International Journal of Mathematical Archive-6(11), 2015, 31-35
IJMA Available online through www.ijma.info ISSN 2229 – 5046

BALANCED DOMINATION NUMBER OF A TREE

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(Received On: 18-10-15; Revised & Accepted On: 12-11-15)

ABSTRACT

Let G= (V, E) be a graph. A Subset D of V is called a dominating set of G if every vertex in V-D is adjacent to atleast one vertex in D. The Domination number γ (G) of G is the cardinality of the minimum dominating set of G. Let G = (V, E) be a graph and let f be a function that assigns to each vertex of V to a set of values from the set {1,2,.......k} that is, f:V(G)→ {1,2,.....k} such that fo r each u,v ∈ V(G), f(u)≠f(v), if u is adjacent to v in G. Then the dominating set D ⊆V (G) is called a balanced dominating set if $\sum_{u \epsilon D} f(u) = \sum_{v \epsilon V-D} f(v)$*. In this paper, this new parameter is going to be analyzed for trees. If T is a tree with order n≥ 3 and l leaves,* $\gamma_{bd}(T) \leq \gamma(T) + l - 1$ *and for s support vertices,* $\gamma_{bd}(T) \leq (n+s)/2$*.*

***Keywords**: balanced domination number, leaf, tree.*

***Mathematics subject classification:** 05C69.*

1. INTRODUCTION

Let G = (V, E) be a graph with vertex set V and edge set E. The degree of v denoted by deg_G (v) is the number of vertices adjacent to v in G. A vertex of degree one is called a leaf and its neighbor is a support vertex.

Let G = (V, E) be a graph and let f be a function that assigns to each vertex of V to a set of values from the set {1,2,.......k} that is, f:V(G) → {1,2,.....k} such that for each u,v ϵ V(G), f(u)≠f(v), if u is adjacent to v in G. Then the set D ⊆V (G) is called a balanced dominating set if $\sum_{u \epsilon D} f(u) = \sum_{v \epsilon V-D} f(v)$

The balanced domination number $\gamma_{bd}(G)$ is the minimum cardinality of the balanced dominating set.

The set D ⊆ V (G) is called strong balanced dominating set if $\sum_{u \epsilon D} f(u) \geq \sum_{v \epsilon V-D} f(v)$.Also the set D ⊆ V(G) is called weak balanced dominating set if $\sum_{u \epsilon D} f(u) \leq \sum_{v \epsilon V-D} f(v)$

The sum of the values assigned to each vertex of G is called the total value of G. that is,
Total value = f (V) = $\sum_{v \epsilon V(G)} f(v)$.

Definition 1.1: The distance d(x, y) between two vertices x and y is the length of the shortest path from x to y considering all possible paths in G from x to y.

Definition 1.2: The eccentricity of vertex v is ecc(v) = max{d(v, w); w $\in V$}.The radius of G is rad(G) = min {ecc(v);v $\in V$}.The diameter of G is diam(G)= max{ecc(v);v$\in V$}.

Definition 1.3: If one vertex of a tree is singled out as a starting point and all the branches fan out from the vertex, we call such a tree a rooted tree.

Definition 1.4: In a rooted tree, the parent of a vertex is the vertex connected to it on the path to the root; every vertex except the root has a unique parent. A child of a vertex v is a vertex of which v is the parent.

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Definition 1.5: A dominating set S is an independent dominating set if no two vertices are adjacent that is, S is an independent set. The independent domination number i (G) of a graph G is the minimum cardinality of an independent dominating set.

Theorem 1.6: Let G be a graph with n vertices. Then G has a balanced dominating set iff f (V) = $\sum_{v\epsilon V(G)} f(v)$ is even. Proved in [6].

Theorem 1.7: Let G be a graph with n vertices. Then G has no balanced dominating set iff f (V) = $\sum_{v\epsilon V(G)} f(v)$ is odd. Proved in [6].

2. UPPER BOUNDS

Theorem 2.1: For any nontrivial tree T, if $T=P_n$ then $\gamma_{bd}(T) \leq 2i(T)$.

Proof: Let $T=P_n$.

We partition the vertices of T into two disjoint i (T)-sets D and D'.

Therefore, $\gamma_{bd}(T) \leq |D \cup D'|$
$\leq |D| + |D'|$
$\leq 2i(T)$.

