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STABILITY OF MHD DEAN FLOW WITH RADIAL HEATING

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ABSTRACT

In this paper we have presented a linear instability analysis on the effect of axial magnetic field on the Dean flow between two narrow spaced concentric circular cylinders. It is observed that when the temperature of the inner cylinder is more than the outer cylinder, the flow is stable. The non-conducting walls are found to be more destabilizing than the perfectly conducting walls.

Keyword: *Dean flow, axial magnetic field, radial heating.*

NOMENCLATURE

a	Dimensionless wave number
a_c	Critical wave number
d	Gap width between the cylinders
N	Temperature gradient parameter
Pr	Prandtl number
P_m	Magnetic Prandtl number
Re	Reynolds number
R_1	Radius of the inner cylinder
R_2	Radius of the outer cylinder
S	Sum of fluid pressure and magnetic pressure
T	Temperature of the fluid
T_1	Temperature of the inner cylinder
T_2	Temperature of the outer cylinder
V	velocity in the azimuthal direction in the basic state
V_m	average azimuthal velocity
u_r, u_θ, u_z	Fluid velocities in r, θ and z directions
H_r, H_θ, H_z	Magnetic field strength in r, θ and z directions
λ	Axial wave length
σ	Growth rate
ρ	Fluid density
ν	Kinematic viscosity
α	Thermal diffusivity of the fluid
e	Magnetic diffusivity
μ_e	Magnetic permeability
Q	Hartmann number
x	Non-dimensional radial distance
η	Ratio of the radii
θ	Non-dimensional temperature
Λ	Dean number
Λ_c	Critical Dean number

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1. INTRODUCTION

Dean [1] and Reid [2] investigated the flow instability in a curved channel formed by two concentric circular cylinders, when the space between the cylinders is small and the disturbance is axisymmetric. The fluid is driven by a constant azimuthal pressure gradient around the circumference of the annulus. This is known as the 'Dean flow' in the literature. Gibson and Cook [3] analyzed the instability of this flow with respect to both axisymmetric and asymmetric disturbances in an annulus for both finite and narrow gap widths.

Chandrasekhar [4] investigated the influence of a uniform axial magnetic field on the flow instability between two concentric circular cylinders, with axisymmetric disturbance in a narrow gap. Recently, Mahapatra *et al.* [5] studied the inhibition of magnetic field on the stability of Dean flow for axisymmetric disturbance with arbitrary gap width. They also determined the asymptotic behavior of the critical wave number and Dean number for large Hartmann number, when the walls of the cylinders are non-conducting.

The effect of radial temperature gradient on the flow instability has been studied by many researchers on the Couette flow between circular cylinders in the presence or absence of axial magnetic field. For example, Takhar *et al.* [6] studied the flow instability of Taylor-Couette flow between rotating cylinders in presence of both axial magnetic field and radial temperature gradient. Ali *et al.* [7] presented the effect of radial temperature gradient on the Dean flow for narrow gap limit, while Deka and Paul [8] studied for permeable cylinders in presence of constant heat flux at the inner cylinder.

However, the effect of the radial temperature gradient on the stability of the hydromagnetic Dean flow in presence of axial magnetic field is limited in the literature. Recently, Dholey [9] studied the hydromagnetic Dean flow in a wide gap with radial heating for conducting cylinders. Finite difference method was applied to find the critical values at the onset of instability. Walowit *et al.* [10] studied the isothermal Dean flow for non-magnetic case and obtained the critical values at the onset of instability by Galerkin method. Their results are within 1% of Chandrasekhar's [4] result. Also, Ali *et al.* [7] applied finite difference method and obtained critical values with similar error. Here we present a linear instability analysis on the effect of uniform axial magnetic field on the Dean flow in a narrow gap for axisymmetric disturbance. We shall determine the critical values by Harris and Reid [11] method and present the subsequent discussion for both conducting and non-conducting cylinders.

2. LINEAR STABILITY EQUATIONS

Let r, θ and z denote the usual cylindrical polar coordinates, and let u_r, u_θ, u_z and H_r, H_θ, H_z denote the components of velocity and magnetic field intensity, respectively. We consider two infinitely long concentric circular stationary cylinders with radius R_1 (inner cylinder), R_2 (outer cylinder). The cylinders are kept at two different temperatures T_1 (inner cylinder) and T_2 (outer cylinder) with the z-axis as their common axis. Note that the annulus is partially filled with the liquid with a partition in the annulus at the air-liquid interface to maintain the constant azimuthal pressure gradient, $(\partial P/\partial\theta)_0$.

