

MORE ON THE COMPLEX REFLECTION ARRANGEMENT  $\mathcal{A}(G_{25})$

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ABSTRACT

In this paper we study the complex reflection arrangement  $\mathcal{A}(G_{25})$  and we found the bases, circuits, broken circuits, no broken circuits base, the triple arrangements, poincaré polynomial, and the characteristic polynomial of  $\mathcal{A}(G_{25})$ .

**Key word:** Arrangement, bases, circuit, broken circuit.

1. INTRODUCTION

An arrangement of hyperplanes is a finite collection of codimension one subspace in a finite dimensional vector space over  $\mathbb{R}(\mathbb{C})$ . We fix a linear order on  $\mathcal{A}$  and we write  $\mathcal{A} = \{H_1, \dots, H_n\}$ . Let  $L = L(\mathcal{A})$  be the intersection poset of  $\mathcal{A}$ .  $L$  is the set of non-empty intersections of hyperplanes in  $\mathcal{A}$  ordered by reverse inclusion. By convention  $L$  includes  $V$  as its unique minimal element. Then  $L$  is a ranked poset, with  $r(X) = \text{codim}(X)$ , and all maximal elements have the same rank. The rank of  $\mathcal{A}$  is then the rank of any maximal element of  $L$ , denoted by  $r$ . The arrangement  $\mathcal{A}$  is called centered with center  $T(\mathcal{A})$  if  $T(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} H \neq \emptyset$ . If  $\mathcal{A}$  is centered, then the coordinate may be chosen so that each hyperplane contains the origin and hence  $\mathcal{A}$  is central. We call  $\mathcal{A}$  centerless if the intersection of the finite family  $\mathcal{A}$  is empty, i.e.  $\bigcap_{H \in \mathcal{A}} H = \emptyset$ .

2. PRELIMINARY DEFINITIONS

**Definition (2.1)** [5][6]: A finite poset  $L$  is said to be geometric lattice if it satisfies the following conditions:

- (1)  $L$  is a lattice (hence has a minimum  $\hat{0}$  and a maximum  $\hat{1}$ ).
- (2) Every  $X \in L$  is a join of atoms.
- (3) If  $X, Y \in L$  cover  $X \wedge Y$  then  $X \vee Y$  covers both  $X$  and  $Y$ .

**Definition (2.2)** [1]: Let  $L$  be a geometric lattice of rank  $r$  and let  $L_1$  be the set of atoms. A set  $B = \{b_1, b_2, \dots, b_n\} \subseteq L_1$  is said to be a **base** of  $L$  if  $n = r$  and,  $\bigvee B = \hat{1}$ .

**Note:**  $\bigvee B$  the intersection of all elements of  $B$

i.e.  $(\bigvee B = b_1 \cap b_2 \cap \dots \cap b_n)$ .

**Definition (2.3)** [10]: Call  $B \subseteq L$  **independent set**, if  $r(\bigvee B) = |B|$  and **dependent** if  $r(\bigvee B) < |B|$ , where  $|B|$  is the cardinal number of  $B$ .

**Definition (2.4)** [2][8]: A **circuit**  $C \subseteq \mathcal{A}$  is a minimal dependent sub arrangement (with respect to the inclusion) and for a given total order  $\preceq$  of the hyperplanes of  $\mathcal{A}$ ,  $\bar{C} = C \setminus \{H\}$  is said to be a **broken circuit of  $\mathcal{A}$** , where  $H$  is the smallest hyperplane in  $C$  with respect to the ordering  $\preceq$ . The sub-arrangement  $B \subseteq \mathcal{A}$  which contains no broken circuit is denoted by **NBC base of  $\mathcal{A}$**  and if  $r(B) = k$  we denoted it by  **$k$ -NBC base**. Let  $X \in L$ , we call the NBC base of  $\mathcal{A}$  a **NBC base of  $X$**  if  $\bigcap_{H \in B} H = X$ .

