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MAGIC HYPERCUBE AND ITS COVER POLYNOMIAL

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ABSTRACT

The four dimensional objects have been already introduced in [2],the standard tesseract in Euclidean 4-space is given as the convex hull of the points (±1,±1,±1 ±1).That is it consists of the points { $(x_1, x_2, x_3, x_4) \in R^4$, $-1 \le x_i \le 1$}. The vertex cover polynomial of a graph G of order n was introduced in [3] it is defined as the polynomial $C(G, x) = \sum_{i=\beta(G)}^{|v(G)|} c(G, i)x^i$, where c(G , i) is the number of vertex covering sets of G of size i, and β(G) is the covering number of G. In this paper we formed a magic 4-dimensional cube and we find the vertex cover polynomial related to the vertices and faces.

***Keywords**: Four dimension, Vertex Covering Set, Vertex Covering Number, Vertex Cover Polynomial.*

INTRODUCTION 1

A generalization of the cube to dimension greater than three, is a 'hypercube'. The tesseract is the four dimensional hypercube or 4-cube. A tesseract is bounded by eight hyper planes ($x_i = \pm 1$). Each pair of non parallel hyper plane intersects to form 24 square faces in a tesseract. Three cubes and three squares intersect at each edge. There are four cubes, six squares, and four edges meeting at every vertex. Over all it consists of 8 cubes, 24 squares, 32 edges, and 16 vertices. Let G = (V, E) be a simple graph. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V / uv \in E\}$ and the closed neighborhood of v is the set $N[v]=N(v) \cup \{v\}$,the open neighborhood of S is N(S) = N(v) and the closed neighborhood of S is $N[S]=N(S) \cup S$. A set $S \subseteq V$ is a vertex covering of G if every edge $uv \in E$ is adjacent to atleast one vertex in S. The vertex covering number β(G) is the minimum cardinality of the vertex covering sets in G.A vertex covering set with cardinality β(G) is called a β-set. Let C(G, i) be the family of vertex covering sets of G with cardinality i and let c(G, i)= |C(G, i) |. The polynomial, $C(G, x) = \sum_{i=\beta(G)}^{|v(G)|} c(G, i)\, x^i$ is defined as the vertex cover polynomial of G. In earlier, many properties of the vertex cover polynomials have been studied.

Definition: 2.1 A tesseract is four dimensional cube in principle obtained by combining two cubes, that is two copies of similar cubes in which the corresponding vertices are connected by edges.

Definition: 2.2 A graph G(V, E) is said to be bipartite the vertex set V can be partitioned into two sets V_1 and V_2, such that every edges of G is adjacent with one end in V_1 and other end in V_2.

Definition: 2.3 A set $I \subseteq V$ is called an independent set if no two elements of I are adjacent in G.

Definition: 2.4 Let C(G, i) be the family of vertex covering sets of G with cardinality i and let c(G, i) = |C(G, i) |. The vertex cover polynomial of G is defined as $C(G, x) = \sum_{i=\beta(G)}^{|v(G)|} c(G, i)\, x^i$.

Definition: 2.5 A set $S \subseteq V$ is said to be a covering of G if every edge of G is incident with at least one element in S.

Definition: 2.6 A set $S \subseteq V$ is an independent set of G, then V – S is a covering of G.

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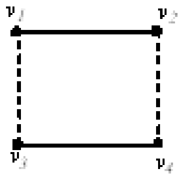
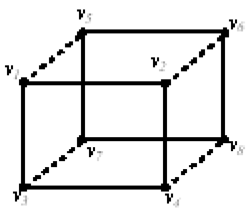
Formation of 4 dimensional cube:

zero dimension
(Figure :1)

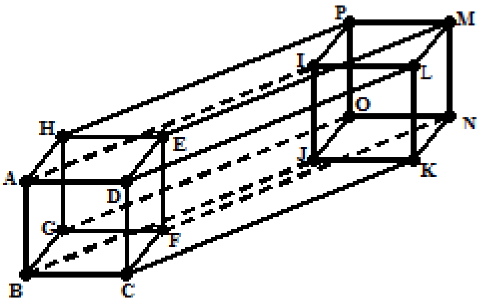
One dimension
(Figure :2)
(Two points v_1 and v_2 are connected by a line that is two copies of v are connected by an edge).



Two dimension
(Figure 3)
(Two parallel line segments v_1v_2 and v_3v_4 can be connected to become a square)
(two copies of v_1v_2 are connected by lines of same length)



Three dimension
(Figure : 4)
(Two copies of a square is connected by lines to form a cube)



Four dimension
(Figure : 5)
Two parallel cubes have to be connected to become a hyper cube.
(Two copies of the cube and the corresponding vertices are connected by lines of same length).

A tesseract (hypercube) in principle is obtained by combining two cubes, and the scheme is similar to the construction of a cube from two squares, and two copies of the lower dimensional cube are connected to the corresponding vertices. Each edge of a tesseract is of same length. Tesseract is a bipartite graph. The bipartite graph of the above tesseract based on its vertices is represented in figure 6.

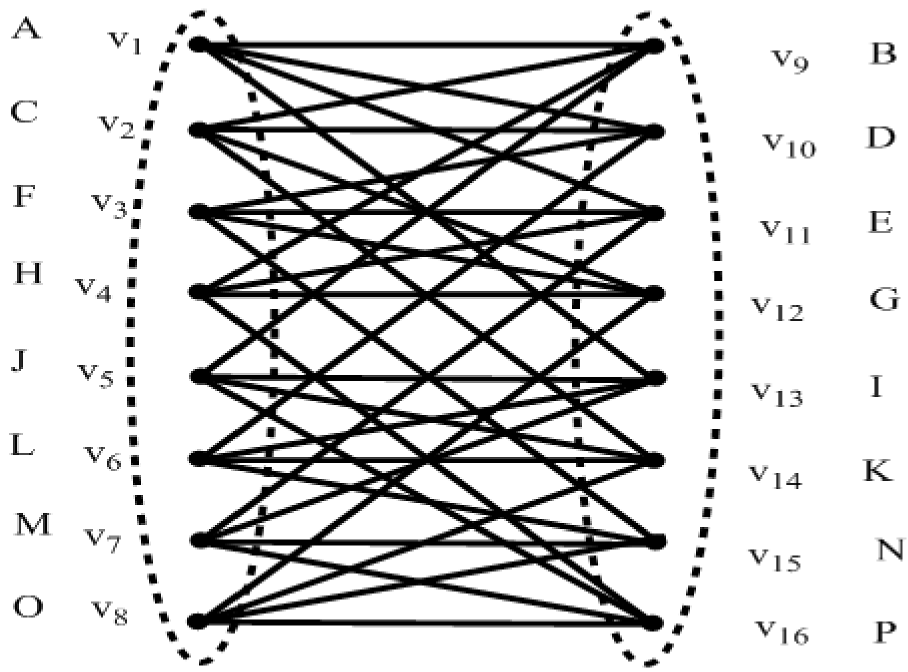


Figure: 6

Let $S_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and $S_2 = \{v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$ be the vertices of G.

Covering polynomial of the tesseract, based on its vertices S_1 and S_2:

Covering sets with cardinality 8 are:

$C(G, 8) = \{\mathbf{S_1}, \mathbf{S_2}\}$

Therefore, $c(G,8) = 2$.

Covering sets with cardinality 9 are

$C(G, 9) = \{\{S_1 \cup \{v_j\} / v_j \in S_2\}; \{S_2 \cup \{v_i\} / v_i \in S_1\}\}$

It can be selected in $2 \times 8C_1$ ways

Therefore, $c(G, 9) = 16$.