The equations of motion and energy equation for an incompressible, viscous electrical conducting fluid in the presence of a uniform magnetic field in the axial direction admit of steady solution as,

$$\left.\begin{aligned} &u_r = 0, u_z = 0, \quad u_\theta = V(r) = \frac{1}{2\rho\nu}\left(\frac{\partial P}{\partial \theta}\right)_0 \left(r\ln r + Cr + \frac{D}{r}\right) \\ &H_r = H_\theta = 0, \quad H_z = H = \text{constant} \\ &T = T_2 - \frac{T_2 - T_1}{\ln(R_1/R_2)}\ln\left(\frac{r}{R_2}\right) \end{aligned}\right\} \tag{1}$$

with

$$C = \frac{R_2^2\ln R_2 - R_1^2\ln R_1}{R_1^2 - R_2^2}, \quad D = \frac{R_2^2 R_1^2}{R_2^2 - R_1^2}\ln\frac{R_2}{R_1} \tag{2}$$

The pressure distribution given by, $(\partial S/\partial r = \rho V^2/r)$ disappears from the stability problem and hence not determined explicitly. Here, S is the sum of fluid pressure P and the magnetic pressure $(H^2/8\pi)$.

To study the instability of this flow we superimpose a general disturbance on the basic solution, substitute in the hydromagnetic governing equations (*cf.* Chandrasekhar [4]) and neglect quadratic terms in the usual way. Since the coefficients in the resultant disturbance equations depend only on r, it is possible to look for solutions of the form:

$$u_\theta = \{V(r) + \mathrm{v}(r)\}e^{\sigma t}\cos\lambda z \tag{3}$$

where v(r) is the azimuthal component of the small disturbance velocity, and with similar expressions for the other components of velocity, pressure, temperature and the components of magnetic field intensity. It is assumed that the axial wave length λ be real. Thus for the marginal state (σ = 0), substituting (3) and other disturbed quantities in incompressible MHD Navier Stokes equations (*cf.* Chandrasekhar [4] and Takhar *et al.* [6]), and after some analytical works, we obtain the following normal-mode disturbance equations:

$$\left[\left(DD^{*}-\lambda^{2}\right)^{2}+\frac{\lambda^{2}\mu_{e}H^{2}}{4\pi\rho\nu e}\right]u=2\frac{e\lambda V}{\nu rH}\left(DD^{*}-\lambda^{2}\right)g-\frac{\alpha V^{2}\lambda^{2}}{r\nu}T \tag{4}$$

$$\left[\left(DD^{*}-\lambda^{2}\right)^{2}+\frac{\lambda^{2}\mu_{e}H^{2}}{4\pi\rho\nu e}\right]g=\frac{\lambda H}{e\nu}\left(D^{*}V\right)u \tag{5}$$

$$\left(DD^{*}-\lambda^{2}\right)T=-\frac{1}{\alpha r}\frac{T_{2}-T_{1}}{\ln(R_{1}/R_{2})}u \tag{6}$$

where,

$$\left.\begin{aligned} &D\equiv\frac{d}{dr},\quad D^{*}\equiv\frac{d}{dr}+\frac{1}{r}\ ,\quad D_{*}\equiv\frac{d}{dr}-\frac{1}{r},\\ &DD^{*}\equiv\frac{d^{2}}{dr^{2}}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^{2}},\quad D^{*}D\equiv\frac{d^{2}}{dr^{2}}+\frac{1}{r}\frac{d}{dr}\end{aligned}\right\} \tag{7}$$

We now introduce the following non-dimensional quantities,

$$x=\frac{r-R_{1}}{d},\quad \xi=\frac{r}{R_{2}},\quad \eta=\frac{R_{1}}{R_{2}},\quad a=\lambda d,\quad \bar{u}=\frac{u}{V_{m}} \tag{8}$$

and the non-dimensional operators,

$$D\equiv\frac{1}{d}\frac{d}{dx}\equiv\frac{1}{d}\bar{D},\quad D^{*}\equiv\frac{1}{d}\left(\frac{d}{dx}-\frac{1-\eta}{\xi}\right)\equiv\frac{1}{d}\bar{D}^{*} \tag{9}$$

where $d = R_2 - R_1$ is the gap width between the cylinders.