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### 3. THE REFLECTION ARRANGEMENT $\mathcal{A}(G_{25})$

In the classification of Shephard and Todd [3]  $G_{25}$  is the complex reflection group of order 648 generated by reflections of order "3" and its reflection arrangement is given by:

$$Q(\mathcal{A}(G_{25})) = xyz \prod_{0 \leq i, j \leq 2} (x + \omega^i y + \omega^j z), \text{ where } \omega = \frac{2\pi i}{e^3}$$

Let  $H_i = \ker \alpha_{H_i}$ ,  $1 \leq i \leq 12$  such that:

$H_1 : x = 0$	$H_5 : x + y + \omega z = 0$	$H_9 : x + \omega y + \omega^2 z = 0$
$H_2 : y = 0$	$H_6 : x + y + \omega^2 z = 0$	$H_{10} : x + \omega^2 y + z = 0$
$H_3 : z = 0$	$H_7 : x + \omega y + z = 0$	$H_{11} : x + \omega^2 y + z = 0$
$H_4 : x + y + z = 0$	$H_8 : x + \omega y + \omega z = 0$	$H_{12} : x + \omega^2 y + \omega^2 z = 0$

Thus  $\mathcal{A}(G_{25})$  contains "12" hyperplanes which are:

$$\{H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9, H_{10}, H_{11}, H_{12}\}.$$

#### 3.1. Bases Of $L(\mathcal{A}(G_{25}))$

The set of atoms of  $L(\mathcal{A}(G_{25}))$  is  $L_I = \{H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9, H_{10}, H_{11}, H_{12}\}$  and the sets  $B_i \subseteq L_I$ ,  $i = 1, \dots, 185$  are bases since  $|B_i| = 3$  for all  $i = 1, \dots, 185$  and  $r = 3$ ,  $\forall B_i = \hat{I}$ ,  $i = 1, \dots, 185$ .

The bases of  $L(\mathcal{A}(G_{25}))$  are:

$\{H_1, H_2, H_3\} = B_1$	$\{H_2, H_3, H_5\} = B_{48}$	$\{H_3, H_6, H_9\} = B_{96}$	$\{H_5, H_7, H_{10}\} = B_{142}$
$\{H_1, H_2, H_4\} = B_2$	$\{H_2, H_3, H_6\} = B_{49}$	$\{H_3, H_6, H_{10}\} = B_{97}$	$\{H_5, H_7, H_{11}\} = B_{143}$
$\{H_1, H_2, H_5\} = B_3$	$\{H_2, H_3, H_7\} = B_{50}$	$\{H_3, H_6, H_{11}\} = B_{98}$	$\{H_5, H_7, H_{12}\} = B_{144}$
$\{H_1, H_2, H_6\} = B_4$	$\{H_2, H_3, H_8\} = B_{51}$	$\{H_3, H_6, H_{12}\} = B_{99}$	$\{H_5, H_8, H_9\} = B_{145}$
$\{H_1, H_2, H_7\} = B_5$	$\{H_2, H_3, H_9\} = B_{52}$	$\{H_3, H_7, H_{10}\} = B_{100}$	$\{H_5, H_8, H_{10}\} = B_{146}$
$\{H_1, H_2, H_8\} = B_6$	$\{H_2, H_3, H_{10}\} = B_{53}$	$\{H_3, H_7, H_{11}\} = B_{101}$	$\{H_5, H_8, H_{12}\} = B_{147}$
$\{H_1, H_2, H_9\} = B_7$	$\{H_2, H_3, H_{11}\} = B_{54}$	$\{H_3, H_7, H_{12}\} = B_{102}$	$\{H_5, H_9, H_{11}\} = B_{148}$
$\{H_1, H_2, H_{10}\} = B_8$	$\{H_2, H_3, H_{12}\} = B_{55}$	$\{H_3, H_8, H_{10}\} = B_{103}$	$\{H_5, H_9, H_{12}\} = B_{149}$
$\{H_1, H_2, H_{11}\} = B_9$	$\{H_2, H_4, H_5\} = B_{56}$	$\{H_3, H_8, H_{11}\} = B_{104}$	$\{H_5, H_{10}, H_{12}\} = B_{150}$
$\{H_1, H_2, H_{12}\} = B_{10}$	$\{H_2, H_4, H_6\} = B_{57}$	$\{H_3, H_8, H_{12}\} = B_{105}$	$\{H_5, H_{11}, H_{12}\} = B_{151}$
$\{H_1, H_3, H_4\} = B_{11}$	$\{H_2, H_4, H_8\} = B_{58}$	$\{H_3, H_9, H_{10}\} = B_{106}$	$\{H_6, H_7, H_8\} = B_{152}$
$\{H_1, H_3, H_5\} = B_{12}$	$\{H_2, H_4, H_9\} = B_{59}$	$\{H_3, H_9, H_{11}\} = B_{107}$	$\{H_6, H_7, H_9\} = B_{153}$
$\{H_1, H_3, H_6\} = B_{13}$	$\{H_2, H_4, H_{11}\} = B_{60}$	$\{H_3, H_9, H_{12}\} = B_{108}$	$\{H_6, H_7, H_{10}\} = B_{154}$
$\{H_1, H_3, H_7\} = B_{14}$	$\{H_2, H_4, H_{12}\} = B_{61}$	$\{H_4, H_5, H_7\} = B_{109}$	$\{H_6, H_7, H_{12}\} = B_{155}$
$\{H_1, H_3, H_8\} = B_{15}$	$\{H_2, H_5, H_6\} = B_{62}$	$\{H_4, H_5, H_8\} = B_{110}$	$\{H_6, H_8, H_9\} = B_{156}$
$\{H_1, H_3, H_9\} = B_{16}$	$\{H_2, H_5, H_7\} = B_{63}$	$\{H_4, H_5, H_9\} = B_{111}$	$\{H_6, H_8, H_{10}\} = B_{157}$
$\{H_1, H_3, H_{10}\} = B_{17}$	$\{H_2, H_5, H_9\} = B_{64}$	$\{H_4, H_5, H_{10}\} = B_{112}$	$\{H_6, H_8, H_{11}\} = B_{158}$
$\{H_1, H_3, H_{11}\} = B_{18}$	$\{H_2, H_5, H_{10}\} = B_{65}$	$\{H_4, H_5, H_{11}\} = B_{113}$	$\{H_6, H_8, H_{12}\} = B_{159}$
$\{H_1, H_3, H_{12}\} = B_{19}$	$\{H_2, H_5, H_{12}\} = B_{66}$	$\{H_4, H_5, H_{12}\} = B_{114}$	$\{H_6, H_9, H_{10}\} = B_{160}$
$\{H_1, H_4, H_5\} = B_{20}$	$\{H_2, H_6, H_7\} = B_{67}$	$\{H_4, H_6, H_7\} = B_{115}$	$\{H_6, H_9, H_{11}\} = B_{161}$
$\{H_1, H_4, H_6\} = B_{21}$	$\{H_2, H_6, H_8\} = B_{68}$	$\{H_4, H_6, H_8\} = B_{116}$	$\{H_6, H_{10}, H_{11}\} = B_{162}$
$\{H_1, H_4, H_7\} = B_{22}$	$\{H_2, H_6, H_{10}\} = B_{69}$	$\{H_4, H_6, H_9\} = B_{117}$	$\{H_6, H_{10}, H_{12}\} = B_{163}$
$\{H_1, H_4, H_9\} = B_{23}$	$\{H_2, H_6, H_{11}\} = B_{71}$	$\{H_4, H_6, H_{10}\} = B_{118}$	$\{H_6, H_{11}, H_{12}\} = B_{164}$
$\{H_1, H_4, H_{10}\} = B_{24}$	$\{H_2, H_7, H_8\} = B_{72}$	$\{H_4, H_6, H_{11}\} = B_{119}$	$\{H_7, H_8, H_{10}\} = B_{165}$
$\{H_1, H_4, H_{11}\} = B_{25}$	$\{H_2, H_7, H_9\} = B_{73}$	$\{H_4, H_6, H_{12}\} = B_{120}$	$\{H_7, H_8, H_{11}\} = B_{166}$
$\{H_1, H_5, H_6\} = B_{26}$	$\{H_2, H_7, H_{11}\} = B_{74}$	$\{H_4, H_7, H_8\} = B_{121}$	$\{H_7, H_8, H_{12}\} = B_{167}$
$\{H_1, H_5, H_7\} = B_{27}$	$\{H_2, H_7, H_{12}\} = B_{75}$	$\{H_4, H_7, H_9\} = B_{122}$	$\{H_7, H_9, H_{10}\} = B_{168}$
$\{H_1, H_5, H_8\} = B_{28}$	$\{H_2, H_8, H_9\} = B_{76}$	$\{H_4, H_7, H_{11}\} = B_{123}$	$\{H_7, H_9, H_{11}\} = B_{169}$
$\{H_1, H_5, H_{11}\} = B_{29}$	$\{H_2, H_8, H_{10}\} = B_{77}$	$\{H_4, H_7, H_{12}\} = B_{124}$	$\{H_7, H_9, H_{12}\} = B_{170}$
$\{H_1, H_5, H_{12}\} = B_{30}$	$\{H_2, H_8, H_{12}\} = B_{78}$	$\{H_4, H_8, H_9\} = B_{125}$	$\{H_7, H_{10}, H_{11}\} = B_{171}$
$\{H_1, H_6, H_8\} = B_{31}$	$\{H_2, H_9, H_{10}\} = B_{79}$	$\{H_4, H_8, H_{10}\} = B_{126}$	$\{H_7, H_{10}, H_{12}\} = B_{172}$
$\{H_1, H_6, H_9\} = B_{32}$	$\{H_2, H_9, H_{11}\} = B_{80}$	$\{H_4, H_8, H_{11}\} = B_{127}$	$\{H_7, H_{11}, H_{12}\} = B_{173}$
$\{H_1, H_6, H_{10}\} = B_{33}$	$\{H_2, H_{11}, H_{12}\} = B_{81}$	$\{H_4, H_8, H_{12}\} = B_{128}$	$\{H_8, H_9, H_{10}\} = B_{174}$
$\{H_1, H_6, H_{12}\} = B_{34}$	$\{H_3, H_4, H_7\} = B_{82}$	$\{H_4, H_9, H_{10}\} = B_{129}$	$\{H_8, H_9, H_{11}\} = B_{175}$
$\{H_1, H_7, H_8\} = B_{35}$	$\{H_3, H_4, H_8\} = B_{83}$	$\{H_4, H_9, H_{11}\} = B_{130}$	$\{H_8, H_9, H_{12}\} = B_{176}$
$\{H_1, H_7, H_9\} = B_{36}$	$\{H_3, H_4, H_9\} = B_{84}$	$\{H_4, H_9, H_{12}\} = B_{130}$	$\{H_8, H_{10}, H_{11}\} = B_{177}$
$\{H_1, H_7, H_{10}\} = B_{37}$	$\{H_3, H_4, H_{10}\} = B_{85}$	$\{H_4, H_{10}, H_{11}\} = B_{131}$	$\{H_8, H_{10}, H_{12}\} = B_{178}$
$\{H_1, H_7, H_{12}\} = B_{38}$	$\{H_3, H_4, H_{11}\} = B_{86}$	$\{H_4, H_{10}, H_{12}\} = B_{132}$	$\{H_8, H_{11}, H_{12}\} = B_{179}$

