International Journal of Mathematical Archive-6(10), 2015, 254-266
Available online through www.ijma.info ISSN 2229 – 5046

SUFFICIENT CONDITIONS FOR THE EXISTENCE OF FIXED POINTS OF WEAK GENERALIZED GERAGHTY CONTRACTIONS IN COMPLETE METRIC SPACES

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India *(Received On: 09-10-15; Revised & Accepted On: 30-10-15)*

ABSTRACT

We introduce the notion of (α,φ,β)*-weak generalized Geraghty contractions via triangular* α *-admissible mappings to prove some sufficient conditions for the existence of fixed points of such maps in complete metric space, where* φ *is an altering distance function and* $\beta \in S$ *where* $S=\{\beta:(0,\infty)\to[0,1)$ *satisfying* $\beta(t_n)\to 1 \Rightarrow t_n \to 0\}$. *Examples are provided to illustrate our results.*

Key words: *Fixed points, Weak contraction, Altering distance function, Geraghty type contraction. triangular* α *- admissible, complete metric space,* (α,φ,β) *- weak generalized Geraghty contraction.*

Mathematics Subject Classification (2010)*: 54H25,47H10.*

1. INTRODUCTION AND PRELIMINARIES

Most of the fixed point theorems in nonlinear analysis usually start with Banach [9] contraction principle. A huge amount of literature is witnessed on applications, generalizations and extensions of this principle carried out by several authors in different directions like weakening the hypothesis and considering different mappings. Fixed point theory is an essential tool in the study of various varieties of problems in control theory, economic theory, nonlinear analysis and global analysis.

But all the generalizations may not be from this principle. In 1989, Bakktin [8] introduced the concept of a b-metric space as a generalization of a metric space. In 1993, Czerwik[11] extended many results related to the b - metric space. In 1994, Matthews [20] introduced the concept of partial metric space in which the self distance of any point of space may not be zero. In 1996, O'Neill [28] generalized the concept of partial metric space by admitting negative distances. In 2013, Shukla [34] generalized both the concepts of b-metric and partial metric space by introducing the notation of partial b-metric spaces. Many authors recently studied the existence of fixed points of self maps in different types of metric spaces [16,35,23,32,37]. Some authors [4,20,24,30,31] obtained some fixed point theorems in b-metric spaces. Some authors proved $\alpha-\psi$ versions of certain fixed point theorems in different types of metric spaces [3,16]. Recently Samet *et.al* [29] and Jalal Hassanzadeasl [14] obtained fixed point theorems for $\alpha-\psi$ contractive mappings. Mustafa [24] gave a generalization of Banach contraction principle in complete ordered partial b-metric space by introducing the notion of a generalized $\alpha-\psi$ weakly contractive mapping. G.V.R. Babu *et.al* [5] proved coupled fixed point theorems by using (α,φ,β)-weak generalized Geraghty contraction. In 2012, Mohammad Mursaleen *et.al* [22] proved coupled fixed point theorems for α - ψ contractive type mappings in partially ordered metric spaces.

In this paper we modify the concepts of G. V. R. Babu *et.al* [6] to study sufficient conditions for the existence of fixed points of weak generalized Geraghty contractions in a complete metric space. In fact, we obtain sufficient conditions for the existence of fixed points of weak generalized Geraghty contractions in a complete metric space. A supporting example is also given. Further an open problem is also given at the end of this paper.

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Notation: G. V. R. Babu. *et.al* [6] used the following notation in their paper. (X,d) denotes a metric space. Let $T:X\to X$ be a self map of X and Fix (T) denotes the set of all fixed points of T. We denote $S=\{\beta:(0,\infty)\to[0,1)/\beta(t_n)\to 1\Rightarrow t_n\to 0\}$, and $\Phi=\{\varphi:[0,\infty)\to[0,\infty)/\varphi$ is non-decreasing, continuous and $\varphi(t)=0\Leftrightarrow t=0\}$. We call the elements of Φ as altering distance functions. Further, we use the following notation: for sub sequences $\{x_{h_n}\}$ and $\{x_{k_n}\}$ in X with $x_{h_n}\neq x_{k_n}$, we write

$$d_n=d(x_{h_n},x_{k_n}),\ \Delta_n=\frac{d(T(x_{h_n}),T(x_{k_n}))}{d_n}\ \text{and}\ \Delta_n^{\phi}=\frac{\phi(d(T(x_{h_n}),T(x_{k_n}))}{\phi(d_n)},\quad \forall\, n.$$

We denote the set of all real numbers by R, the set of all non-negative reals by R^+ and the set of all natural numbers by N.