Now, in view of (7) and (8), equation (6) becomes,

$$\left(\bar{D}\bar{D}^{*}-a^{2}\right)T=-\frac{V_{m}d}{\alpha\xi}\frac{(1-\eta)(T_{2}-T_{1})}{\ln\eta}\bar{u} \tag{10}$$

Now with the non-dimensional quantity for temperature, $\theta=T\alpha/\{dV_m(T_2-T_1)\}$ the above equation become,

$$\left(\bar{D}\bar{D}^{*}-a^{2}\right)\theta=-\frac{1}{\xi}\frac{(1-\eta)}{\ln\eta}\bar{u} \tag{11}$$

For narrow gap between the cylinders ($\eta\to1$, $\xi\to1$), $\bar{D}\equiv\bar{D}^{*}\equiv D\equiv d/dx$ and the azimuthal velocity V = $6V_mG(x)$ (see Chandrasekhar [4]), where V_m is the average velocity in the azimuthal direction. Using (7), (8) and introducing non-dimensional variable, $\bar{g} = e\nu/(6\lambda d^4 HV_m)g$ assuming magnetic Prandtl number to be small enough and after dropping bars, equations. (4), (5) and (11) become,

$$\left[\left(D^{2}-a^{2}\right)^{2}+Qa^{2}\right]u=\Lambda a^{2}G(x)\left[\left(D^{2}-a^{2}\right)g-NG(x)\theta\right] \tag{12}$$

$$\left[\left(D^{2}-a^{2}\right)^{2}+Qa^{2}\right]g=(1-2x)u \tag{13}$$

$$\left(D^{2}-a^{2}\right)\theta=u \tag{14}$$

where $N=\{\alpha Pr(T_2-T_1)\}/2$ is the temperature gradient parameter, $Pr = \nu/\alpha$ is Prandtl number, $Q=\mu_e H^2 d^2/(4\pi\rho\nu e)$ is the Hartmann number and $G(x) = x(1-x)$.

Also, $\Lambda=72(V_m d/\nu)^2(d/R_2)=72Re^2(d/R_2)$ where $Re = V_m d/\nu$ is the azimuthal Reynolds number. Therefore, the Dean number, $Re(d/R_2)^{1/2}=(\Lambda/72)^{1/2}$.

The boundary conditions for both non-conducting and conducting walls respectively are,

$$u=Du=g=\left(D^{2}-a^{2}\right)g=\theta=0\quad\text{at}\quad x=0,\ 1 \tag{15}$$

$$u=Du=Dg=\left(D^{2}-a^{2}\right)g=\theta=0\quad\text{at}\quad x=0,\ 1 \tag{16}$$

The ordinary differential equations (12)-(14) with boundary condition (15) or (16) determine an eigen value problem of the form,

$$F(N, Q, a, \Lambda) = 0 \tag{17}$$

and is a four parameter eigenvalue problem. For given values of *N*, *Q*, this relationship gives the curve of neutral stability. We are interested in the least positive values of the Dean number $Re(d/R_2)^{1/2}=(\Lambda/72)^{1/2}$ and corresponding value of the wave number *a*, termed as critical Dean number $[Re_c(d/R_2)^{1/2}=(\Lambda_c/72)^{1/2}]$ and critical wave number (a_c); for these are the values which defines the onset of instability. We solve the eigenvalue problem using Harris and Reid [11] method. This method has been also used by for example Mahaptra *et al.* [5] and Deka and Paul [8] for similar instability problems.

3. RESULTS AND DISCUSSION

We are interested in determining the threshold of stationary instabilities; the flow between stationary circular cylinders, driven by a constant azimuthal pressure gradient, when the cylinders are kept at two different temperatures and an axial magnetic field is present. From smaller to large values of *Q*, indicating the strength of the magnetic field is considered. Also, both positive and negative values of *N* are chosen, because *N* depends on the temperature difference of the cylinder walls. Thus, if the temperature of the outer cylinder is higher than that of the inner, then $T_2 - T_1 > 0$ so that $N > 0$. Also, $N < 0$ when $T_2 < T_1$. Hence, we have considered several values of *N* ranging from -1.0 to 1.0 in our study. Ali *et al.* [7] studied the instability of Dean flow in presence of radial heating, which is a special case of the present problem with $Q = 0$. Also, our problem coincides with the work of Walowit *et al.* [10], when $Q = 0$ and $N = 0$.