$\{H_1, H_8, H_9\} = B_{39}$	$\{H_3, H_4, H_{12}\} = B_{87}$	$\{H_4, H_{11}, H_{12}\} = B_{133}$	$\{H_9, H_{10}, H_{11}\} = B_{180}$
$\{H_1, H_8, H_{10}\} = B_{40}$	$\{H_3, H_5, H_7\} = B_{88}$	$\{H_5, H_6, H_7\} = B_{134}$	$\{H_9, H_{10}, H_{12}\} = B_{181}$
$\{H_1, H_8, H_{11}\} = B_{41}$	$\{H_3, H_5, H_8\} = B_{89}$	$\{H_5, H_6, H_8\} = B_{135}$	$\{H_9, H_{11}, H_{12}\} = B_{182}$
$\{H_1, H_9, H_{11}\} = B_{42}$	$\{H_3, H_5, H_9\} = B_{90}$	$\{H_5, H_6, H_9\} = B_{136}$	$\{H_2, H_{10}, H_{12}\} = B_{183}$
$\{H_1, H_9, H_{12}\} = B_{43}$	$\{H_3, H_5, H_{10}\} = B_{91}$	$\{H_5, H_6, H_{10}\} = B_{137}$	$\{H_2, H_{11}, H_{12}\} = B_{184}$
$\{H_1, H_{10}, H_{11}\} = B_{44}$	$\{H_3, H_5, H_{11}\} = B_{92}$	$\{H_5, H_6, H_{11}\} = B_{138}$	$\{H_4, H_{10}, H_{12}\} = B_{185}$
$\{H_1, H_{10}, H_{12}\} = B_{45}$	$\{H_3, H_5, H_{12}\} = B_{93}$	$\{H_5, H_6, H_{12}\} = B_{139}$	
$\{H_1, H_{11}, H_{12}\} = B_{46}$	$\{H_3, H_6, H_7\} = B_{94}$	$\{H_5, H_7, H_8\} = B_{140}$	
$\{H_2, H_3, H_4\} = B_{47}$	$\{H_3, H_6, H_8\} = B_{95}$	$\{H_5, H_7, H_9\} = B_{141}$	