Definition 1.1: *(B.Samet.et.al.[29]) Let $T:X\to X$ be a self map and $\alpha:X\times X\to R$ be a function. Then T is said to be α - admissible if $\alpha(x,y)\geq 1\Rightarrow\alpha(Tx,Ty)\geq 1$.*

Definition 1.2: *(E. Karapinar. et.al. [18]) An α -admissible map T is said to be triangular α - admissible if $\alpha(x,z)\geq 1$ and $\alpha(z,y)\geq 1\Rightarrow\alpha(x,y)\geq 1$.*

For more details and examples on α -admissible and triangular α -admissible maps, one can refer [17], [18] and [29].

Lemma 1.3: *(E. Karapinar. et.al. [18])*
Let $T:X\to X$ be triangular α -admissible map. Assume that there exists $x_1\in X$ such that $\alpha(x_1,Tx_1)\geq 1$. Define the sequence x_{n+1} by $x_{n+1}=Tx_n$, $n=1,2,\ldots$. Then we have $\alpha(x_n,x_m)\geq 1$ for all $m,n\in N$ with $n<m$.

Definition 1.4: *(S. H. Cho. et.al.[13]) Let (X,d) be a metric space, and let $\alpha:X\times X\to R$ be a function. A map $T:X\to X$ is called an α - Geraghty type contraction if there exists $\beta\in S$ such that*
$\alpha(x,y)\ d(Tx,Ty)\ \leq\ \beta(d(x,y))\ d(x,y)$ for all $x,y\in X$. (1.4.1)

Definition 1.5: *(G. V. R. Babu. et.al.[6]) Let (X,d) be a metric space and $T:X\to X$ be a self map. If there exist $\alpha:X\times X\to R,\varphi\in\Phi$ and $L\geq 0$ such that*
$$\alpha(x,y)\ \phi((d(Tx,Ty)))\ <\ \phi((M(x,y)))\ +\ L.N(x,y) \quad (1.5.1)$$
for all $x,y\in X, x\neq y$
where
$$M(x,y)=\ max\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}[d(x,Ty)+d(y,Tx)]\},$$
$N(x,y)=\ min\{d(x,Tx),d(x,Ty),d(y,Tx)\}$ then T is said to be an almost generalized α - contractive map w.r.to the altering distance function φ.

Definition 1.6: *(G. V. R. Babu. et.al.[6]) Let (X,d) be a metric space let $T:X\to X$ be a self map. If there exist $\alpha:X\times X\to R,\ \beta\in S,\ \phi\in\Phi$ and $L\geq 0$ such that*
$$\alpha(x,y)\ \phi((d(Tx,Ty)))\leq\beta(\phi(M(x,y)))\ \phi(M(x,y))+L.N(x,y) \quad (1.6.1)$$
for all $x,y\in X$,
where
$$M(x,y)=max\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}[d(x,Ty)+d(y,Tx)]\},$$
$N(x,y)=min\{d(x,Tx),d(x,Ty),d(y,Tx)\}$ then we say that T is (α,φ,β) - weak generalized Geraghty contractive map.

Theorem 1.7: *(G. V. R. Babu. et.al.[6]) Let T be a self map on a complete metric space X. Let $\alpha : X \times X \to R$ be a function. Assume that there exist $\varphi \in \Phi$, and $L \geq 0$ such that*

$$\alpha(x, y)\varphi(d(Tx,Ty)) < \varphi(M(x, y)) + L.N(x, y) \tag{1.7.1}$$

for all $x, y \in X, x \neq y$

where

$$M(x, y) = max\{d(x, y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\},$$

$$N(x, y) = min\{d(x,Tx), d(x,Ty), d(y,Tx)\}.$$

Further, assume that

(i) T is α - admissible, and

(ii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and set $x_n = Tx_{n-1}$ for $n = 1,2,3,...$. Then the sequence $\{x_n\}$ converges to z and z is a unique fixed point of T in X if and only if for any two sub-sequences $\{x_{h_n}\}$ and $\{x_{k_n}\}$ of $\{x_n\}$ with $x_{h_n} \neq x_{k_n}$, we have that $\Delta_n^{\varphi} \to 1 \Rightarrow d_n \to 0$, provided that T is continuous at z.