To conduct our analysis, we compute calculations by the Runge–Kutta Fehlberg integration scheme with an automatic step size control with Harris and Reid method of unit disturbance. First, we compute for the case $N = 0$ and $Q = 0$ and obtain values of the pair (a_c, Λ_c) as (3.96, 92912) while Chadrasekhar [4], Walowit *et al.* [10], Ali *et al.* [7] and Dholey [9] obtained as (3.96, 92975), (3.96, 92794), (3.95, 92782) and (3.95, 92788) respectively. Also, we have computed the critical values for the case $N = 0$ and $Q \neq 0$ and presented in Table 1. The compared values with Chandrasekhar [4] are in good agreement.

Table-1: Comparison between the results of Chandrasekhar [4] (B) with the present results (A) for $N = 0$. The boldfaced data are for conducting walls

N	a_c		$Re_c(d/R_2)^{1/2}$	
	A	B	A	B
0	3.96	3.95	35.93	35.92
30	4.26	4.269	50.87	50.85
50	3.65	3.641	54.95	54.94
100	3.185	3.178	71.56	71.56
100	4.57	4.531	81.50	81.50
1000	0.866	0.866	227.31	227.32

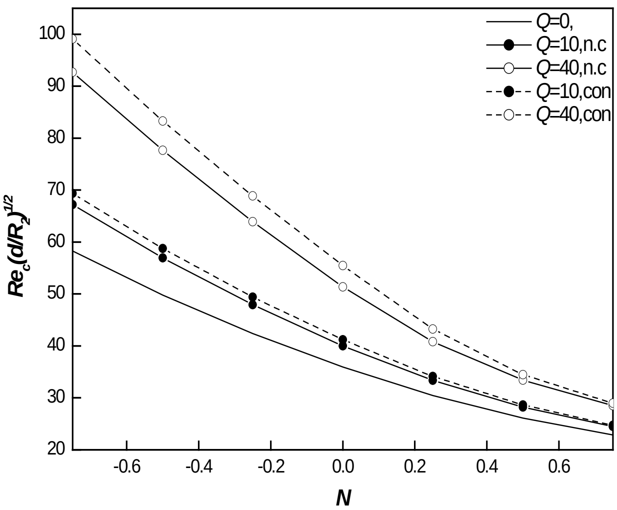

Fig. 1: Variation of $Re_c(d/R_2)^{1/2}$ against N with Q = 0, 10, 40 for both conducting(con) and non conducting(n.c) walls

3.1 Effect of radial temperature gradient:

To study the effect of the radial temperature gradient, the numerical results are shown in Table 2 for N in the range [-1, 1] with different values of Q for both perfectly conducting and non-conducting walls and further illustrated with the help of figures. Figure 1 is drawn against N for Q = 0, 10 and 40. The critical Dean number increases as Q increases as N decreases from a positive value to a negative value. This indicates that radial heating exerts a stabilizing influence of

the flow, when the temperature of the inner cylinder is more than the outer cylinder. This behavior is cross verified in Fig. 2 drawn for critical Dean number against Q for N = -0.25, 0 and 0.25 for both conducting and non-conducting walls. Figure 3 shows the spacing of the vortices for positive as well as negative values of N when Q increases from a lower value to higher values. Obviously, the value of the critical wave number, a_c strongly decreases for increasing magnetic field, when the cylinders are non- conducting and N = 0. Also, when the cylinders are conducting and the cylinders are isothermal, a_c increases (see Chandrasekhar [4]). Therefore, for experimental purpose, due to the smaller sizes of the wave numbers, the insulated cylinders are considered in a laboratory as for larger sizes of the wave numbers, it is difficult to run experiment in a laboratory (see Rüdiger *et al.* [12]). Furthermore, in our present study, by applying radial heating, we have observed that the critical wave number shows a novel behavior. It is interesting to see that a_c decreases as Q increases for both conducting and non-conducting cylinders, when the temperature of inner cylinder is lower than the outer cylinder ($N > 0$). However, for $N < 0$, the wave numbers increases with Q. In all the cases ($N > 0$, $N = 0$, $N < 0$), the wave numbers are larger for conducting walls in comparison to non-conducting walls.