### 3.2. The Circuits and Broken Circuits of $\mathcal{A}(G_{25})$

We will give all the minimal dependent subsets of  $L_I$  (set of atoms) with  $\leq$  in the following table:

$\{H_1, H_4, H_8\} = C_1$	$\{H_2, H_5, H_{11}\} = C_{13}$	$\{H_3, H_{10}, H_{11}\} = C_{25}$
$\{H_1, H_4, H_{12}\} = C_2$	$\{H_2, H_6, H_9\} = C_{14}$	$\{H_3, H_{10}, H_{12}\} = C_{26}$
$\{H_1, H_5, H_9\} = C_3$	$\{H_2, H_6, H_{12}\} = C_{15}$	$\{H_3, H_{11}, H_{12}\} = C_{27}$
$\{H_1, H_5, H_{10}\} = C_4$	$\{H_2, H_7, H_{10}\} = C_{16}$	$\{H_4, H_5, H_6\} = C_{28}$
$\{H_1, H_6, H_7\} = C_5$	$\{H_2, H_8, H_{11}\} = C_{17}$	$\{H_4, H_7, H_{10}\} = C_{29}$
$\{H_1, H_6, H_{11}\} = C_6$	$\{H_2, H_9, H_{12}\} = C_{18}$	$\{H_5, H_8, H_{11}\} = C_{30}$
$\{H_1, H_7, H_{11}\} = C_7$	$\{H_3, H_4, H_5\} = C_{19}$	$\{H_5, H_9, H_{10}\} = C_{31}$
$\{H_1, H_8, H_{12}\} = C_8$	$\{H_3, H_4, H_6\} = C_{20}$	$\{H_6, H_7, H_{11}\} = C_{32}$
$\{H_1, H_9, H_{10}\} = C_9$	$\{H_3, H_5, H_6\} = C_{21}$	$\{H_6, H_9, H_{12}\} = C_{33}$
$\{H_2, H_4, H_7\} = C_{10}$	$\{H_3, H_7, H_8\} = C_{22}$	$\{H_7, H_8, H_9\} = C_{34}$
$\{H_2, H_4, H_{10}\} = C_{11}$	$\{H_3, H_7, H_9\} = C_{23}$	$\{H_{10}, H_{11}, H_{12}\} = C_{35}$
$\{H_2, H_5, H_8\} = C_{12}$	$\{H_3, H_8, H_9\} = C_{24}$	

All these subsets which we listed above are called circuits. Since  $r(VB) < |B|$ .

