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ON THE PRINCIPLE OF THE EXCHANGE OF STABILITIES FOR THE TRIPLY DIFFUSIVE CONVECTION PROBLEM IN COMPLETELY CONFINED FLUIDS

J. PRAKASH*[1], S. MANAN[2], V. SINGH[3]

[1]Associate Professor, Department of Mathematics and Statistics,
Himachal Pradesh University, Summer Hill, Shimla-171005, India.

[2, 3]Research Scholar, Department of Mathematics and Statistics,
Himachal Pradesh University, Summer Hill, Shimla-171005, India.

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ABSTRACT

The triply diffusive instability problem for fluid completely confined in an arbitrary region with rigid bounding surfaces is considered. A sufficient condition for the validity of the principle of the exchange of stabilities is derived, which to the best of our knowledge do not appear to have been derived in the literature on triply diffusive convection with the same degree of generality.

Keywords: *Triply diffusive convection, The principle of the exchange of stabilities, Completely confined fluids, Concentration Rayleigh number.*

2015 Subject Classification: *76 E06.*

1. INTRODUCTION

Convective motions can occur in a stably stratified fluid when there are two components contributing to the density which diffuse at different rates. This phenomenon is called double-diffusive convection. To determine the conditions under which these convective motions will occur, the linear stability of two superposed concentration (or one of them may be temperature gradient) gradients has been studied by Stern [1], Veronis [2], Nield [3], Baines and Gill [4] and Turner [5].

The case of two component system has been considered only. However, it has been recognized later (Griffiths [6], Turner [7]) that there are many situations wherein more than two components are present. Examples of such multiple diffusive convection fluid systems include the solidification of molten alloys, geothermally heated lakes, magmas and their laboratory models and sea water. For the detailed overview of the work done on triply/ multiple diffusive convection one may refer to Griffiths [6], Pearlstein *et al.* [8], Lopez *et al.* [9], Ryzhkov and Shevtsova [10,11], Rionero [12,13], Prakash *et al.* [14,15]. These researchers found that small concentrations of a third component with a smaller diffusivity can have a significant effect upon the nature of diffusive instabilities and direct salt finger and oscillatory modes are simultaneously unstable under a wide range of conditions, when the density gradients due to components with the greatest and smallest diffusivity are of same signs.

All these researchers have confined themselves to horizontal layer geometry, perhaps, due to the complexity involved in the analysis of the hydrodynamic problems with arbitrary geometries. However, there are a few researchers (Sherman and Ostrach [16], Gupta *et al.* [17], Gupta and Dhiman [18], Gupta *et al.* [19]) who have extended the classical work to more general hydrodynamic stability problems with arbitrary boundaries. In the present communication, which is motivated by the desire to extend the works of Sherman and Ostrach [16], Gupta et *al.* [17] to more complex problems, namely, triply diffusive convection problems for completely confined fluids, a sufficient condition for the validity of the exchange principle is derived. The results of Sherman and Ostrach [16] for Rayleigh-Benard problem, Gupta *et al.* [17] for double-diffusive convection problem (both for completely confined fluids) are obtained as a consequence.

Corresponding Author: J. Prakash[1]*
[1]Associate Professor, Department of Mathematics and Statistics,
Himachal Pradesh University, Summer Hill, Shimla-171005, India.

2. MATHEMATICAL FORMULATION AND ANALYSIS

Consider a Boussinesq fluid statically confined in an arbitrary completely enclosed region (as shown in Fig.1) which is maintained at a uniform temperature and concentration gradient parallel to the body force acting on a fluid by applying certain prescribed thermal and concentration boundary conditions on the bounding walls. The problem under investigation is to examine the stability of this physical configuration when the heat and the two concentrations make opposing contributions to the vertical density gradient. It is further assumed that the cross diffusion effects can been neglected.