Theorem 1.8: *(G. V. R. Babu. et.al.[6]) Let (X,d) be a complete metric space, $\alpha : X \times X \to R$ be a function and let $T : X \to X$ be a self map. Suppose the following conditions hold:*

(i) T is (α,φ,β) - weak generalized Geraghty contraction;

(ii) T is triangular α - admissible,

(iii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and set $x_n = Tx_{n-1}$ for $n = 1,2,3,...$

(iv) either (a) T is continuous (or) (b) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then there exists a sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k.

Then T has a fixed point x in X, provided that β is continuous on $(0,\infty)$

2. MAIN RESULT

In this section we obtain sufficient conditions for the existence of fixed points of weak generalized Geraghty contractions in a complete space. We begin this section with the following lemma, which can be easily established.

Lemma 2.1: *Let (X,d) be a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ and*

(i) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \varepsilon$

(ii) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$

(iii) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \varepsilon$

(iv) $\lim_{k\to\infty} d(x_{m_k}, x_{n_{k+1}}) = \varepsilon$

Notation: We adopt the following notation throughout this paper. (X,d) denotes a metric space. Let $T : X \to X$ be a self map of X and Fix (T) denotes the set of all fixed points of T. We denote $S = \{\beta : (0,\infty) \to [0,1) / \beta(t_n) \to 1 \Rightarrow t_n \to 0\}$, and $\Phi = \{\varphi : [0,\infty) \to [0,\infty) / \varphi$ is non-decreasing, continuous and $\varphi(t) = 0 \Leftrightarrow t = 0\}$. We call the elements of Φ as altering distance functions. Further, we use the following notation: for any sequences $\{a_n\}$ and $\{b_n\}$ in X with $a_n \neq b_n$, we write $d_n = d(a_n,b_n), \Delta_n = \frac{d(T(a_n),T(b_n))}{d_n}$ and $\Delta_n^{\phi} = \frac{\phi(d(T(a_n),T(b_n))}{\phi(d_n)}$, $\forall n$ We denote the set of all real numbers by R, the set of all non-negative reals by R^+ and the set of all natural numbers by N.

Theorem 2.2: *Let T be a self map on a complete metric space X. Let $\alpha : X \times X \to R$ be a continuous function and $\alpha(x,x) > 1 \; \forall \; x \in X$. Assume that there exist $\varphi \in \Phi$, and $L \geq 0$ such that*

$$\alpha(x,y)\varphi(d(Tx,Ty)) < \varphi(M(x,y)) + L.N(x,y) \tag{2.2.1}$$

for all $x, y \in X, x \neq y$ where

$$M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\},$$

and

$$N(x,y) = min\{d(x,Tx), d(x,Ty), d(y,Tx)\} \tag{2.2.2}$$

Further, assume that
(i) T is α - admissible, and
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Set $x_n = Tx_{n-1}$ for $n = 1,2,3,\ldots$.
(iii)forany two sequences $\{a_n\}$ and $\{b_n\}$ of X with $a_n \neq b_n$, we have that $\Delta_n^{\varphi} \to 1$

$$\Rightarrow d_n \to 0 \tag{2.2.3}$$

Then the sequence $\{x_n\}$ converges to a point $z \in X$ and z is a fixed point of T in X. Further if y, z are fixed point of T in X, then $\alpha(y,z) < 1$ or $y = z$