Covering sets with Cardinality 10 are

$C(G, 10) = \{\{S_1 \cup \{v_i, v_j\} / v_i, v_j \in S_2\}; \{S_2 \cup \{v_i, v_j\} / v_i, v_j \in S_1\}\}$

It can be selected in $2 \times 8C_2$ ways.

Therefore, $c(G, 10) = 2 \times 8C_2$.

Covering sets with Cardinality 11 are

$C(G,11) = \{\{S_1 \cup \{v_i, v_j, v_k\} / \{v_i, v_j, v_k\} \in S_2\}; \{S_2 \cup \{v_i, v_j, v_k\} / \{v_i, v_j, v_k\} \in S_1\};$
$\{S_1 - \{v_i\} \cup N(v_i) / v_i \in S_1\}, \{S_2 - \{v_i\} \cup N(v_i) / v_i \in S_2\}\}$

Therefore, $c(G, 11) = 2(8C_3 + 8C_1)$

Covering sets with Cardinality12 are

$C(G,12) = \{\{S_1 \cup \{v_i, v_j, v_k, v_l\} / \{v_i, v_j, v_k, v_l\} \in S_2\}; \{S_2 \cup \{v_i, v_j, v_k, v_l\} / \{v_i\, v_j\, v_k\, v_l\} \in S_1\};$
$\{S_1 - \{v_i\} \cup N(v_i) \cup \{v_j\} / \{v_i\} \in S_1, \{v_j\} \in S_2\}; \{S_2 - \{v_i\} \cup N(v_i) \cup \{v_j\} / \{v_i\} \in S_2, \{v_j\} \in S_1\}$.
$\{S_1 - \{v_i, v_j\} \cup N(v_i, v_j) / (v_i, v_j)$ are not an opposite diagonal vertices of hypercube,
$(v_i, v_j) \neq (v_1, v_8), (v_2, v_7)\ (v_3, v_5)\ (v_4, v_6)$ and $N(v_i, v_j) = S_2\}\}$.

Therefore, $c(G, 12) = 2 \times 8C_4 + 2 \times 8C_1 \times 4C_1 + 24$.

Covering sets with cardinality13 are same as the independent sets with cardinality 3.

$I(G, 3) = \{\{\{v_i, v_j, v_k\} / \{v_i, v_j, v_k\} \in S_1\}; \{v_i, v_j, v_k\} / \{v_i, v_j, v_k\} \in S_2\};$
$\{\{v_i, v_j\} \cup \{v_k\} / \{v_i, v_j\} \in S_1, \{v_k\} \in S_2$ and $v_k \notin N(v_i, v_j)\},$
$\{\{v_i, v_j\} \cup \{v_k\} / \{v_i, v_j\} \in S_2; \{v_k\} \in S_1$ and $v_k \notin N(v_i, v_j)\}\}$.

Therefore $c(G,13) = 2 \times [8C_3 + 24 \times 2C_1]$

Covering sets with cardinality 14 is same as the independent sets with cardinality 2 are

$I(G,2) = \{\{\{v_i, v_j\} / \{v_i, v_j\} \in S_1\}; \{\{v_i, v_j\} / \{v_i, v_j\} \in S_2\}; \{\{v_i\} \cup \{v_j\} / v_i \in S_1$ and $v_j \notin N(v_i) \in S_2\}\}$

Therefore, $|I(G, 2)| = c(G, 14) = 2[8C_2 + 4 \times 4C_1]$

Covering sets with cardinality 15 and **16** are:

$c(G, 15) = 16$ and $c(G, 16) = 1$

Therefore, vertex cover polynomial is

$$\begin{aligned} \mathbf{C(G, x)} &= 2x^8 + 8C_1x^9 + 8C_2x^{10} + (8C_3 + 8C_1)\, x^{11} + (8C_4 + 8C_1 \times 4 + 12)\, x^{12} \\ &\quad + (8C_3 + 8C_1 \times 3)\, x^{13} + (8C_2 + 4 \times 4C_1)\, x^{14} + (8\, x^{15}] + x^{16} \\ &= 2x^8 + 16x^9 + 56x^{10} + 128x^{11} + 228x^{12} + 208x^{13} + 88x^{14} + 16x^{15} + x^{16}. \end{aligned}$$

Lemma: 2.7 The coefficient of the vertex cover polynomial $x^{-8}[C(G, x)]$ is log- concave.

Proof: Clearly $a_i^2 \geq a_{i-1}.\, a_{i+1},\ \forall\ i=1, 2,\ldots,.9$.

Therefore, $x^{-8}[C(G, x)]$ is log -concave.

Magic Tesseract:

Now our aim is to form a tesseract, in such a way that facial sums of all cubes are equal.

Let the vertices of the two 3-dimensional cubes to be denoted by ABCDEFGH and IJKLMNOP respectively. From figure: 5, the 24 faces of the tesseract are denoted by the vertices v_1, v_2, v_3, . . . v_{24} as follows:

ABCD – v_1	DELM – v_9	ABGH– v_{17}
CDKL – v_2	CFNK – v_{10}	AHIP – v_{18}
ADIL – v_3	KLMN – v_{11}	BGJO – v_{19}
ABIJ – v_4	EFGH – v_{12}	IJOP – v_{20}
BCJK – v_5	GHOP – v_{13}	BCFG – v_{21}
IJKL – v_6	EHMP– v_{14}	JKNO – v_2
CDEF – v_7	FGNO – v_{15}	ADEH – v_{23}
EFMN – v_8	MNOP– v_{16}	ILMP – v_{24}

By our given notation, the faces of eight cubes and the corresponding vertices are as follows:

Cube 1	**Cube 2**	**Cube 3**	**Cube 4**	**Cube 5**	**Cube 6**	**Cube 7**	**Cube 8**
ABCD– v_1	CDEF– v_7	EFGH– v_{12}	ABGH-v_{17}	BCFG–v_{21}	ADEH–v_{23}	ABCD– v_1	IJKL– v_6
CDKL– v_2	EFMN– v_8	GHOP– v_{13}	ABIJ – v_4	CFKN–v_{10}	ADIL– v_3	CDEF– v_7	KLMN-v_{11}
ADIL– v_3	DELM– v_9	EH MP–v_{14}	AHIP –v_{18}	BCJK –v_5	DELM–v_9	EFGH–v_{12}	MNOP-v_{16}
ADIJ– v_4	CDKL– v_2	EFMN– v_8	GHOP–v_{13}	BGJO –v_{19}	EHMP–v_{14}	ABGH-v_{17}	IJOP –v_{20}
BCJK– v_5	CFKN–v_{10}	FGNO–v_{15}	BGJO–v_{19}	FGNO–v_{15}	AHIP –v_{18}	BCFG–v_{21}	JKNO–v_{22}
IJKL– v_6	KLMN-v_{11}	MNOP–v_{16}	IJOP –v_{20}	JKNO– v_{22}	ILMP –v_{24}	ADEH–v_{23}	ILMP –v_{24}

Table: 1

Three mutually perpendicular set of four surfaces of each cube is denoted as the following:

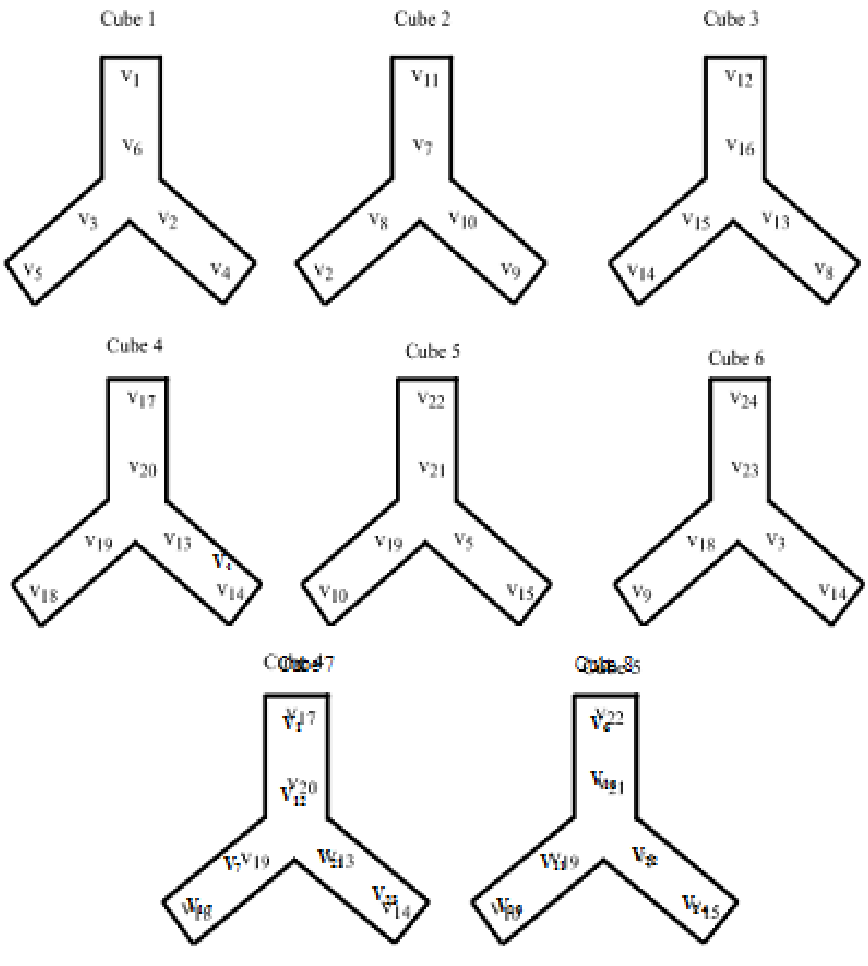


Figure: 7

In cube 1, the three mutually perpendicular four surfaces are (i) v_1,v_6,v_2,v_4 (ii) v_1,v_6,v_3,v_5 (iii)v_5,v_3,v_2,v_4. In similar each cube consists of three mutually perpendicular set of four surfaces is given in figure: 7.

In our tesseract, the common faces of newly formed six cubes other than the basic two cubes are shown below:

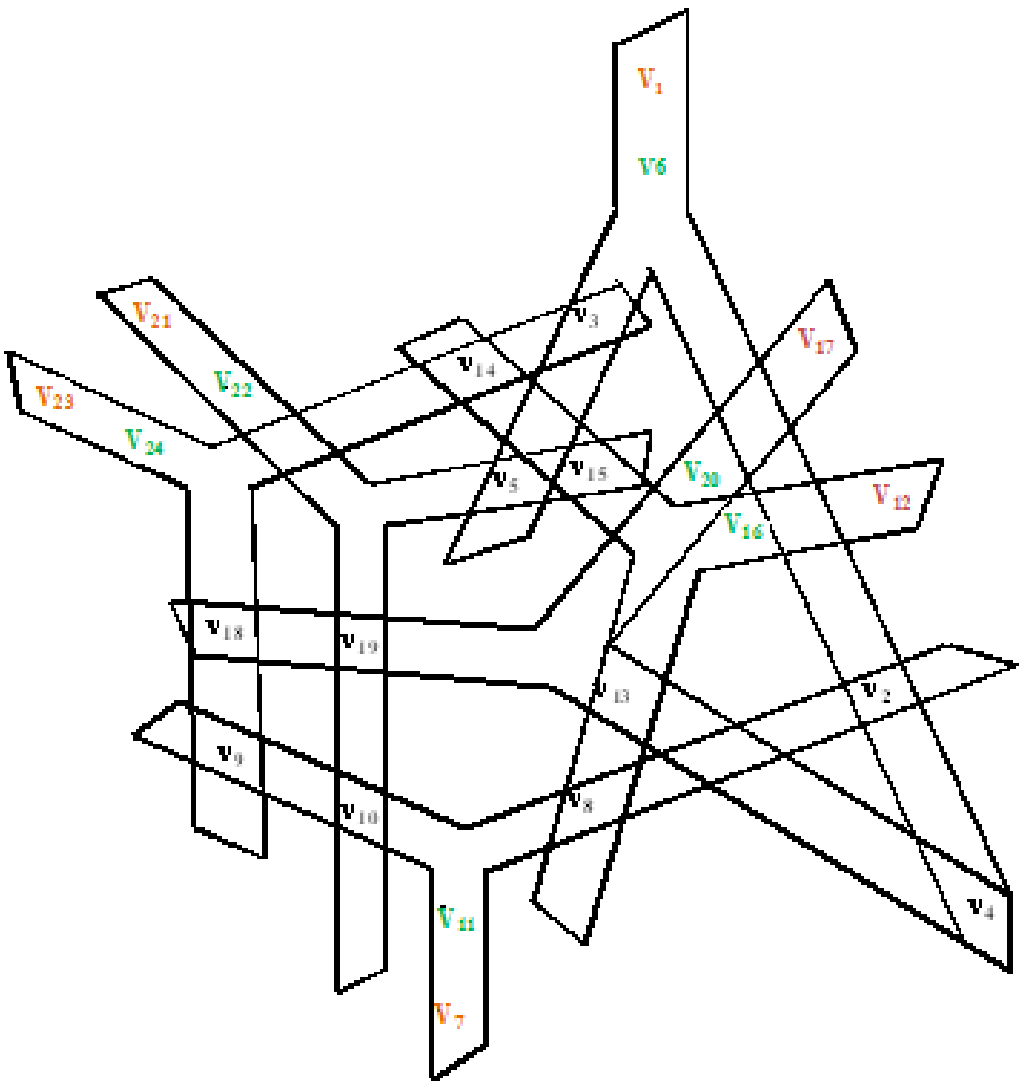


Figure: 8

Now we choose 24 even values from 2 to 48 to fill on the faces of the tesseract such that the sum of the facial values of each cubes are equal to 150.

Cubes							
1	24	22	26	28	42	8	150
2	24	30	6	38	32	20	150
3	36	30	16	40	10	18	150
4	12	46	26	14	34	18	150
5	4	48	16	28	34	20	150
6	36	22	44	14	32	2	150
7	12	48	6	40	42	2	150
8	4	46	44	38	10	8	150

Table: 2

The faces of newly formed six cubes are filled by the values given in table: 2 is represented In figure: 9 as below:

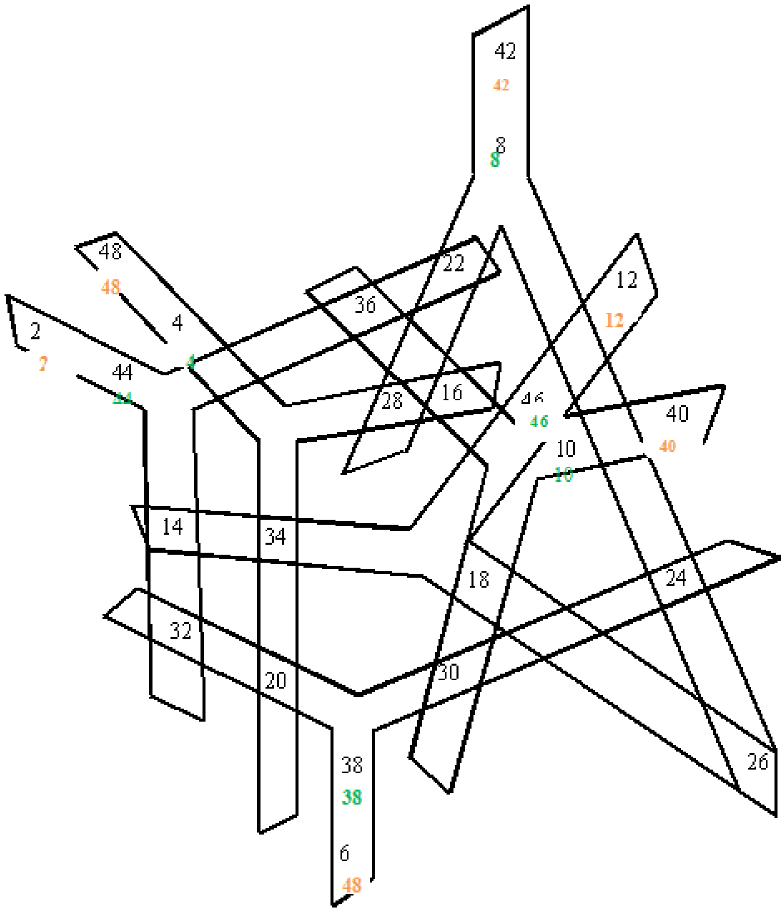


Figure: 9

Each cube and its six facial values are shown as below in figure: 10

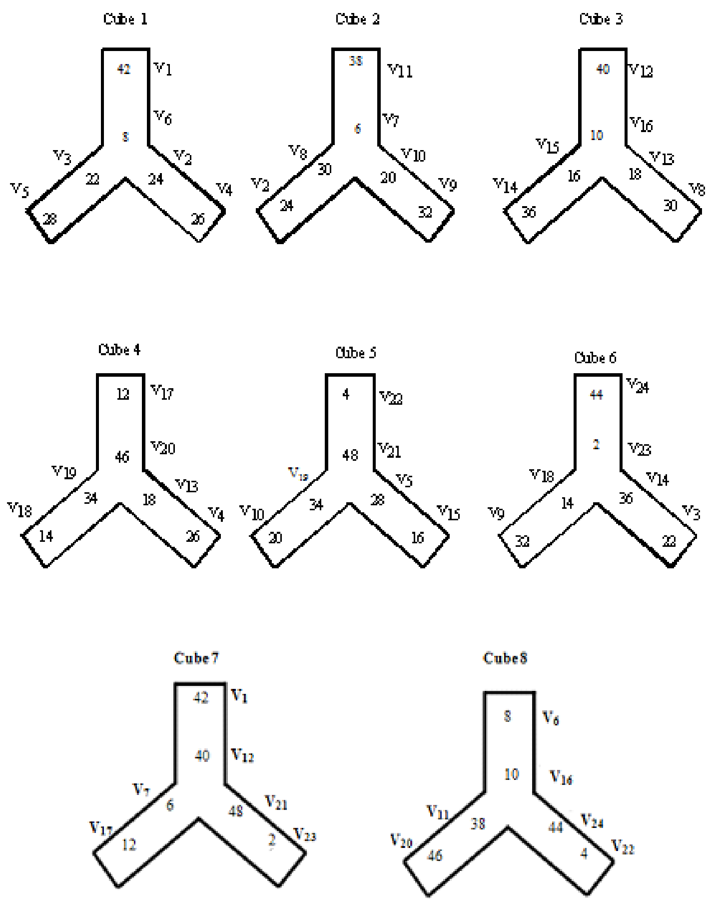


Figure: 10

Now the faces of the tesseract ate consider as the vertices and its adjacency is represented in figure: 11 is shown below:

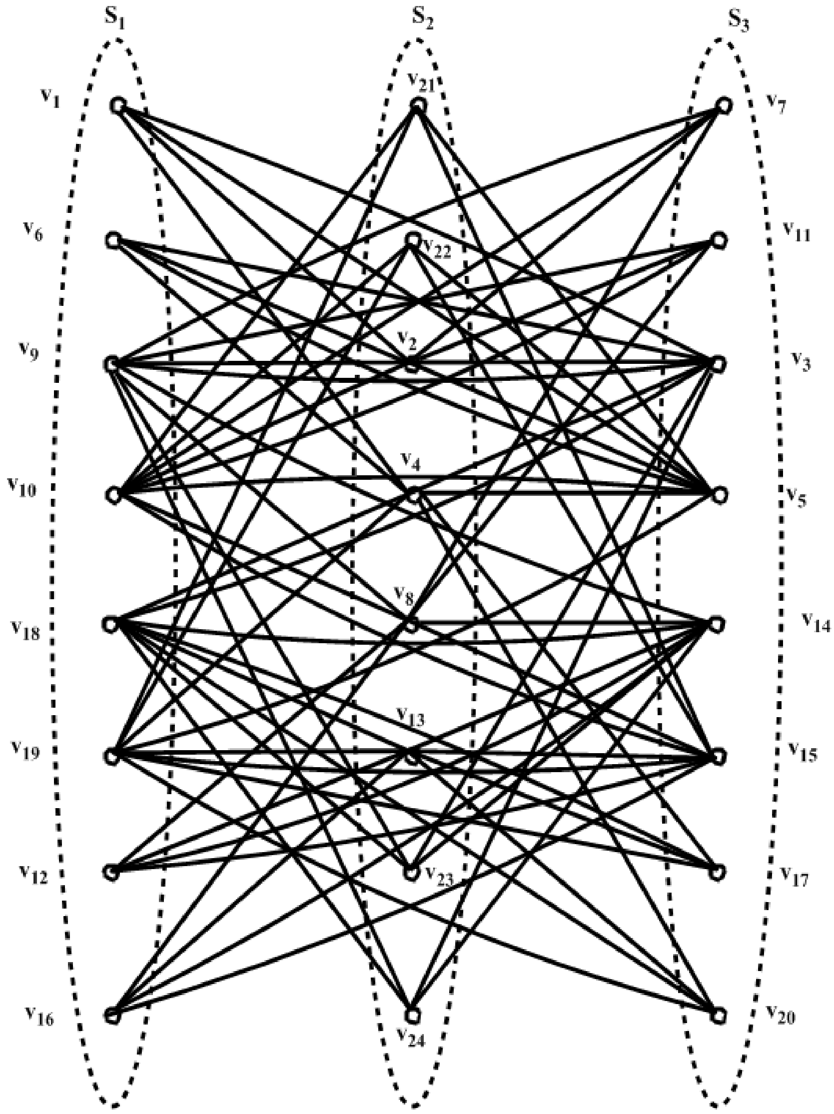


Figure: 11

The vertex cover polynomial based on its faces considered as vertices.

Now the vertices of G can be partitioned into three sets S_1, S_2 and S_3 as given below:

Let $S_1 = \{v_1, v_6, v_9, v_{10}, v_{18}, v_{19}, v_{12}, v_{16}\}$
$S_2 = \{v_{21}, v_{22}, v_2, v_4, v_8, v_{13}, v_{23}, v_{24}\}$.
$S_3 = \{v_7, v_{11}, v_3, v_5, v_{14}, v_{15}, v_{17}, v_{20}\}$

I. Covering set with cardinality 24 is

`$\{S_1 \cup S_2 \cup S_3\}$; Therefore, c(G, 24) = 1

II. Covering sets with cardinality 23 are

$\{\{S_1 \cup S_2 \cup S_3\} - \{v_i\}$ / for every clement $v_i \in S_1 \cup S_2 \cup S_3\}$
Therefore, c(G, 23) = 24

III. Covering sets with cardinality 22 of G is same as the Independent sets with cardinality 2 are

(i) Select $\{\{v_i, v_j\}/ \{v_i, v_j\} \in S_1\}$. it can be selected in $8C_2$ ways.