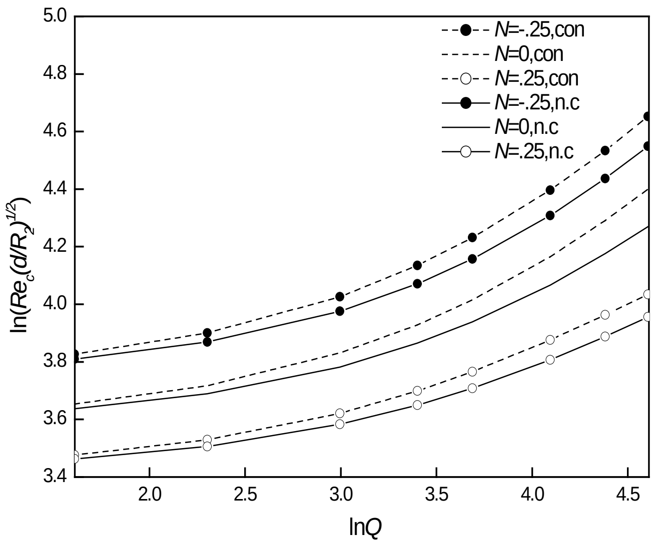

Fig.2: Variation of $Re_c(d/R_2)^{1/2}$ against Q with N = -0.25, 0 and 0.25 for both conducting (con) and non conducting(n.c) wall

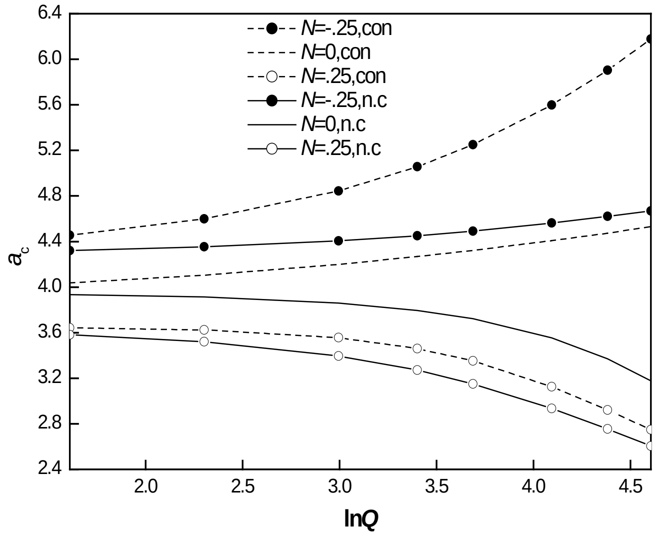

Fig. 3: Variation of wave number a_c against Q with N = -0.25, 0 and 0.25 for both conducting(con) and non-conducting(n.c) walls.

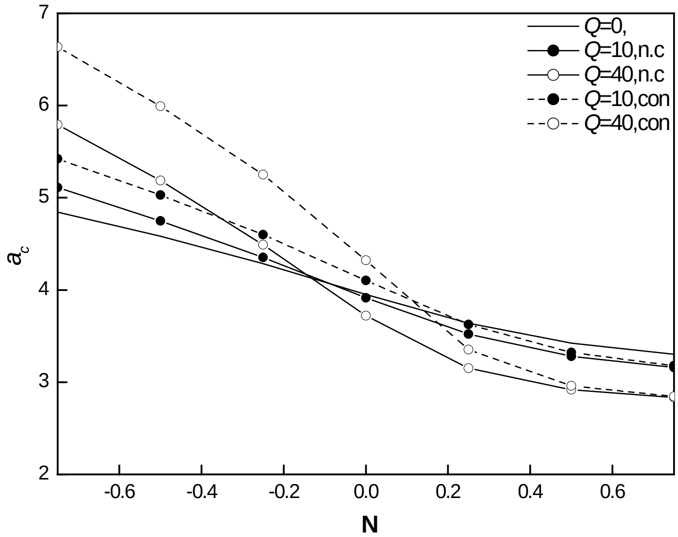

Fig.4: Variation of wave number a_c with against N for Q = 0, 10, 40 for both conducting (con) and non-conducting(n.c) walls.