Proof: Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. We define $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for each $n = 0,1,2,3,\ldots$ If $x_n = x_{n+1}$ for some $n \in N$, then $x_n = Tx_n$ and hence x_n is a fixed point of T. Hence, without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n \in N$. By using the α - admissibility of T, we have $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \;\Rightarrow \alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) = \geq 1$. Now, by mathematical induction, it is easy to see that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$. By taking $x = x_{n-1}$ and $y = x_n$ in the inequality (2.2.1), we get

$$\varphi(d(x_n, x_{n+1})) = \varphi(d(Tx_{n-1}, Tx_n)) \leq \alpha(x_{n-1}, x_n)\varphi(d(Tx_{n-1}, Tx_n)) \leq \varphi(M(x_{n-1}, x_n)) + LN(x_{n-1}, x_n) \tag{2.2.4}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{1}{2}[d(x_{n-1,Tx_n}) + d(x_n, Tx_{n-1})]\} \\ &= max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]\} \\ &= max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \end{aligned} \tag{2.2.5}$$

$$\begin{aligned} N(x_{n-1}, x_n) &= min\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_n), d(Tx_{n-1}, x_n)\} \\ &= min\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_n), d(x_n, x_n)\} \\ &= 0 \end{aligned} \tag{2.2.6}$$

If $max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ for some $n \in N$ then from (2.2.4),(2.2.5) and (2.2.6), we have $\varphi(d(x_n, x_{n+1})) < \varphi(M(x_{n-1}, x_n)) = \varphi(d(x_n, x_{n+1}))$, a contradiction.

Thus, we have $max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$ for all $n \in N$ and hence

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \text{ for all } n \in N. \tag{2.2.7}$$

Thus it follows that $\{d(x_n, x_{n+1})\}$ is a non-negative, decreasing sequence of real numbers.
Suppose that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r, r \geq 0$
Now we prove that $r = 0$.
Assume that $r > 0$.

By choosing $a_n = x_n, b_n = x_{n+1}$,

$$\Delta_n^{\varphi} = \frac{\varphi(d(T(x_n),T(x_{n+1}))}{\varphi(d_n)} \forall n,$$

On taking limit as $n \to \infty$,

then we have $$\lim_{n\to\infty} \Delta_n^{\phi} = \lim_{n\to\infty} \frac{\phi(d(x_{n+1},x_{n+2}))}{\phi(d(x_n,x_{n+1}))} = \frac{\phi(\lim_{n\to\infty} d(x_{n+1},x_{n+2}))}{\phi(\lim_{n\to\infty} d(x_n,x_{n+1}))} = \frac{\phi(r)}{\phi(r)} = 1.$$

Hence by our assumption $\varphi(d_n) \to 0$ as $n \to \infty$ i.e., $\lim_{n\to\infty} \varphi(d(x_n,x_{n+1})) = 0$

$\Rightarrow \varphi(r) = 0$

$\Rightarrow r = 0$, contradicts our assumption that $r > 0$

$\therefore r = \lim_{n\to\infty} d(x_n,x_{n+1}) = 0$

Now, we show that $\{x_n\}$ is a Cauchy sequence in X.

Suppose that $\{x_n\}$ is not a Cauchy sequence.

Then there exist $\varepsilon > 0$, and sub-sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \varepsilon$ and $d(x_{m_k -1}, x_{n_k}) < \varepsilon$ and we have the following identities.

(1) $\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \varepsilon$

(2) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$

Hence $$\lim_{k\to\infty} \frac{\varphi(d(Tx_{m_k},Tx_{n_k}))}{\varphi(d(x_{m_k},x_{n_k}))} = \lim_{k\to\infty} \frac{\phi(d(x_{m_k+1},x_{n_k+1}))}{\phi(d(x_{m_k},x_{n_k}))} = \frac{\varphi(\varepsilon)}{\varphi(\varepsilon)} = 1.$$

Now, by our assumption $\varphi(d_n) \to 0$ as $n \to \infty$ i.e.,

$\lim_{n\to\infty} \varphi(d(x_{m_k}, x_{n_k})) = 0$

$\Rightarrow \lim_{n\to\infty} d(x_{m_k}, x_{n_k}) = 0$, a contradiction to (2).

Therefore $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Now, we show that z is a fixed point of T .

Then the following are the cases.

Case- (1): Suppose there is a sequence $\{n_k\} \ni x_{n_k} = z$, we suppose $x_{n_{k-1}} \neq z$

$\because x_{n_{k-1}} \to z$, $\alpha(z,z) > 1$ and α is continuous,

$\therefore \alpha(x_{n_{k-1}}, z) > 1$ for large k.