Hence $\gamma_{bd}(T) \leq 2i(T)$.

Theorem 2.2: If $T=K_{1,\, n-1}$ then $\gamma_{bd}(T) = \begin{cases} \frac{\Delta}{2} & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$.

Proof:

Case-1: n is odd

For $K_{1,\, n-1}$, we have two partition, that is, D_1 having one vertex of value 1 and D_2 having n-1 vertices of value 2.

$$\sum_{v\epsilon V(G)} f(v) = 2+n-1 = n+1.$$

$D_1 U (D_2/2) - 1$ form a balanced dominating set of T.

Therefore, $\gamma_{bd}(T) = |D_1 \, U \frac{D_2}{2} - 1|$
$= |D_1| + |\frac{D_2}{2}| - 1$
$= 1 + \frac{n-1}{2} - 1$
$= \frac{n-1}{2}$

$\gamma_{bd}(T) = \frac{n-1}{2} = \frac{\Delta}{2}$.

Case-2: n is even
$\sum_{v\epsilon V(G)} f(v) = 2+n-1 = n+1$.

Therefore, $\sum_{v\epsilon V(G)} f(v)$ is odd.

Therefore T has no balanced dominating set.

Hence $\gamma_{bd}(T) = 0$.

Theorem 2.3: If T is a tree of order atleast three with *l leaves* then

$$\gamma_{bd}(T) \leq \gamma(T) + l - 1.$$

Proof: To establish the upper bound, we proceed by induction on the order of T. It is obvious for n ∈ {3, 4, 5}. Let n ≥ 6.

Assume that for any tree T' of order $3 \leq n' < n$ having l' leaves,

$$\gamma_{bd}(T') \leq \gamma(T') + l' - 1.$$

Let T be a tree of order n with l leaves. Let S and D be $\gamma_{bd}(T)$-set and γ(T)- set respectively. If T is a star $K_{1,\, n-1}$, we have $\gamma_{bd}(T) = \begin{cases} \frac{\Delta}{2} & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$.

Therefore $\gamma_{bd}(T)$=l/2 and γ(T) =1.hence $\gamma_{bd}(T) \leq \gamma(T) + l - 1$.

Hence we may assume diam (T) ≥ 4.

If any support vertex say t, is adjacent to two or more leaves,

Then T' be the tree obtained from T by removing a leaf adjacent to t.

$\gamma_{bd}(T')=0$ *or* $\gamma_{bd}(T') \leq \gamma_{bd}(T)$, γ(T') = γ(T) and l'=l-1.

Applying inductive hypothesis to T', we get $\gamma_{bd}(T') \leq \gamma(T') + l' - 1$.

Hence $\gamma_{bd}(T) \leq \gamma(T) + l - 1$.

We can assume that every support vertex of T is adjacent to exactly one leaf.

We now root the tree at a vertex r at maximum eccentricity diam (T) ≥ 4.

Let u be a support vertex of maximum distance from r and v be the parent of u in the rooted tree. Then $\deg_T(u)$ =2.

Let w be the parent of v and x be the parent of w. By our choice of u, every child of v is either a leaf or a support vertex of degree two.

Consider the following two cases:

Case-1: The child of v is a support vertex of degree two. V has a child besides u, say y, that is a support vertex. T' can be obtained by removing a leaf from the support vertex y.

Then $\gamma_{bd}(T') = \gamma_{bd}(T)$ - 1, γ(T') = γ(T) and l'=l.

Applying inductive hypothesis to T', we get $\gamma_{bd}(T') \leq \gamma(T') + l' - 1$.

Hence $\gamma_{bd}(T) \leq \gamma(T) + l - 1$.

Case-2: The child of v is a leaf. V is a support vertex and has no child besides u of degree two. T' can be obtained by removing a leaf from the support vertex u.

Then $\gamma_{bd}(T') = \gamma_{bd}(T)$ - 1, γ(T') = γ(T)-1 and l'=l.

Applying inductive hypothesis to T', we get $\gamma_{bd}(T') \leq \gamma(T') + l' - 1$.

Hence $\gamma_{bd}(T) \leq \gamma(T) + l - 1$.