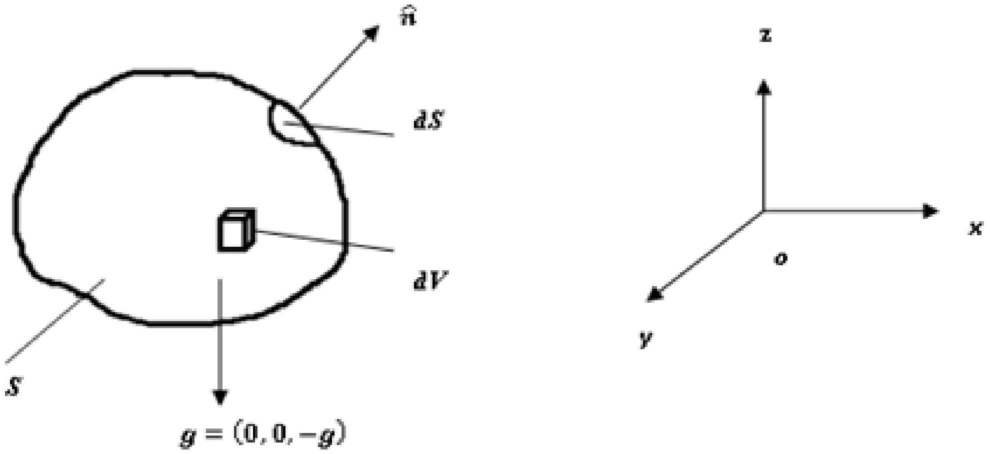

Fig. 1Physical configuration

The governing linearized perturbation equations in non dimensional form for the problem with time dependence of the form $\exp(pt)$ $(p = p_r + ip_i)$ being complex in general) are given by [17]

$$\frac{p}{\sigma}\vec{U} = \nabla^2\vec{U} - \text{grad}(P) + R\theta\hat{k} - R_1\varphi_1\hat{k} - R_2\varphi_2\hat{k}, \tag{1}$$

$$\nabla^2\theta - p\theta = -\vec{U}.\hat{k}, \tag{2}$$

$$\tau_1\nabla^2\varphi_1 - p\varphi_1 = -\vec{U}.\hat{k}, \tag{3}$$

$$\tau_2\nabla^2\varphi_2 - p\varphi_2 = -\vec{U}.\hat{k}, \tag{4}$$

$$\text{div}\vec{U} = 0, \tag{5}$$

where $\vec{U}, P, \theta, \varphi_1$ and φ_2 denote respectively the perturbed velocity, pressure, temperature, concentration of the first component and concentration of the second component. R is the thermal Rayleigh number, R_1 and R_2 are the the concentration Rayleigh numbers for two concentration components respectively. σ is the Prandtl number, τ_1 and τ_2 are the Lewis numbers for two concentration components respectively and $\hat{k}$ is a unit vector in the vertical direction. The equations have been written in dimensionless forms by using the scale factors $\frac{d^2}{\kappa}, \frac{\kappa}{d}, \frac{\rho\nu\kappa}{d^2}, \beta d, \beta_1 d$ and $\beta_2 d$ for time, velocity, pressure, temperature, and the two concentrations respectively, where d a characteristic length, κ the thermal diffusivity, ρ the density, ν the kinematic viscosity, β the constant temperature gradient, β_1 and β_2 are the constant concentration gradients for two concentration components.

Eqs. (1) – (5) are to be solved in a simply connected subset V of R^3 with boundary S subject to the following homogeneous time independent boundary conditions:

$\vec{U} = 0 = \theta = \varphi_1 = \varphi_2$ on S (rigid bounding surface with fixed temperature and mass concentrations). (6)

Eqs. (1) – (5) together with boundary conditions (6) describe an eigenvalue problem for p for prescribed values of the other parameters and the system is stable, neutral or unstable according as p_r is negative, zero or positive. Further if $p_r = 0$ implies $p_i = 0$, then the principle of the exchange of stabilities (PES) is valid otherwise we will have overstability.

Now we prove the following theorem:

Theorem: If $(p, \vec{U}, \theta, \varphi_1, \varphi_2), p = p_r + ip_i, p_r \geq 0$, is a solution of Eqs. (1) - (6) with $R > 0, R_1 > 0, R_2 > 0$ and $\frac{R_1 \sigma l^4}{\tau_1^2 \Lambda^2} + \frac{R_2 \sigma l^4}{\tau_2^2 \Lambda^2} \leq 1$, then $p_i = 0$, where l is the smallest distance between two parallel planes that just contain V and $\Lambda(> 2)$ is a constant. In particular PES is valid if $\frac{R_1 \sigma l^4}{\tau_1^2 \Lambda^2} + \frac{R_2 \sigma l^4}{\tau_2^2 \Lambda^2} \leq 1$.