Now,

$$\begin{aligned}\varphi(d(z,Tz)) &= \varphi(d(x_{n_k},Tz)) \\ &= \varphi(d(Tx_{n_{k-1}},Tz)) \\ &\le \alpha(x_{n_{k-1}},z)\varphi(d(Tx_{n_{k-1}},Tz)) \\ &< \varphi(M(x_{n_{k-1}},z)) + \ L.N(x_n,z) \end{aligned} \quad (2.2.8)$$

But

$$\begin{aligned} M(x_{n_{k-1}},z) &= ma\ \{d(x_{n_{k-1}},z), d(x_{n_{k-1}},x_{n_k}),\ d(z,Tz), \frac{1}{2}[d(x_{n_{k-1}},Tz)+d(x_{n_k},z)]\} \\ &= max\{d(x_{n_{k-1}},x_{n_k}),\ d(x_{n_k},x_{n_{k+1}})\} \\ &= d(x_{n_{k-1}},x_{n_k}) \end{aligned}$$

$$\therefore \varphi(d(z,Tz)) < \varphi(d(x_{n_{k-1}},x_{n_k})) = 0$$

$$\therefore \varphi(d(z,Tz)) = 0$$

$$\Rightarrow d(z,Tz) = 0$$

$$\therefore z = Tz$$

Case-(2): Suppose $x_n \ne z$ and $z \ne Tz$

$$\begin{aligned}\therefore \varphi(d(x_{n+1},Tz)) &= \varphi(d(Tx_n,Tz)) \\ &\le \alpha(x_n,z)\varphi(d(Tx_n,Tz)) \quad \text{(since } \alpha \text{ is continuous and } \alpha(z,z) > 1) \\ &< \varphi(M(x_n,z)) \ + \ L.N(x_n,z) \end{aligned} \quad (2.2.9)$$

But

$$M(x_n,z) = max\{d(x_n,z), d(x_n,x_{n+1}), d(z,Tz), \frac{1}{2s}[d(x_{n+1},z)+d(x_n,Tz)]\} = d(z,Tz) \text{ for large } n$$

$$N(x_n,z) = min\{d(x_n,z), d(x_n,x_{n+1}), d(z,Tz)\} = 0 \text{ for large } n$$

$$\therefore\ \varphi(d(x_{n+1},Tz)) \le \alpha(x_n,z)\varphi(d(Tx_n,Tz)) < \varphi(d(z,Tz)) + L.0$$

On letting $n \to \infty$, we get

$$\varphi(d(z,Tz)) \le \alpha(z,z)\varphi(d(z,Tz)) \le \varphi(d(z,Tz))$$

$\Rightarrow \alpha(z,z) = 1$, a contradiction.

Hence $z = Tz$.

Consequently z is a fixed point of T in X.

Suppose y, z are two fixed points of T in X

$$\Rightarrow Ty = y, Tz = z \quad (2.2.10)$$

If $\alpha(y,z) < 1$, then there is nothing to prove.

Suppose $\alpha(y,z) \ge 1$ Assume that $y \ne z$.

Then

$$\begin{aligned}\varphi(d(Ty,Tz)) &\leq \alpha(y,z)\varphi(d(Ty,Tz))\\ &< \varphi(M(y,z)) + L.N(y,z)\end{aligned} \tag{2.2.11}$$

But $M(y,z) = max\{d(y,z), d(y,Ty), d(z,Tz), \frac{1}{2}[d(y,Tz)+d(Ty,z)]\} = d(y,z)$

$N(y,z) = min\{d(y,z), d(y,Ty), d(z,Tz)\} = 0$

$\therefore\ \varphi(d(Ty,Tz)) \leq \alpha(y,z)\varphi(d(Ty,Tz)) < \varphi(d(y,z))$

$\Rightarrow \varphi(d(y,z)) \leq \alpha(y,z)\varphi(d(y,z)) < \varphi(d(y,z))$, a contradiction.

Hence $y = z$.

The following corollary can be easily established.

Corollary 2.3: *Let T be a self map on a complete metric space X. Let $\alpha : X \times X \to R$ be a continuous function. Assume that there exist $\varphi \in \Phi$, such that*

$$\alpha(x,y)\varphi(d(Tx,Ty)) < \varphi(M(x,y)) + L.N(x,y) \tag{2.3.1}$$

for all $x, y \in X, x \neq y$ where

$$M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(Tx,y)]\},$$

$$N(x,y) = min\{d(x,Tx), d(x,Ty), d(y,Tx)\}.$$

Further, assume that

(i) T is α - admissible, and

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and set $x_n = Tx_{n-1}$ for $n = 1,2,3,\ldots$.