(ii) For the fixed element $v_i \in S_1$ and $v_j \in S_2$,$\{\{v_i\} \cup \{v_j\}$ / deg $(v_i) = 4$; $v_j \in S_2\}$, It can be selected in $4 \times 6C_1$ ways.

(iii) $\{\{v_i\} \cup \{v_j\}$ / deg $(v_j) = 8$; $v_j \in S_2\}$, It can be selected in $4 \times 4C_2$ ways.

For all three categories,

$c(G, 22) = 3[8C_2 + 4 \times 6C_1 + 4 \times 4C_2]$

IV. Covering sets with cardinality 21 of G is same as the independent sets with cardinality 3 are as follows

(i) Select all three elements from the same set $\{\{v_i\ v_j, v_k\}/\{v_i, v_j, v_k\} \in S_1\}$

It can be selected in $8C_3$ ways and for all the three sets $3 \times 8C_3$ ways.

(ii) Selection of two vertices from one set and one element from any one among the remaining sets.

Select two elements from S_1 and one element from S_2:

$\{\{v_i, v_j\}\cup\{v_k\}/\{v_i, v_j\} \in S_1, \{v_k\} \in S_2$ and $d(v_i) = d(v_j) = 4$ & $N(v_j) = N(v_k)\}$ it can be selected in $2 \times 6C_1$ ways

Select one element from S_1 and two elements from S_2:

$\{\{v_i\}\cup\{v_j, v_k\}/\{v_i\} \in S_1, \{v_j, v_k\} \in S_2$ and $d(v_i) = d(v_k) = 4$& $N(v_j) \neq N(v_k)\}$

it can be selected in $4 \times 4C_1$ ways

$\{\{v_i\}\cup\{v_j, v_k\} /\{v_i\} \in S_1, \{v_j, v_k\} \in S_2; d(v_i) = 4, d(v_k) = 8\}$

$\{\{v_i\}\cup\{v_j, v_k\} /\{v_i\} \in S_1, \{v_j, v_k\} \in S_2; d(v_j) = 4$ and $d(v_k) = 8\}$

it can be selected in $4 \times 4 \times 3C_1$ ways

$\{\{v_i\}\cup\{v_j, v_k\} /\{v_i\} \in S_1, \{v_j, v_k\} \in S_2, d(v_i) = d(v_k) = 8\}$

it can be selected in $4 \times 2C_1$ ways

All six sets have total number of choices

$$= 6\left[\overline{2 \times 6C_1} + \overline{4 \times 4C_1} + \overline{4 \times 4 \times 3C_1} + \overline{4 \times 2C_1}\right]$$

(iii) Selection of one element from each set

$\{\{v_i\}\cup\{v_j\}\cup\{v_k\}/\{v_i\} \in S_1, \{v_j\} \in S_2, \{v_k\} \in S_3, d(v_i) = d(v_j) = d(v_k) = 4\}$

it can be selected in $16 \times 5C_1$ ways

$\{\{v_i\}\cup\{v_j\}\cup\{v_k\}/\{v_i\} \in S_1, \{v_j\} \in S_2, \{v_k\} \in S_3, d(v_i) = d(v_j) = 4$ and $d(v_k) = 8\}$

it can be selected in $16 \times 2C_1$ ways

$\{\{v_i\} \cup\{v_j\} \cup\{v_k\}/\{v_i\} \in S_1, \{v_j\} \in S_2, \{v_k\} \in S_3, d(v_i\} = d(v_j) = d(v_k) = 8\}$

it can be selected in $8 \times 1C_1$ ways

Total number of choices to select one element from each set is

$$\left[\overline{16 \times 5C_1} + \overline{16 \times 2C_1} + \overline{18 \times 1C_1}\right]$$

Therefore,

$$c(G, 20) = 3 \times 8C_3 + 6\left[\overline{2 \times 6C_1} + \overline{4 \times 4C_1} + \overline{4 \times 4 \times 3C_1} + \overline{4 \times 2C_1}\right] + \left[\overline{16 \times 5C_1} + \overline{16 \times 2C_1} + \overline{18 \times 1C_1}\right]$$

IV. Covering sets with cardinality 20 is same as the independent sets with cardinality 4

(i) Selection of four elements from any one among three sets

$\{\{v_i, v_j, v_k\} /\{v_i, v_j, v_k\} \in S_1\}$

It can be selected in $8C_4$ ways. For all the three sets it can be selected in $3 \times 8C_4$ ways.

(ii) Selection of three elements from any one of the set and one element from any one among other two. Now the different ways to select one from S_1 and three from S_2.

$\{\{v_i\}\cup\{v_j, v_k, v_l\}/\{v_i\} \in S_1, \{v_j, v_k, v_l\} \in S_2, d(v_j) = d(v_k) = d(v_k) = 4, v_i \notin N(v_i, v_j, v_k\}\}$

it can be selected in $4 \times 4C_1$ ways.

$\{\{v_i\} \cup\{v_j, v_k, v_l\}/\{v_i\} \in S_1, \{v_j, v_k, v_l\} \in S_2, d(v_j) = d(v_k) = 4; d(v_l)=8; N(v_z) = N(v_k)\}$

it can be selected in $8 \times 3C_1$ ways.

$\{\{v_i\} \cup\{v_j, v_k, v_l\}/\{v_i\} \in S_1, \{v_j, v_k, v_l\} \in S_2, d(v_j) = d(v_k) = 4; d(v_l)=8; N(v_j) \neq N(v_k)\}$

it can be selected in $4 \times 4 \times 2C_1$ ways.

$\{\{v_i\}\cup\{v_j, v_k, v_l\}/d\{v_j\}=4; d(v_k) = d(v_l)=8, v_i \notin N\{v_j, v_k, v_l\} \in S_1\}$

it can be selected in $4 \times 2C_1$ ways

$\{\{v_i\} \cup\{v_j, v_k, v_l\} / d(v_j) = 4; d(v_k) = d(v_l) = 8$

$v_i \notin N\{v_j, v_k, v_l\} \in S_1$ and $N(v_j) = N(v_j) = N(v_k) = S_1 - \{v_i\}\}$

it can be selected in 4×1 ways. for all the six sets the total number of choices

$$6\left[\overline{4 \times 4C_1} + \overline{8 \times 3C_1} + \overline{4 \times 4 \times 2C_1} + \overline{4 \times 2C_1} + \overline{4 \times 1}\right]$$

(iii) Among the three sets, select two sets, and take two elements from each

$\{\{v_i, v_j\}\cup\{v_k, v_l\}/\{v_i, v_j\}\in S_1;\{v_k, v_l\}\in S_2, d(v_k)=d(v_l) = 4 \text{ and } N(v_k) = N(v_l)\}$

it can be selected in $2 \times 6C_2$ ways.

$\{\{v_i, v_j\}\cup\{v_k, v_l\}/\{v_i, v_j\}\in S_1;\{v_k, v_l\}\in S_2, d(v_k) = d(v_l) = 4 \text{ and } N(v_k) \neq N(v_l)\}$

it can be selected in $4 \times 4C_2$ ways.