Proof: We rewrite system of Eqs. (1) – (4) in the following alternate forms:

$$\frac{p}{\sigma}\vec{U} + \text{grad}(P) - \nabla^2\vec{U} - R\theta\hat{k} + R_1\varphi_1\hat{k} + R_2\varphi_2\hat{k} = 0, \tag{7}$$

$$-R\left[\nabla^2\theta - p\theta + \vec{U}.\hat{k}\right] = 0, \tag{8}$$

$$R_1\left[\tau_1\nabla^2\varphi_1 - p\varphi_1 + \vec{U}.\hat{k}\right] = 0, \tag{9}$$

$$R_2\left[\tau_2\nabla^2\varphi_2 - p\varphi_2 + \vec{U}.\hat{k}\right] = 0, \tag{10}$$

Forming the dot product of Eq. (7) with $\vec{U}^*$ ($*$ denotes complex conjugation) and integrating over the domain V, we obtain

$$\frac{p}{\sigma}\int_V(\vec{U}.\vec{U}^*)dV + \int_V\left[(\text{grad}P).\vec{U}^*\right]dV - \int_V\left(\vec{U}^*.\nabla^2\vec{U}\right)dV - R\int_V\left[(\theta\hat{k}).\vec{U}^*\right]dV + R_1\int_V\left[(\varphi_1\hat{k}).\vec{U}^*\right]dV + R_2\int_V\left[(\varphi_2\hat{k}).\vec{U}^*\right]dV = 0. \tag{11}$$

Subsequently, for convenience in writing, we omit V and the infinitesimal dV volume from the integral sign and the integrand, respectively.

Multiplying Eqs. (8) - (10) by $\theta^*, \varphi_1^*, \varphi_2^*$, respectively, integrating over the domain V, we get

$$-R\int\theta^*\left[\nabla^2\theta - p\theta + \vec{U}.\hat{k}\right] = 0, \tag{12}$$

$$R_1\int\varphi_1^*\left[\tau_1\nabla^2\varphi_1 - p\varphi_1 + \vec{U}.\hat{k}\right] = 0, \tag{13}$$

$$R_2\int\varphi_2^*\left[\tau_2\nabla^2\varphi_2 - p\varphi_2 + \vec{U}.\hat{k}\right] = 0. \tag{14}$$

Now, adding Eqs. (12) – (14) to Eq. (11), we have

$$\frac{p}{\sigma}\int(\vec{U}.\vec{U}^*) + \int(\text{grad}P).\vec{U}^* - \int\left(\vec{U}^*.\nabla^2\vec{U}\right) - R\int\theta^*(\nabla^2 - p)\,\theta + R_1\int\varphi_1^*(\tau_1\nabla^2 - p)\,\varphi_1 + R_2\int\varphi_2^*(\tau_2\nabla^2 - p)\,\varphi_2 - RI + R_1I_1 + R_2I_2 = 0, \tag{15}$$

where $I = 2Re\left[\int(\theta\hat{k}.\vec{U}^*)\right], I_1 = 2Re\left[\int(\varphi_1\hat{k}.\vec{U}^*)\right], I_2 = 2Re\left[\int(\varphi_2\hat{k}.\vec{U}^*)\right]$, and Re denotes the real part.

Using Gauss' theorem and boundary conditions (6), we have

$$\int(\text{grad}P).\vec{U}^* = \int_S P\,\vec{U}^*.\hat{n}dS - \int P\text{div}\,\vec{U}^* = 0, \tag{16}$$

$$\int\left(\vec{U}^*.\nabla^2\vec{U}\right) = -\int\left(\text{curlcurl}\vec{U}.\vec{U}^*\right) = -\int\text{curl}\vec{U}.\text{curl}\vec{U}^* - \int_S\left(\text{curl}\vec{U}\right)\times\vec{U}^*.\hat{n}dS = -\int\text{curl}\vec{U}.\text{curl}\vec{U}^*, \tag{17}$$

$$\int(\theta^*\nabla^2\theta) = \int_S(\theta^*\nabla\theta).\hat{n}dS - \int\nabla\theta.\nabla\theta^* = -\int\nabla\theta.\nabla\theta^*, \tag{18}$$

$$\int(\varphi_1^*\nabla^2\varphi_1) = \int_S(\varphi_1^*\nabla\varphi_1).\hat{n}dS - \int\nabla\varphi_1.\nabla\varphi_1^* = -\int\nabla\varphi_1.\nabla\varphi_1^*, \tag{19}$$

$$\int(\varphi_2^*\nabla^2\varphi_2) = \int_S(\varphi_2^*\nabla\varphi_2).\hat{n}dS - \int\nabla\varphi_2.\nabla\varphi_2^* = -\int\nabla\varphi_2.\nabla\varphi_2^*, \tag{20}$$

where $\hat{n}$ is a unit outward drawn normal at any point on S. Using integral relations (16) – (20) in Eq. (15), we have

$$\frac{p}{\sigma}\int(\vec{U}.\vec{U}^*) + \int\text{curl}\vec{U}.\text{curl}\vec{U}^* + R\int(\nabla\theta.\nabla\theta^* + p|\theta|^2) - R_1\int(\tau_1\nabla\varphi_1.\nabla\varphi_1^* + p|\varphi_1|^2) - R_2\int(\tau_2\nabla\varphi_2.\nabla\varphi_2^* + p|\varphi_2|^2) - RI + R_1I_1 + R_2I_2 = 0. \tag{21}$$