(iii) for any two sequences $\{a_n\}$ and $\{b_n\}$ of X with $a_n \neq b_n$,

We have that $\Delta_n^{\varphi} \to 1 \Rightarrow d_n \to 0$ (2.3.2)

Then the sequence $\{x_n\}$ is a Cauchy sequence. Suppose $\{x_n\}$ converges to z and $\alpha(z,z) > 1$.

Then z is a fixed point of T in X.

Theorem 2.4: *Let (X,d) be a complete metric space, $\alpha : X \times X \to R$ be a continuous function and let $T : X \to X$ be a self map. Suppose the following conditions hold:*

(i) T is (α, φ, β) - weak generalized Geraghty contraction i.e.,

$$\alpha(x,y)\varphi(d(Tx,Ty)) \leq \beta(\varphi(M(x,y)))\varphi(M(x,y)) + L.N(x,y) \tag{2.4.1}$$

where $L \geq 0$ for all $x, y \in X$, where

$$M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(Tx,y)]\},$$

$$and\ N(x,y) = min\{d(x,Tx), d(x,Ty), d(y,Tx)\}.$$

(ii) T is triangular α - admissible,

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and set $x_n = Tx_{n-1}$ for $n = 1,2,3,\ldots$ .Then $\{x_n\}$ is a Cauchy sequence.

Suppose $\{x_n\}$ converges to x and $\alpha(x,x) > 1$. Then x is a fixed point of T in X.

Proof: As in the theorem 2.2, Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. We define $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for each $n = 1,2,3,...$ If $x_n = x_{n+1}$ for some $n \in N$, then $x_n = Tx_n$ and hence x_n is a fixed point of T. Hence, without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n \in N$. By using the α - admissibility of T, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$$

$$\Rightarrow \alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1.$$

Now, by mathematical induction, it is easy to see that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$.

By taking $x = x_{n-1}$ and $y = x_n$ in the inequality (2.4.1),

We get

$$\begin{aligned} \phi(d(x_n, x_{n+1})) &= \phi(d(Tx_{n-1}, Tx_n)) \\ &\leq \alpha(x_{n-1}, x_n)\phi(d(Tx_{n-1}, Tx_n)) \\ &\leq \alpha(x_{n-1}, x_n)\phi(d(Tx_{n-1}, Tx_n)) \end{aligned} \quad (2.4.2)$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{1}{2}[d(x_{n-1,Tx_n}) + d(x_n, Tx_{n-1})]\} \\ &= max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]\} \\ &= max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \end{aligned} \quad (2.4.3)$$

$$\begin{aligned} N(x_{n-1}, x_n) &= min\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_n), d(Tx_{n-1}, x_n)\} \\ &= min\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_n), d(x_n, x_n)\} \\ &= 0 \end{aligned} \quad (2.4.4)$$

If $max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ for some $n \in N$ then from (2.4.2),(2.4.3) and (2.4.4), we have $\varphi(d(x_n, x_{n+1})) < \varphi(M(x_{n-1}, x_n)) = \varphi(d(x_n, x_{n+1}))$, a contradiction.

Thus, we have $max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$ for all $n \in N$ and hence

$d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all $n \in N$. (2.4.5)

Thus it follows that $\{d(x_n, x_{n+1})\}$ is a non-negative, decreasing sequence of real numbers. Suppose that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r, r \geq 0$

Now we prove that $r = 0$.