$\{\{v_i, v_j\}\cup\{v_k, v_l\}/\{v_i, v_j\}\in S_1;\{v_k, v_l\}\in S_2, d(v_k) = 4 \text{ and } d(v_l) =$

it can be selected in $4 \times 4 \times 3C_2$ ways.

$\{\{v_i, v_j\}\cup\{v_k, v_l\}/\{v_i, v_j\}\in S_1;\{v_k, v_l\}\in S_2, d(v_k) = d(v_l) = 8\}$

it can be selected in 4×1 ways.

Total choices for the above category are

$$\left[\overline{2 \times 6C_2} + \overline{4 \times 4C_2} + \overline{4 \times 4 \times 3C_2} + \overline{4 \times 1}\right]$$

for all the three sets, $3\left(\overline{2 \times 6C_2} + \overline{4 \times 4C_2} + \overline{4 \times 4 \times 3C_2} + \overline{4 \times 1}\right)$

(iv) Selection of 2 elements from one set and exactly one from the remaining two

$\{\{v_i\}\cup\{v_j, v_k\}\cup\{v_l\}/\{v_i\in S_1;\{v_j, v_k\}\in S_2;\{v_k\}\in S_3, d(v_i) = d(v_j) = d(v_k) = 4 \text{ \& } N(v_j) = N(v_k)\}$

it can be selected in $8 \times 5C_1$ ways.

$\{\{v_i\}\cup\{v_j, v_k\}\cup\{v_l\}/\{v_i\in S_1;\{v_j, v_k\}\in S_2;\{v_k\}\in S_3, d(v_i) = d(v_j) = d(v_k) = 4 \text{ \& } N(v_j) \neq N(v_k)\}$

it can be selected in $16 \times 4C_1$ ways.

$\{\{v_i\}\cup\{v_j, v_k\}\cup\{v_l\}/\{v_i\}\in S_1;\{v_j, v_k\}\in S_2; \{v_k\}\in S_3$, any one of v_i or v_j or v_k is of degree 8$\}$

it can be selected in $4 \times 10 \times 2C_1$ ways.

Therefore, all the three sets the total number of choices in the above category is

$3\,[8 \times 5C_1 + 16 \times 4C_1 + \overline{4 \times 10 \times 2C_1}\,]$

Therefore, $c(G, 20) = \overline{3 \times 8C_4} + 6\,[(4 \times 4C_1) + (8 \times 3C_1) + (4\times4\times2C_1) + (4\times2C_1) + (4\times1)]$

$+3\left[\overline{2 \times 6C_2} + \overline{4 \times 4C_2} + \overline{4 \times 4 \times 3C_2} + \overline{4 \times 1}\right]$

$+ 3\,[8 \times 5C_1 + 16 \times 4C_1 + \overline{4 \times 10 \times 2C_1}\,]$.

V) Covering sets with cardinality 19 is same as the independent sets with cardinality 5.

(i) Selection of five independent elements from any one of the set $\{S_i - \{v_j, v_k, v_l\} / i = 1, 2, 3\}$ and it can be selected in $3 \times 8C_5$ ways.

(ii) Selection of 4 elements from one set and one element from any one of the remaining two sets

$\{\{v_i\}\cup\{v_j, v_k, v_l, v_m\}/\{v_i\in S_1;\{v_j, v_k, v_m\}\in S_2; d(v_i) = 4;\{v_j, v_k, v_l, v_m\}\notin N(v_i)\}$

it can be selected in $4 \times 6C_4$ ways.

For an six different choices $6 \times 4 \times 6C_4$ ways.

$\{\{v_i\}\cup\{v_j, v_k, v_l, v_m\}/\{v_i\}\in S_1;\{v_j, v_k, v_l, v_m\}\in S_2, d(v_i) = 8;\{v_j, v_k, v_l, v_m\}\notin N(v_i)\}$

it can be choose $4 \times 4C_0$ ways

For all six sets it can be selected in $6 \times 4 \times 4C_0$ ways.

(iii) Selection 3 elements from one set and 2 elements from any one among the other two sets for a fixed set S_1 and S_2.

$\{\{v_i, v_j\}\cup\{v_k, v_l, v_s\}/\{v_i, v_j\}\in S_1;\{v_k, v_l, v_s\}\in S_2, d(v_i) = d(v_j) = 4 \text{ \& } N(v_i) = N(v_j)\}$

it can be selected in $2 \times 6C_3$ ways

$\{\{v_i, v_j\}\cup\{v_k, v_l, v_s\}/\{v_i, v_j\}\in S_1;\{v_k, v_l, v_s\}\in S_2; d(v_i) = d(v_j) = 4 \text{ \& } N(v_i) \neq N(v_j)\}$

it can be selected in $4 \times 4C_3$ ways.

$\{\{v_i, v_j\}\cup\{v_k, v_l, v_s\}/\{v_i, v_j\}\in S_1;\{v_k, v_l, v_s\}\in S_2, d(v_i) = 4 \text{ and } d(v_j) = 8\}$

it can be selected in 4×4 ways

For all sets S_1, S_2 & S_3, The number of choices for the above category

$$6\left[\overline{2 \times 6C_3} + \overline{4 \times 4C_3} + \overline{4 \times 4}\right]$$

(vi) Selection of three elements from one set and each one element from the remaining two set

$\{\{v_i\}\cup\{v_j, v_k, v_l\}\cup\{v_s\} / \{v_i\}\in S_1; \{v_j, v_k, v_l\}\in S_2; \{v_s\}\in S_3$
$d(v_i) = d(v_j) = d(v_k) = d(v_l) = d(v_s) = 4\}$

it can be selected in $4 \times 4 \times 4C_1$ ways.

For a fixed set $S_{1,}$

$\{\{v_i\}\cup\{v_j, v_k, v_l\}\cup\{v_s\} / v_i \in S_1; \{v_j, v_k, v_l\}\in S_2; v_s\in S_3$

$d(v_i) = d(v_j) = d(v_k) = 4; d(v_l) = 8 \ \& \ v_l \notin N(v_i)\}$.

it can be selected in $4 \times 4 \times 2C_1$ ways

Total number of set in the above category for all three sets

$$3\left[\overline{4 \times 4 \times 4C_1} + \overline{4 \times 4 \times 2C_1}\right] \text{ ways}$$

(v) Each two elements from any two sets and one element from the remaining set for a fixed set S_2

Let $\{v_k\} \in S_2$.

$\{\{v_i, v_j\} \cup\{v_k,\}\cup\{v_s, v_t\}\} = 4 \times 5C_2$ sets if $N(v_i) = N(v_j) / d(v_i) = d(v_j) = d(v_k) = 4$

$4 \times 4C_2$ sets if $N(v_i) \neq N(v_j) / d(v_i) = d(v_j) = d(v_k) = 4$

4×4 sets if $d(v_i) = 4$ $d(v_j) = 4$; $d(v_j) = 8$; $d(v_k) = 4$ and $N(v_j) \neq N(v_k)$

2×2 sets if $d(v_i) = 4$ $d(v_j) = 4$; $d(v_k) = 8$ and $N(v_i) = N(v_j)$

Therefore, all sets S_1, S_2 & S_3, total number of different sets are

$$3\left[\overline{4 \times 5C_2} + \overline{4 \times 4C_2} + \overline{4 \times 4} + \overline{4 \times 1}\right]$$

Therefore,

$$c(G, 20) = \overline{3 \times 8C_5} + \overline{6 \times 4 \times 6C_4} + \overline{6 \times 4 \times 1}$$
$$+6\left[\overline{2 \times 6C_3} + \overline{4 \times 4C_3} + \overline{4 \times 4}\right] + 3\left[\overline{4 \times 4 \times 4C_1} + \overline{4 \times 4 \times 2C_1}\right]$$
$$+3\left[\overline{4 \times 5C_2} + \overline{4 \times 4C_2} + \overline{4 \times 4} + \overline{4 \times 1}\right].$$

VI. Covering sets with cardinality 18 is same as the independent sets with cardinality 6.

(i) Selection of six elements from any one of the sets

$\{S_i - \{v_i, v_j\} / \{v_i, v_j\}\in S_i / i = 1, 2, 3\}$

it can be selected in $3 \times 6C_2$ ways

(ii) Selection of five elements from one set and the remaining one element from any one of the other two sets

$\{\{v_j\}\cup\{$any five elements from six elements of $S_j - N(v_i)\}/N(v_i)\in S_i$ & deg $(v_i) = 4\}$

For a fixed set S_j four elements which satisfies the above condition.

