8 $=\|u-u_n\|$	Error Durand's rule	Error Weddle rule
0	0.000e+000	0.000e+000	0.000e+000
0.048	4.253e-006	1.673e-005	2.071e-005
0.095	2.744e-005	2.517e-006	4.134e-005
0.143	4.103e-005	6.970e-006	6.167e-005
0.19	5.447e-005	2.071e-006	8.642e-005
0.238	4.357e-005	2.772e-006	1.109e-004
0.286	8.134e-005	7.547e-006	1.352e-004
0.333	8.549e-005	1.225e-005	1.593e-004
0.381	1.078e-004	1.687e-005	1.834e-004
0.429	1.210e-004	2.143e-005	2.074e-004
0.476	1.343e-004	2.595e-005	2.317e-004
0.524	1.850e-004	3.044e-005	2.564e-004
0.571	1.616e-004	3.494e-005	2.816e-004
0.619	2.440e-004	3.950e-005	3.077e-004
0.667	1.905e-004	4.416e-005	3.350e-004
0.714	2.072e-004	4.897e-005	3.637e-004
0.762	2.243e-004	5.401e-005	3.943e-004
0.810	3.878e-004	5.935e-005	4.271e-004
0.857	2.630e-004	6.507e-005	4.627e-004
0.905	4.761e-004	7.127e-005	5.015e-004
0.952	3.106e-004	7.805e-005	5.443e-004
1	3.390e-004	8.555e-005	5.916e-004

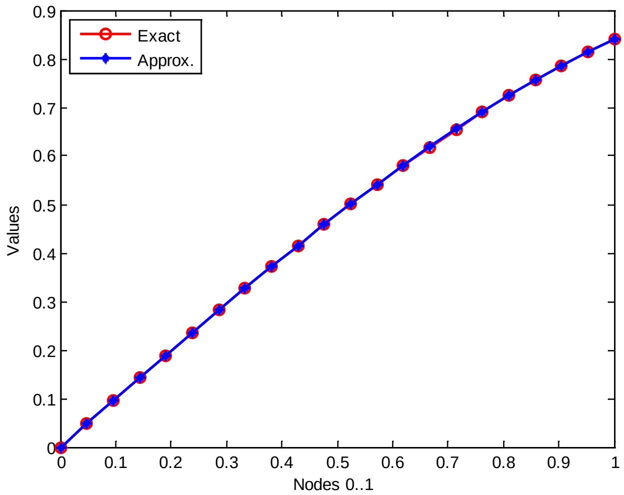

Figure 2.1: compares the exact solution $u(x) = sinx$ with the approximate Simpson's 3/8 solution with n=7

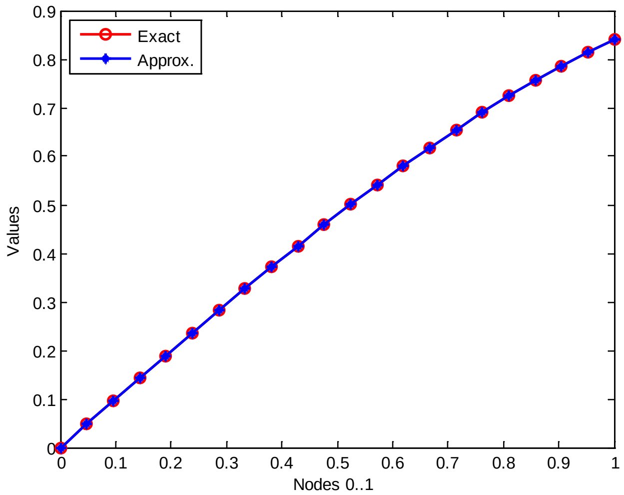

Figure 2.2: shows the exact solution $u(x) = sinx$ with the approximate Durand's rule when n= 7

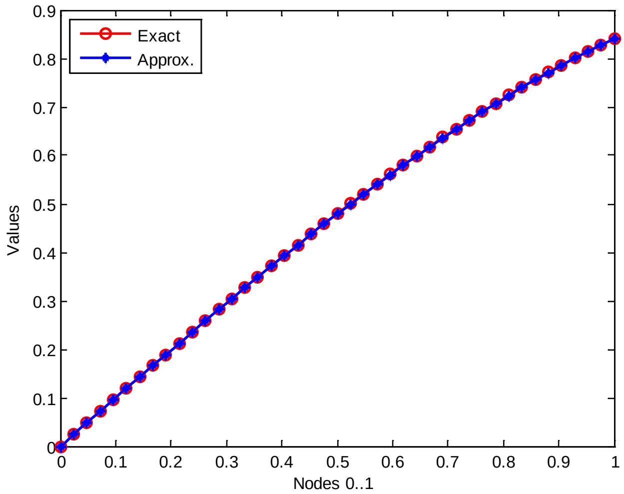

Figure 2.3: *shows the exact solution u(x)= sinx with the approximate Weddle solution when n= 7.*

CONCLUSIONS

Which obtain from the illustrative example, we conclude that:

1. The proposed numerical methods are efficient and accurate to estimate the solution of these equations.
2. In most cases, the (LMVFIE's) are usually difficult to solve analytically, so we can solved by approximation method.
3. In this work the tables are appointed the common points of comparison between the three methods.
4. The Durand's method gives better accuracy than other methods.
5. When n increase, we notice the values h decrease and the error decrease.
6. This methods can be applied to nonlinear (MVFIE’s).

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