Assume that $r > 0$.

$$\therefore \ \alpha(x_n, x_{n+1})\ \phi(d(Tx_n, Tx_{n+1})) \ \leq \ \beta(\phi(M(x_n, x_{n+1})))\ \phi(M(x_n, x_{n+1})) + L.N(x_n, x_{n+1})$$

$$\Rightarrow \ \varphi(d(Tx_n, Tx_{n+1})) \leq \beta(\varphi(M(x_n, x_{n+1})))\varphi(d(x_n, x_{n+1}) + L.N(x_n, x_{n+1})$$

$$\Rightarrow \ \varphi(d(x_{n+1}, x_{n+2})) \leq \beta(\varphi(M(x_n, x_{n+1})))\varphi(d(x_n, x_{n+1}) + L.N(x_n, x_{n+1})$$

On letting $n \to \infty$

$$\begin{aligned}\lim_{n\to\infty} \phi(d(x_{n+1}, x_{n+2})) &\le \liminf_{n\to\infty} \beta(\phi(d(x_n, x_{n+1})))\phi(d(x_n, x_{n+1})) \\ &\le \limsup_{n\to\infty} \beta(\phi(d(x_n, x_{n+1})))\phi(d(x_n, x_{n+1})) \\ &\le \lim_{n\to\infty} \phi(d(x_n, x_{n+1}))\end{aligned}$$

$$\therefore \varphi(r) \le \liminf_{n\to\infty} \beta(\varphi(d(x_n, x_{n+1})))\varphi(r) \le \limsup_{n\to\infty} \beta(\varphi(d(x_n, x_{n+1}))\varphi(r) \le \varphi(r)$$

$$\Rightarrow \varphi(r) = 0 \text{ or } \lim_{n\to\infty} \beta(\varphi(d(x_n, x_{n+1}))) = 1$$

$$\therefore \lim_{n\to\infty} \beta(\varphi(d(x_n, x_{n+1}))) = 1$$

$$\Rightarrow \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

$\Rightarrow r = 0$, a contradiction for our assumption of $r > 0$.

$\therefore r = 0$

Hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$

Let us suppose $\{x_n\}$ is not a Cauchy sequence.

Then there exist $\varepsilon > 0$, and sub-sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \ge \varepsilon$ and $d(x_{m_k-1}, x_{n_k}) < \varepsilon$ and we have the following identities.

(i) $\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \varepsilon$ (ii) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$

Also since T is triangular α - admissible and $\alpha(x_n, x_{n+1}) \ge 1 \forall n \in N$,

$\therefore \alpha(x_{m_k}, x_{n_k}) \ge 1$

Hence

$$\begin{aligned}\phi(d(x_{m_k+1}, x_{n_k+1})) &= \phi(d(Tx_{m_k}, Tx_{n_k})) \\ &\le \alpha(x_{m_k}, x_{n_k})\phi(d(Tx_{m_k}, Tx_{n_k})) \\ &\le \beta(\phi(M(x_{m_k}, x_{n_k}))\phi(M(x_{m_k}, x_{n_k})) + LN(x_{m_k}, x_{n_k})\end{aligned} \tag{2.4.6}$$

where

$$\begin{aligned}M(x_{m_k}, x_{n_k}) &= ma\ \{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1}), \frac{1}{2}[d(x_{m_k}, x_{n_k+1}) + d(x_{m_k+1}, x_{n_k})]\} \\ &= d(x_{m_k}, x_{n_k})\end{aligned} \tag{2.4.7}$$

$$\begin{aligned}N(x_{m_k}, x_{n_k}) &= min\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1})\} \\ &= 0 \text{ for large } k\end{aligned} \tag{2.4.8}$$

Therefore from (2.4.6),(2.4.7) and (2.4.8), we have

$$\varphi(d(x_{m_k+1}, x_{n_k+1})) \le \beta(\varphi(d(x_{m_k}, x_{n_k}))\varphi(d(x_{m_k}, x_{n_k})) + LN(x_{m_k}, x_{n_k}) \tag{2.4.9}$$

Allowing $k \to \infty$

$$\lim_{k\to\infty} \phi(d(x_{m_k+1}, x_{n_k+1})) \le \liminf_{k\to\infty} \beta(\phi(d(x_{m_k}, x_{n_k})))\phi(d(x_{m_k}, x_{n_k}))$$
$$\le \limsup_{k\to\infty} \beta(\phi(d(x_{m_k}, x_{n_k})))\phi(d(x_{m_k}, x_{n_k}))$$
$$\le \lim_{k\to\infty} \phi(d(x_{m_k}, x_{n_k}))$$

$$\therefore \phi(\varepsilon) \le \liminf_{k\to\infty} \beta(\phi(d(x_{m_k}, x_{n_k})))\phi(\varepsilon)$$
$$\le \limsup_{k\to\infty} \beta(\phi(d(x_{m_k}, x_{n_k}))\phi(\varepsilon)$$
$$\le \phi(\varepsilon)$$