Figure 4 shows the critical wave number (a_c) evolutions versus N. It is obvious that in non-magnetic case (Q = 0), a_c shows a smooth behavior and increases for $N < 0$ and decreases for $N >0$. However, in the presence of magnetic field, there is a strong increase for $N < 0$, while a weak decrease for $N > 0$. The observation of abrupt changes of the critical wave number was not unique to hydromagnetic Couette flow. The complicated figures of a_c are difficult to be physically explained, since the critical wave number always shows an unsmooth curve when heat is added in the fluid system. In the study of hydrodynamic Couette flow between rotating cylinders, Kong and Liu [13] and Soundalgekar *et al.* [14] observed such an unsmooth behavior.

Table-2: Values of a_c and $Re_c(d/R_2)^{1/2}$ for assign values of N, Q for both conducting and non- conducting walls. The values in the brackets are for non-conducting walls.

N	Q	a_c	$Re_c(d/R_2)^{1/2}$
1.0	0	3.236	20.458
	30	2.888(2.883)	24.360(24.111)
	50	2.721(2.719)	26.372 (26.091)
	100	2.442(2.439)	30.472 (30.186)
0.75	0	3.302	22.880
	30	2.941(2.927)	27.725 (27.276)
	50	2.759(2.750)	30.161 (29.635)
	100	2.460 (2.453)	35.030 (34.475)
0.50	0	3.423	26.123
	30	3.072(3.026)	32.724(31.809)
	50	2.858 (2.825)	36.004 (34.858)
	100	2.507 (2.487)	42.374 (41.052)
0.25	0	3.640	30.447
	30	3.461 (3.271)	40.431 (38.496)
	50	3.238 (3.039)	45.820 (42.987)
	100	2.748 (2.604)	56.504 (52.262)
0	0	3.950	35.920
	30	4.270 (3.795)	50.854 (47.707)
	50	4.368 (3.641)	59.974 (54.938)
	100	4.531 (3.178)	81.499 (71.560)
-0.25	0	4.285	42.392
	30	5.057 (4.451)	62.547 (58.643)
	50	5.430 (4.528)	75.081 (69.131)
	100	6.178 (4.669)	104.856 (94.631)
-0.50	0	4.583	49.796
	10	5.030 (4.749)	58.732(56.924)
	30	5.708 (5.048)	75.373 (70.845)
	40	5.992 (5.186)	83.306 (77.666)

-0.75	0	4.845	58.294
	10	5.422 (5.110)	69.369 (67.221)
	30	6.285 (5.586)	89.588 (84.418)
	40	6.634 (5.794)	99.074 (92.707)
-1.0	0	5.094	68.145
	10	5.802 (5.461)	81.527 (79.019)
	20	6.354 (5.790)	93.793 (89.439)
	30	6.813 (6.079)	105.327 (99.462)

3.2 Inhibiting influence of magnetic field on stability:

Table 2 and fig. 2 show that for a given value of *N*, an increase in Hartmann number *Q* results an increase in the critical Dean number. Thus the magnetic field has an inhibiting effect on the onset of instability. The physical reason for this is that in addition to dissipation of mechanical energy due to viscosity arising out of shearing of fluid between the cylinders, there will be dissipation of magnetic energy due to Joule heating. In a conducting fluid where the effect of electrical resistance is important, fluid of small magnetic Prandtl number P_m as in our present problem, the effect of the magnetic force can be regarded as giving rise to a strong magnetic viscosity in the presence of a magnetic field. Thus a magnetic field exerts a stabilizing influence on the flow. An interesting result which is observed from Table 2 and figure 3 is that for given values of *Q*, and *N*, the critical values of Dean number for conducting walls is greater than the corresponding value for electrically non-conducting walls. Thus we see that the insulating walls are more destabilizing than the conducting walls.

4. CONLUSIONS

A linear instability analysis is presented for narrow-gap hydromagnetic MHD dissipative Couette flow of an incompressible electrically conducting fluid between two concentric circular cylinders in the presence of a uniform axial magnetic field and a radial heating. The motion of the flow is driven by a constant azimuthal pressure gradient around the cylinders. It is shown that for given *N*, the critical Dean number at the onset of instability increases with increase in Hartmann number *Q* for both electrically conducting and electrically insulating walls. It is also found that insulating walls are more destabilizing than the conducting walls. The analysis reveals that the negative radial heating has stabilizing influence on the flow with the corresponding increase in the critical Dean numbers. It is found that when the temperature of the outer cylinder is more than the inner cylinder ($N > 0$), the critical wave numbers decreases, while increases for $N < 0$ as *Q* increases and this is true for both conducting as well as non-conducting walls.

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