Equating the imaginary part of Eq. (21) to zero and since $p_i \neq 0$, we have

$$\frac{1}{\sigma}\int(\vec{U}.\vec{U}^*) + R\int|\theta|^2 = R_1\int|\varphi_1|^2 + R_2\int|\varphi_2|^2. \tag{22}$$

Multiplying Eq. (3) by φ_1^* , integrating over V, utilizing the relation (19), and then equating the real parts of the resulting equation, we obtain

$$\tau_1 \int \nabla\varphi_1.\nabla\varphi_1^* + p_r \int|\varphi_1|^2 = Re\left[\int(\vec{U}.\hat{k})\varphi_1^*\right] \le \int|\vec{U}.\hat{k}||\varphi_1| \le \left\{\int\left|\vec{U}.\hat{k}\right|^2\right\}^{1/2}\{\int|\varphi_1|^2\}^{1/2}. \tag{23}$$

Since $p_r \ge 0$ we have from inequality (23) that

$$\tau_1 \int \nabla\varphi_1.\nabla\varphi_1^* < \left\{\int\left|\vec{U}.\hat{k}\right|^2\right\}^{1/2}\{\int|\varphi_1|^2\}^{1/2},$$

which upon utilizing Poincare inequality (Joseph [20]) viz.

$$\int \nabla\varphi_1.\nabla\varphi_1^* \ge \frac{\Lambda}{l^2}\int|\varphi_1|^2, \tag{24}$$

yields

$$\int|\varphi_1|^2 \le \frac{l^4}{\tau_1^2\Lambda^2}\int\left|\vec{U}.\hat{k}\right|^2 \le \frac{l^4}{\tau_1^2\Lambda^2}\int \vec{U}.\vec{U}^*. \tag{25}$$

Using the same procedure, it follows from Eq. (4) that

$$\int|\varphi_2|^2 \le \frac{l^4}{\tau_2^2\Lambda^2}\int \vec{U}.\vec{U}^*. \tag{26}$$

Utilizing inequalities (25) and (26) in Eq. (22), we have

$$\left(\frac{1}{\sigma} - \frac{R_1 l^4}{\tau_1^2\Lambda^2} - \frac{R_2 l^4}{\tau_2^2\Lambda^2}\right)\int \vec{U}.\vec{U}^* + R\int|\theta|^2 < 0, \tag{27}$$

which clearly implies that $\frac{R_1\sigma l^4}{\tau_1^2\Lambda^2} + \frac{R_2\sigma l^4}{\tau_2^2\Lambda^2} > 1.$

Hence, if $\frac{R_1\sigma l^4}{\tau_1^2\Lambda^2} + \frac{R_2\sigma l^4}{\tau_2^2\Lambda^2} \le 1$, then we must have $p_i = 0.$

This proves the theorem.

The above theorem states from the physical point of view that for the problem of Triply diffusive convection for the completely confined fluids, an arbitrary neutral or unstable mode of the system is definitely nonoscillatory in character and in particular the principle of the exchange of stabilities is valid if $\frac{R_1\sigma l^4}{\tau_1^2\Lambda^2} + \frac{R_2\sigma l^4}{\tau_2^2\Lambda^2} \le 1.$

SPECIAL CASES

It follows from theorem that an arbitrary neutral or unstable mode is nonoscillatory in character and in particular PES is valid for:

1. Rayleigh - Benard convection problem in completely confined fluids $(R_1 = R_2 = 0)$ (Sherman and Ostrach [16])
2. Thermohaline convection $(R_2 = 0)$ if $\frac{R_1\sigma l^4}{\tau_1^2\Lambda^2} \le 1.$ (Gupta *et al.* [17])
3. Thermohaline convection of Stern [1] type if $(R < 0, R_1 < 0, R_2 = 0)$ if $\frac{|R|\sigma l^4}{\Lambda^2} \le 1.$ (Gupta *et al.* [17])
4. Triply diffusive convection analogous to Stern [1] type $(R < 0, R_1 < 0, R_2 < 0)$ if $\frac{|R|\sigma l^4}{\Lambda^2} \le 1.$

Proof: Putting $R = -|R|$, $R_1 = -|R_1|, R_2 = -|R_2|$ in Eq. (1) and adopting the procedure exactly similar to the one used in proving theorem we obtain the desired result.

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