$$\Rightarrow \varphi(\varepsilon) = 0 \text{ or } \lim_{k\to\infty} \beta(\varphi(d(x_{m_k}, x_{n_k}))) = 1$$

$$\therefore \lim_{k\to\infty} \beta(\varphi(d(x_{m_k}, x_{n_k}))) = 1$$

$$\Rightarrow \beta \in S$$

$$\therefore \lim_{k\to\infty} \varphi(d(x_{m_k}, x_{n_k})) = 0$$

$\Rightarrow \quad \varphi(\varepsilon) = 0$, a contradiction.

Therefore $\{x_n\}$ is a Cauchy sequence in X.

Let it converges to $x \in X$.

$\therefore \lim_{n\to\infty} x_n = x$ and $\alpha(x, x) \ge 1$

$$\Rightarrow \alpha(x_n, x) \ge 1 \,\forall\, n \in N$$

$$\therefore \phi(d(x_{n+1}, Tx)) = \phi(d(Tx_n, Tx))$$
$$\le \alpha(x_n, x)\phi(d(Tx_n, Tx))$$
$$\le \beta(\varphi(M(x_n, x)))\varphi(M(x_n, x)) \quad \text{where } M(x_n, x) = d(x, Tx) \text{ for large } n.$$

Supposing $d(x, Tx) \ne 0$

Allowing $n \to \infty$, $\varphi(d(x,Tx)) \le \beta(\varphi(d(x,Tx)))\varphi(d(x,Tx)) < \varphi(d(x,Tx))$

which is a contradiction.

$\therefore \quad \varphi(d(x,Tx)) = 0 \Rightarrow d(x,Tx) = 0$

$\therefore x = Tx$

Hence x is a fixed point of T.

Note: G.V.R.Babu *et.al* [6] used continuity of β while proving the theorem 1.8, here we successfully avoided the ontinuity of β.

Now we give an example in support of theorem 2.2

Example 2.5: *Let* $X = [0, \frac{1}{2}]$ with usual metric d .

Clearly, (X, d) is a metric space.

Define $T : X \to X$ by $Tx = \frac{1}{4}$

Define $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = 2\{1 - \frac{1}{4}(x + y)\}$

Define the altering distance function
$\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = t$ if $t \geq 0$

Let $x, y \in [0, \frac{1}{2}]$

$\Rightarrow \alpha(x, y) = 2\{1 - \frac{1}{4}(x + y)\} > 1$ and

$$\alpha(Tx, Ty)) = 2\{1 - \frac{1}{4}(Tx + Ty)\} = 2\{1 - \frac{1}{4}(1 - (x + y))\} = 2\{\frac{3}{4} + \frac{1}{4}(x + y))\} > 1$$

$\therefore T$ is α - admissible and taking $x_0 = 0$

$$\Rightarrow \alpha(x_0, Tx_0) = 2\{1 - \frac{1}{16}\} = \frac{15}{8} > 1$$

Now $\alpha(x, y)\varphi(d(Tx, Ty)) = \alpha(x, y)d(Tx, Ty) = 0$

Further

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(Tx, y)]\} \quad \text{for } x \neq y$$

$$= max\{| x - y |, | x - \frac{1}{4} |, | y - \frac{1}{4} |, \frac{1}{2} | x + y - \frac{1}{2} |\} > 0$$

$\Rightarrow \varphi(M(x, y)) > 0$ and $N(x, y) \geq 0$

$\therefore \alpha(x, y)\varphi(d(Tx, Ty)) < \varphi(M(x, y)) + L.N(x, y)$ for any $L \geq 0$

$\therefore$ hypothesis of theorem 2.2 is satisfied.

$\because T\frac{1}{4} = \frac{1}{4}$ Hence $\frac{1}{4}$ is a fixed point of T

Is the theorem 2.2 true if the conditions on α are removed ?

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The fourth author is grateful to management of Mrs.AVNCollege, Visakhapatnam and Mathematics Department of Andhra University for giving necessary permission and providing necessary facilities to carry on this research.

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Source of support: Nil, Conflict of interest: None Declared