Example 2.4:

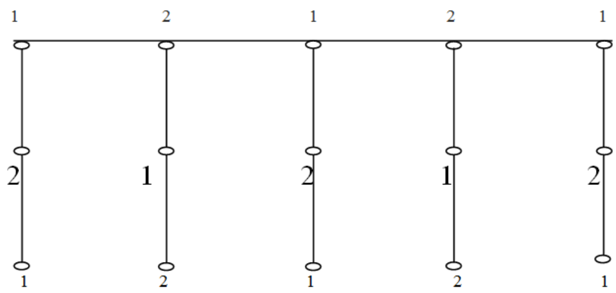

Figure-1

γ_{bd} (T) = 6

$l = 5$, $\gamma(T)$ =5

$\gamma(T)$+l – 1=5+5-1=9

$\gamma_{bd}(T) \leq \gamma(T) + l - 1$.

Note 2.5: The bound of the theorem 3 is sharp.

Example 2.6:
P_8

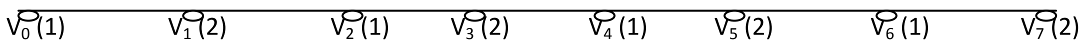

Figure-2

γ_{bd} $(P_8) = 4$

$l = 2$, $\gamma(P_8) = 3$

$\gamma(P_8)+l - 1=3+2-1=4$

$\gamma_{bd}(T) = \gamma(T) + l - 1.$

Theorem 2.7: If T is a tree of order n ≥ 3 with s support vertices then $\gamma_{bd}(T) \leq (n+s)/2$.

Proof: We proceed by induction on the order n.It is obvious that result is valid if diam (T) ∈ {2, 3} establishing the base case.

Assume that every tree T' of order 3≤ n'< n with s' support vertices satisfies $\gamma_{bd}(T') \leq (n'+s')/2$.

Let T be a tree of order n with s support vertices. We now root T at a vertex r of maximum eccentricity diam (T) ≥ 4.

Let u be a support vertex at maximum distance from r and v its parent in the rooted tree.

Since diam (T) ≥ 4, let w be the parent of v in the rooted tree. Consider the following two cases

Case-1: $\deg_T(w) \geq 3$

Then either w is a support vertex of T or w has a child besides u as a support vertex.

Let $T' = T - T_v$. clearly n'=n-(N [V]) =n-3 and s'=s-1.

There is a γ(T')-set S' containing w. Thus, S'U {u} is a balanced dominating set of T, implying that

$$\gamma_{bd}(T) \leq \gamma_{bd}(T') + 1.$$

Applying the inductive hypothesis to T', it follows that

$$\begin{aligned}\gamma_{bd}(T) &\leq \gamma_{bd}(T') + 1\\ &\leq (n'+s')/2\\ &\leq \frac{n+s-4}{2} + 1\\ &\leq (n+s)/2.\end{aligned}$$

Case-2: $\deg_T(w) = 3$

Let $T' = T - T_v$. clearly n'=n-(N [V]) =n-3 and s'=s-1.

There is a γ(T')-set S' containing w.Thus, S'U {u} is a balanced dominating set of T, implying that

$$\gamma_{bd}(T) \leq \gamma_{bd}(T') + 1.$$

Applying the inductive hypothesis to T', it follows that

$$\begin{aligned}\gamma_{bd}(T) &\leq \gamma_{bd}(T') + 1\\ &\leq (n'+s')/2\\ &\leq \frac{n+s-4}{2} + 1\\ &\leq (n+s)/2.\end{aligned}$$

Hence $\gamma_{bd}(T) \leq (n+s)/2$.

Example 2.8:

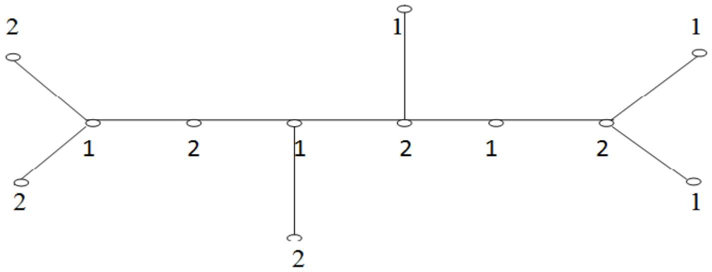

Figure-3

γ_{bd} (T) = 5

$n = 12$, s=4

$\gamma_{bd}(T) \leq$ (n+s)/2.

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Source of support: Nil, Conflict of interest: None Declared