Therefore, the total number of sets are $6 \times 4 \times 6C_5$.

(iii) Selection of four elements from one set and each one element from the remaining two sets

$\{\{v_i\}\cup\{v_k\}\cup\{$any four elements from 5 elements of $S_3 - N(v_i, v_j)\}/v_i\in S_1, v_j\in S_2,$

$d(v_i) = d(v_j) = 4$ and $|N(v_i) \cap N(v_j)| = 1\}$

it can be selected in $4 \times 4 \times 5C_4$ ways.

All the sets the total choices are $3 \times 4 \times 4 \times 5C_4$ ways

(iv) Selection of 4 elements from one set and 2 elements from any one among the other two sets

$\{\{v_i, v_j\}\cup$ any four elements from 6 element of $S_2 - N(v_i, v_j) / \{v_i\ v_j\}\in S_1,$
$d(v_i) = d(v_j) = 4$ and $|N(v_i) = N(v_j)\}$

it can be selected in $2 \times 6C_2$ ways.

$\{\{v_i, v_j\}\cup$ all four elements of $S_2 - N(v_i, v_j) / \{v_i\ v_j\}\in S_1,$

$d(v_i) = d(v_j) = 4$ and $|N(v_i) \neq N(v_j)\}$

it can be selected in $4 \times 4C_4$ ways.

All the six sets which satisfy the above category are $6\left[\overline{2 \times 6C_2} + \overline{4 \times 1}\right]$

(v) Selection of each 3 elements from any two sets for a fixed set S_1

$\{\{v_i, v_j, v_k\}\cup\{S_2 - N\{v_i, v_j, v_k\}/\{v_i, v_j, v_k\}\in S_1, d(v_i) = d(v_j) = d(v_k) = 4\}$

it can be selected in $4 \times 4C_3$ ways.

$\{\{v_i, v_j, v_k\}\cup\{S_2 - N(v_i, v_j, v_k) / \{v_i, v_j, v_k\}\in S_1$; any one of the vertices of v_i, v_j, v_k is of degree eight and other two vertices of degree 4$\}$

it can be selected in 8 ways.

Therefore, total sets which satisfy the above conditions are $3\left[\overline{4 \times 4C_3} + 8\right]$ ways.

(vi) Independent sets with selection of any one set containing three elements another one set containing two elements and one element from the remaining set.

$\{\{v_i, v_j\} \cup \{v_k\} \cup$ {any three elements from the five elements of

$S_3 - N(v_i, v_j) \cup \{v_k\} / \{v_i, v_j\} \in S_1; \{v_k\} \in S_2, d(v_i) = d(v_j) = d(v_k) = 4$ and $N(v_i) = N(v_j)\}$

it can be selected in $4 \times 5C_3$ ways

$\{\{v_i, v_j\} \cup \{v_k\} \cup$ {any three elements from four elements of

$S_3 - N(v_i, v_j) \cup N(v_k) / \{v_i, v_j\} \in S_1; \{v_k\} \in S_2, d(v_i) = d(v_j) = d(v_k) = 4$ and $N(v_i) \neq N(v_j)\}$

it can be selected in $4 \times 4C_3$ ways.

Total number of sets which satisfy the above condition is $6\left[\overline{4 \times 5C_3} + \overline{4 \times 4C_3}\right]$

(vii) Selection of exactly two elements from each set

$\{\{v_i, v_j\} \cup \{v_k, v_l\} \cup \{v_s, v_t\} / d(v_i) = d(v_j) = d(v_k) = d(v_l) = 4$

$\{v_i, v_j\} \in S_1; \{v_k, v_l\} \in S_2; \{v_s, v_t\} \in S_3$ and $N(v_i) = N(v_j)\}$

it can be selected in $4 \times 5C_2$ ways

$\{\{v_i, v_j\} \cup \{v_k, v_l\} \cup \{v_s, v_t\} / d(v_i) = d(v_j) = d(v_k) = d(v_l) = 4$

$\{v_i, v_j\} \in S_1; \{v_k, v_l\} \in S_2; \{v_s, v_t\} \in S_3$ and $N(v_i) \neq N(v_j)\}$

it can be selected in $(4C_2 \times 4C_2 - 4) \times 4C_2$ ways.

$\{\{v_i, v_j\} \cup \{v_k, v_l\} \cup \{v_s, v_t\} / d(v_i) = d(v_j) = d(v_k) = 4, d(v_l) = 8$

$N(v_i) \neq N(v_j); \{v_k, v_l\} \notin N(v_i, v_j)\}$

(viii) it can be selected in $4 \times 4 \times 2$ ways.

Therefore, the total number of sets for the above category $\left[\overline{4 \times 5C_2} + \overline{32 \times 4C_2} + \overline{16 \text{ x } 2}\right]$

Therefore,

$$c(G,18) = \overline{3 \times 6C_2} + \overline{6 \times 4 \times 6C_5} + \left(3 \times 4 \times 4 \times 5C_4\right) + 6\left[\overline{2 \times 6C_2} + \overline{4 \times 1}\right]$$
$$+3\left[\overline{4 \times 4C_3} + 8\right] + 6\left[4 \times 5C_3 + 4 \times 4C_3\right] + \left[\overline{4 \times 5C_2} + \overline{32 \times 4C_2} + \overline{16 \text{ x } 2}\right].$$

VI) Covering sets with cardinality 17 is same as an independent sets with cardinality 7.

(i) Selection of seven independent elements from any one of the set is as follows

$\{\{S_i - v_i\} / v_i \in S_1, \ i = 1, 2, 3\}$

The number of sets which satisfy the above conditions are $3 \times 8C_7$

Selection of 6 elements from one set and one element from the remaining two sets. Let $v_i \in S_1$

$\{\{v_i\} \cup \{S_j - N(v_i) / d(v_i) = 4\}$ in all the six different choices

the total number of sets are 6×4

(ii) Selection of 5 elements from one set and two elements from any one of the remaining two, for the fixed sets S_1 and S_2.