Thus the Broken circuits of  $\mathcal{A}(G_{25})$  are:

$\{H_4, H_8\} = \bar{C}_1$	$\{H_5, H_{11}\} = \bar{C}_{13}$	$\{H_{10}, H_{11}\} = \bar{C}_{25}$
$\{H_4, H_{12}\} = \bar{C}_2$	$\{H_6, H_9\} = \bar{C}_{14}$	$\{H_{10}, H_{12}\} = \bar{C}_{26}$
$\{H_5, H_9\} = \bar{C}_3$	$\{H_6, H_{12}\} = \bar{C}_{15}$	$\{H_{11}, H_{12}\} = \bar{C}_{27}$
$\{H_5, H_{10}\} = \bar{C}_4$	$\{H_7, H_{10}\} = \bar{C}_{16}$	$\{H_5, H_6\} = \bar{C}_{28}$
$\{H_6, H_7\} = \bar{C}_5$	$\{H_8, H_{11}\} = \bar{C}_{17}$	$\{H_7, H_{10}\} = \bar{C}_{29}$
$\{H_6, H_{11}\} = \bar{C}_6$	$\{H_9, H_{12}\} = \bar{C}_{18}$	$\{H_8, H_{11}\} = \bar{C}_{30}$
$\{H_7, H_{11}\} = \bar{C}_7$	$\{H_4, H_5\} = \bar{C}_{19}$	$\{H_9, H_{10}\} = \bar{C}_{31}$
$\{H_8, H_{12}\} = \bar{C}_8$	$\{H_4, H_6\} = \bar{C}_{20}$	$\{H_7, H_{11}\} = \bar{C}_{32}$
$\{H_9, H_{10}\} = \bar{C}_9$	$\{H_5, H_6\} = \bar{C}_{21}$	$\{H_9, H_{12}\} = \bar{C}_{33}$
$\{H_4, H_7\} = \bar{C}_{10}$	$\{H_7, H_8\} = \bar{C}_{22}$	$\{H_8, H_9\} = \bar{C}_{34}$
$\{H_4, H_{10}\} = \bar{C}_{11}$	$\{H_7, H_9\} = \bar{C}_{23}$	$\{H_{11}, H_{12}\} = \bar{C}_{35}$
$\{H_5, H_8\} = \bar{C}_{12}$	$\{H_8, H_9\} = \bar{C}_{24}$	

### 3.3. No Broken Circuits Base Of $L(\mathcal{A}(G_{25}))$

We will give the NBC base of elements of  $L(\mathcal{A}(G_{25}))$  of rank two

The elements of  $L(\mathcal{A}(G_{25}))$  of rank two are:

1) $H_1 \cap H_4 \cap H_8 \cap H_{12} = a_1$	8) $H_3 \cap H_7 \cap H_8 \cap H_9 = a_8$	15) $H_5 \cap H_7 = a_{15}$
2) $H_1 \cap H_5 \cap H_9 \cap H_{10} = a_2$	9) $H_3 \cap H_{10} \cap H_{11} \cap H_{12} = a_9$	16) $H_5 \cap H_{12} = a_{16}$
3) $H_1 \cap H_6 \cap H_7 \cap H_{11} = a_3$	10) $H_1 \cap H_2 = a_{10}$	17) $H_6 \cap H_8 = a_{17}$
4) $H_2 \cap H_4 \cap H_7 \cap H_{10} = a_4$	11) $H_1 \cap H_3 = a_{11}$	18) $H_6 \cap H_{10} = a_{18}$
5) $H_2 \cap H_5 \cap H_8 \cap H_{11} = a_5$	12) $H_2 \cap H_3 = a_{12}$	19) $H_7 \cap H_{12} = a_{19}$
6) $H_2 \cap H_6 \cap H_9 \cap H_{12} = a_6$	13) $H_4 \cap H_9 = a_{13}$	20) $H_8 \cap H_{10} = a_{20}$
7) $H_3 \cap H_4 \cap H_5 \cap H_6 = a_7$	14) $H_4 \cap H_{11} = a_{14}$	21) $H_9 \cap H_{11} = a_{21}$

Now we will give the NBC base of elements of  $L(\mathcal{A}(G_{25}))$  of rank two as follows:

Elements of $L(\mathcal{A}(G_{25}))$ of rank two	The NBC bases	The cardinal number of NBC bases
a <sub>1</sub>	{H <sub>1</sub> ,H <sub>4</sub> },{H <sub>1</sub> ,H <sub>8</sub> },{H <sub>1</sub> ,H <sub>12</sub> }	3
a <sub>2</sub>	{H <sub>1</sub> ,H <sub>5</sub> },{H <sub>1</sub> ,H <sub>9</sub> },{H <sub>1</sub> ,H <sub>10</sub> }	3
a <sub>3</sub>	{H <sub>1</sub> ,H <sub>6</sub> },{H <sub>1</sub> ,H <sub>7</sub> },{H <sub>1</sub> ,H <sub>11</sub> }	3
a <sub>4</sub>	{H <sub>2</sub> ,H <sub>4</sub> },{H <sub>2</sub> ,H <sub>7</sub> },{H <sub>2</sub> ,H <sub>10</sub> }	3
a <sub>5</sub>	{H <sub>2</sub> ,H <sub>5</sub> },{H <sub>2</sub> ,H <sub>8</sub> },{H <sub>2</sub> ,H <sub>11</sub> }	3
a <sub>6</sub>	{H <sub>2</sub> ,H <sub>6</sub> },{H <sub>2</sub> ,H <sub>9</sub> },{H <sub>2</sub> ,H <sub>12</sub> }	3
a <sub>7</sub>	{H <sub>3</sub> ,H <sub>4</sub> },{H <sub>3</sub> ,H <sub>5</sub> },{H <sub>3</sub> ,H <sub>6</sub> }	3
a <sub>8</sub>	{H <sub>3</sub> ,H <sub>7</sub> },{H <sub>3</sub> ,H <sub>8</sub> },{H <sub>3</sub> ,H <sub>9</sub> }	3
a <sub>9</sub>	{H <sub>3</sub> ,H <sub>10</sub> },{H <sub>3</sub> ,H <sub>11</sub> },{H <sub>3</sub> ,H <sub>12</sub> }	3
a <sub>10</sub>	{H <sub>1</sub> ,H <sub>2</sub> }	1
a <sub>11</sub>	{H <sub>1</sub> ,H <sub>3</sub> }	1
a <sub>12</sub>	{H <sub>2</sub> ,H <sub>3</sub> }	1
a <sub>13</sub>	{H <sub>4</sub> ,H <sub>9</sub> }	1
a <sub>14</sub>	{H <sub>4</sub> ,H <sub>11</sub> }	1
a <sub>15</sub>	{H <sub>5</sub> ,H <sub>7</sub> }	1
a <sub>16</sub>	{H <sub>5</sub> ,H <sub>12</sub> }	1
a <sub>17</sub>	{H <sub>6</sub> ,H <sub>8</sub> }	1
a <sub>18</sub>	{H <sub>6</sub> ,H <sub>10</sub> }	1
a <sub>19</sub>	{H <sub>7</sub> ,H <sub>12</sub> }	1
a <sub>20</sub>	{H <sub>8</sub> ,H <sub>10</sub> }	1
a <sub>21</sub>	{H <sub>9</sub> ,H <sub>11</sub> }	1

**Definition (3.1) [11]:** Let  $\mathcal{A}$  be an arrangement.

1. A subset  $\mathcal{B}$  of  $\mathcal{A}$ , is called **sub-arrangement** of  $\mathcal{A}$ .
2. For any subspace  $X \in L_{\mathcal{A}}$  there are two smaller arrangements associated to it.
  - (i)  $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$ , is a sub-arrangement of  $\mathcal{A}$ , called **localization arrangement**.
  - (ii)  $\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} - \mathcal{A}_X \text{ and } X \cap H \neq \emptyset\}$ , is the arrangement within the vector space  $X$  called **restriction arrangement**.

**Definition (3.2) [2]:** Let  $\mathcal{A}$  be an arrangement and let  $H \in \mathcal{A}$ , let  $\mathcal{A}' = \mathcal{A} - \{H\}$  and  $\mathcal{A}^H$ . We call  $(\mathcal{A}, \mathcal{A}', \mathcal{A}^H)$  a **triple of arrangements** and  $H$  the **distinguished hyperplane**.

### 3.4. The Triple Arrangements of $\mathcal{A}(G_{25})$

Given the reflection arrangement

$$\mathcal{A}(G_{25}) = \{H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9, H_{10}, H_{11}, H_{12}\}.$$

Let  $H_1$  be the distinguished hyperplane then

$$\mathcal{A}'(G_{25}) = \{H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9, H_{10}, H_{11}, H_{12}\}.$$

By reverses inclusion as subspace of  $V$  we get  $\mathcal{A}_{H_1} = \{H_1\}$  and

$$\mathcal{A}^{H_1} = \mathcal{A}^{\setminus\{H_1\}}(G_{25}) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$$

where:

$$\begin{aligned} \alpha_1 = H_1 \cap H_2: & \quad y = x = 0 \\ \alpha_2 = H_1 \cap H_3: & \quad x = z = 0 \\ \alpha_3 = H_1 \cap H_4: & \quad x = 0, y = -z \\ \alpha_4 = H_1 \cap H_5: & \quad x = 0, y = -\omega z \\ \alpha_5 = H_1 \cap H_6: & \quad x = 0, y = -\omega^2 z \end{aligned}$$

Thus the triple arrangements of  $\mathcal{A}(G_{25})$  are  $(\mathcal{A}(G_{25}), \mathcal{A}'(G_{25}), \mathcal{A}^{\setminus\{H_1\}}(G_{25}))$ .

**Definition (3.3) [7]:** Let  $\mathcal{A}$  be an arrangement and let  $L = L(\mathcal{A})$ . Define the **Möbius function**  $\mu_{\mathcal{A}} = \mu : L \times L \rightarrow Z$  as follows:

$$\begin{aligned} \mu(X, X) &= 1 && \text{if } X \in L, \\ \sum_{X \subseteq Y} \mu(X, Y) &= 0 && \text{if } X, Y \in L \text{ and } X < Y, \\ \mu(X, Y) &= 0 && \text{otherwise.} \end{aligned}$$

**Definition (3.4)** [12]: Let  $\mathcal{A}$  be an arrangement with lattice  $L$  and Möbius function. Let  $t$  be an indeterminate. Define the *poincaré polynomial* of  $\mathcal{A}$  by

$$\pi(\mathcal{A}, t) = \sum_{X \in L} \mu(X)(-t)^{r(X)}.$$

To compute the poincaré polynomial of  $\mathcal{A}(G_{25})$ , note that if  $X \in L_{\mathcal{A}(G_{25})}$  s.t  $\text{rk}(X) = 1$ , then  $\mu(X) = -1$ , if  $\text{rk}(X) = 2$ , and  $X$  is in two planes then  $\mu(X) = 1$ , and if  $\text{rk}(X) = 2$ , s.t  $X$  in four planes then  $\mu(X) = 3$ . This allows calculation of  $\mu(\{0\})$  which is equal to  $-28$ .

$$\text{Thus } \pi(\mathcal{A}(G_{25})) = 1 + 12t + 39t^2 + 28t^3.$$

The Poincaré polynomial of  $\mathcal{A}^{\wedge}(G_{25})$  can be computed by the same way after removing the distinguished hyperplane  $H_1$ .

$$\text{Thus } \pi(\mathcal{A}^{\wedge}(G_{25})) = 1 + 11t + 34t^2 + 12t^3.$$

Now since  $\pi(\mathcal{A}, t) = \pi(\mathcal{A}^{\wedge}, t) + t\pi(\mathcal{A}^{\vee}, t)$  by [4]

$$\text{Then } \pi(\mathcal{A}^{\vee}(G_{25})) = 1 + 5t + 4t^2.$$

**Definition (3.5)** [9]: Define the characteristic polynomial of  $\mathcal{A}$  by

$$\chi(\mathcal{A}, t) = t^{\ell} \pi(\mathcal{A}, -t^{-1}) = \sum_{X \in L} \mu(X)t^{\dim(X)}.$$

By applying the above equation on  $\mathcal{A}(G_{25})$  we get:

$$\chi(\mathcal{A}(G_{25}), t) = t^3 - 12t^2 + 39t - 28.$$

And the characteristic polynomial of  $\mathcal{A}^{\wedge}(G_{25})$  can be computed using the same equation by removing distinguished hyperplane  $H_1$ .

Thus

$$\chi(\mathcal{A}^{\wedge}(G_{25})) = t^3 - 11t^2 + 34t - 24.$$

So by [4] we get:

$$\chi(\mathcal{A}^{\vee}(G_{25}), t) = t^2 - 5t + 4.$$

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