$\{\{v_i, v_j\} \cup S_2 - (v_i, v_j)\} / (v_i, v_j) \in S_1;$
$D(v_i) = d(v_j) = 4$ and $N(v_i) = N(v_j)\}$
it can be selected in 2 ways

Therefore, the total choice for all in combinations are 6×2 sets

(iii) Selection of 5 elements from one set and each one element from the remaining two sets

$\{\{v_i\} \cup \{v_j\} \cup S_3 - N(v_i, v_j) / v_i \in S_1, v_j \in S_2);$

$d(v_i) = d(v_j) = 4$ & $|N(v_i) \cap N(v_j)| = 1\}$
it can be selected in 8 ways

for all the three sets, the number of sets which satisfy the above conditions are 3×8

(iv) Section of 4 elements from one set 2 elements from another set and one element it from the remaining are

$\{\{v_i\} \cup \{v_j, v_k\} \cup$ {any four elements from the five elements of

$S_3 - N(v_i) \cup N(v_j, v_k)\} / v_i \in S_1, v_j, v_k \in S_2); d(v_i) = d(v_j) = d(v_k) = 4$ & $N(v_j) = N(v_k)\}$

It can be selected in $4 \times 5C_4$ ways

$\{\{v_i\} \cup \{v_j, v_k\} \cup S_3 - N(v_i) \cup N(v_j, v_k)\} / \{v_i \in S_1\}, \{v_j, v_k \in S_2\});$

$d(v_i) = d(v_j) = d(v_k) = 4; |N(v_j) \cap N(v_k)| = 4\}$

The number of sets which satisfy the conditions are 4×4 ways

Therefore, for all six different choices total number of sets which satisfy the above conditions are

$$6\left[\overline{4\times 5C_4} + \overline{4\times 4}\right]$$

(v) Selection of three elements from any two sets and one element from the remaining one

$\{\{v_i, v_j, v_k\} \cup \{v_s\} \cup \{v_l, v_m, v_n\} / \{v_i, v_j, v_k\} \in S_1; \{v_s\} \in S_2;$

$\{v_l, v_m, v_n\} \in S_3, d(v_i) = d(v_j) = d(v_k) = d(v_l) = 4\}$

it can be selected in $4 \times 4C_3 \times 4C_3$ ways.

for all 3 fixed choice total number of sets which satisfy above conditions are

$3\left[\overline{4\times 4C_3 \times 4C_3}\right]$ collections

(vi) Selection of three elements from one set, and each two elements from the remaining two set

$\{\{v_i, v_j\} \cup \{v_k, v_l\} \cup \{v_r, v_s, v_t\} / \{v_i, v_j\} \in S_1; \{v_k, v_l\} \in S_2; \{v_r, v_s, v_t\} \in S_3$

$d(v_i) = d(v_j) = d(v_k) = d(v_l) = 4, N(v_i) = N(v_j)$ & $N(v_k) = N(v_l)\}$

The number of sets for the above category are $4 \times 5C_3$

$\{\{v_i, v_j\} \cup \{v_k, v_l\} \cup \{v_r, v_s, v_t\} / \{v_i, v_j\} \in S_1; \{v_k, v_l\} \in S_2; \{v_r, v_s, v_t\} \in S_3$

$d(v_i) = d(v_j) = d(v_k) = d(v_l) = 4$ and $N(v_i) \neq N(v_j)\}$

The number of sets which satisfy above conditions are $4 \times 4C_2 \times 4C_3$

For all three different sets the total number of choices $3[\overline{4\times 5C_3 + 4 \times 4C_2 \times 4C_3}]$

Therefore ,

$$c(G, 17) = (3\times 8C_7) + (6\times 4) + (6\times 2) + (3\times 8) + 6\,[4\times 5C_4 + 4\times 4] + 3\left[\overline{4\times 4C_3\times 4C_3}\right] + 3\left[\overline{4\times 5C_3} + \overline{4\times 4C_2 \times 4C_3}\right]$$

VII) Covering sets with cardinality sixteen is equal to the Independent set with cardinality 8

(i) Selection of eight elements from each set $\{S_i / i = 1, 2, 3\}$. This can be selected in 3 ways.

(ii) Selection of six elements from one set and 2 elements from one among the other two sets

$\{\{v_i, v_j\} \cup \{S_2 - N(v_i, v_j) / \{v_i, v_j\} \in S_1; d(v_i) = d(v_j) = 4; N(v_i) = N(v_j)\}$

Two sets which satisfy the above condition.

For the all six sets. Total number of choices are 6×2

(iii) Selection of five element from any one, two element from one among the remaining two, one element from the other, the six different categories are

$\{\{v_i\} \cup \{v_j, v_k\} \cup S_3 - N(v_i) \cup N(v_j, v_k)\} / v_i \in S_1, v_j, v_k \in S_2);$

$d(v_i) = d(v_j) = d(v_k) = 4; N(v_j) = N(v_k)\}$

For each $v_i \in S_1$, two elements $\{v_j, v_k\} \in S_2$ satisfies the above condition.

Therefore 4×2 number of sets which satisfy the above condition.

For all the six different choices $6 \times 4 \times 2$ sets have the above property.

(iv) Selection of each four elements from any two sets and one element from the other

$\{\{v_i, v_j\} \cup \{v_k, v_l\} / \{v_i, v_j\} \in S_1, \{v_k, v_l\} \in S_2); d(v_i) = d(v_j) = d(v_k) = d(v_l) = 4$

$N(v_i) = N(v_j)$ & $N(v_k) = N(v_l)\}$

Totally 3 sets which satisfy the above property

(v) Selection of Four elements from one set, three elements from another & one from the remaining one

$\{\{v_i, v_j, v_k, v_l\} \cup \{v_r, v_s, v_t\} \cup \{v_m\} / \{v_i, v_j, v_k, v_l\} \in S_1,$

$\{v_r, v_s, v_t\} \in S_2; \{v_m\} \in S_3;$ all vertices of degree 4}

It can be selected in 16 ways.

For all six different choices 6×16 sets which satisfy the above category.

(vi) Selection of four elements from any one and each two elements from the remaining two sets.

$\{\{v_i, v_j, v_k, v_l\} \cup \{v_r, v_s\} \cup \{v_m, v_n\} / \{v_i, v_j, v_k, v_l\} \in S_1, \{v_r, v_s\} \in S_2; \{v_m, v_n\} \in S_3;$

$d(v_i) = d(v_j) = d(v_k) = d(v_l) = d(v_r) = d(v_s) = d(v_m) = d(v_n) = 4\}$

It can be selected in $1 \times 4C_2 \times 4C_2$ ways

For all the 3 different choice the number of sets satisfy the above category $3 \times 4C_2 \times 4C_2$

(vii) Selection of each 3 elements from any two sets and 2 from the remaining set

$\{\{v_i, v_j, v_k\} \cup \{v_r, v_s, v_t\} \cup \{v_l, v_m\} / \{v_i, v_j, v_k\} \in S_1,$

$\{v_r, v_s, v_t\} \in S_2; \{v_l, v_m\} \in S_3;$ all vertices of degree 4}

It can be selected in $4C_3 \times 4C_3 \times 4C_2$ ways.

(viii) For all 3 different choices the number of sets which satisfy the above condition are

$3 \times 4C_3 \times 4C_3 \times 4C_2$

Therefore, the covering sets with cardinality 16 is

$c(G, 16) = 3 + \overline{6 \times 2} + \overline{6 \times 4 \times 2} + 3 + \overline{6 \times 16}$

$+ \overline{3 \times 4C_2 \times 4C_2} + \overline{3 \times 4C_3 \times 4C_3 \times 4C_2}$]

Therefore, the vertex covering polynomial

$C(G, x) = x^{24} + 24x^{23} + 228x^{22} + 802x^{21} + 1584x^{20} + 1524x^{19} + 1305x^{18} + 900x^{17} + 558x^{16}$ (A)

Lemma: 2.8 The coefficients of the cover polynomial of (A) is x^{-16}[C(G, x)] is log-concave.

Proof: Clearly $a_i^2 \geq a_{i-1}. a_{i+1}, \forall\ i = 1,2,\ldots,9$.

Therefore, x^{-16}[C(G, x)] is log-concave.

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