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36 RUN SPLIT-PLOT DESIGNS CONSTRUCTED FROM MIXED-LEVEL ORTHOGONAL ARRAYS

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ABSTRACT

In this paper, we use orthogonal arrays of mixed levels to construct some new 36 run designs within a split-plot structure, where factors have different number of levels. The D-values of the designs for different variance ratios are also computed.

Keywords: *Orthogonal arrays; Hard-to-change factors; Easy-to-change factors; Split-plot designs; D-values.*

AMS Subject Classification: *Primary 62K20.*

1. INTRODUCTION

In many industrial experiments we often encounter situations where two types of factors exist: hard-to-change (HTC) factors, whose levels are difficult to change, and easy-to-change (ETC) factors, whose levels are easy-to-change. In such situations complete randomization of the design is not appropriate and so split-plot structure is used. HTC factors are randomly applied to large sections of the units, called whole-plots. These whole-plots are then split into smaller units, called sub-plots and ETC factors are randomly applied to these plots. This leads to two error terms, one for the whole-plot treatments and second for the sub-plot treatments and the interaction between whole-plot and sub-plot treatments.

Split-plot designs are extensively used in agricultural and industrial experiments. Some references on split-plot designs are Box and Jones (1992), Bisgaard and Kulahci (2001), Goos and Vandebroek (2001), Kowalski (2002), Kowalski *et al.* (2007), Parker *et al.* (2008) and Jones and Nachtsheim (2009).

When the number of factors is large the regular designs have fairly large number of runs. In some circumstances, it is desirable to reduce the number of runs as much as possible while maintaining the ability to estimate all of the terms in the model. In literature, orthogonal arrays are available also which can be used to construct the designs with comparatively less number of runs for same number of factors.

Orthogonal arrays (OA) are beautiful and useful. They are essential in Statistics and they are used in many areas of research including computer science, medicine, agriculture and manufacturing. Orthogonal arrays were introduced by Rao (1947, 1973). There are two types of orthogonal arrays: symmetrical and asymmetrical (or mixed-level) orthogonal arrays. Here we use asymmetrical orthogonal arrays in which the various factors have different number of levels.

An orthogonal array OA$\left(N, s_1^{m_1}, s_2^{m_2}, \ldots, s_\gamma^{m_\gamma}, t\right)$ of strength t is an N×m matrix, $m = m_1 + m_2 + \ldots + m_\gamma$, in which m_i columns have s_i (≥ 2) symbols or levels such that, for any t columns, all possible combinations of symbols appear equally often in the matrix. For $\gamma = 1$, all factors have the same number of levels. These arrays are called symmetrical orthogonal arrays. For $\gamma > 1$, they are called asymmetrical (or mixed-level) orthogonal arrays.

There are situations, in particular, in industrial research where we need a design with various factors having different number of levels. Schoen *et al.* (2011) have constructed D-optimum split-plot designs with two HTC and seven ETC factors, at different levels, to redesign the production process for making coffee cream maintaining the same viscosity as the old one for the coffee-cream experiment.

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Here, we have used the columns of the orthogonal arrays OA(36, $2^{11} \times 3^{12}$, 2) given by Zhang *et al.* (2001) to construct the 36 run designs under split-plot structure. We have constructed the designs having seven factors with one, two and three HTC factors. Here HTC and ETC factors have different number of levels. We have then obtained D-values of the designs constructed. Since there is less number of runs involved to estimate the terms in the model, these designs are more economical.

The organization of this paper is as follows. In Section 2, we give the model and formula to compute the D-values. We describe the method of construction of the designs in Section 3.We illustrate the method with the help of examples. In Section 4, we conclude this paper with some discussion. The D-values and column combinations for the designs with one HTC and six ETC and three HTC and four ETC factors for different variance ratios are given in the Annexure I. The complete list of D-values and all column combinations is available with the Authors.

2. MODEL AND D-VALUES

Usually, in industrial experiments the goal of an experimenter is to estimate all the main effects and also all two factor interactions: among HTC factors, between HTC and ETC factors and among ETC factors, while keeping the number of runs as small as possible. So the purpose of the designs constructed here is to achieve all goals.

Consider a split-plot design involving n HTC factors denoted by z_i (i =1, 2,…, n) and m ETC factors denoted by x_j (j =1, 2,…, m). The Model, considered here, for the u^{th} (u = 1, 2, …, N) run is

$$y_u = \mu + \sum_{i=1}^{n} a_i z_{iu} + \sum_{i=1}^{n} b_j x_{ju} + \sum_{\substack{i \neq i=1 \\ i<i'}}^{n} a_{ii'} z_{iu} z_{i'u} \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} z_{iu} x_{ju} + \sum_{\substack{j \neq j=1 \\ j<j'}}^{m} b_{jj'} x_{ju} x_{j'u} + \delta_u + \varepsilon_u \tag{2.1}$$

where, y_u is the response, μ is the general mean effect, a's, b's and c's are regression coefficients, δ_u is whole-plots error and ε_u is sub-plots error with σ_δ^2 as whole-plots error variance and σ_ε^2 as sub-plots error variance. Here,
$\mathrm{Var}(y_u) = \sigma_\delta^2 + \sigma_\varepsilon^2$, for all u.

$$\mathrm{Cov}(y_u, y_{u'}) = \begin{cases} \sigma_\delta^2, & \text{for } u \neq u' \\ 0, & \text{otherwise} \end{cases}$$

Various optimality criteria for the study of split-plot designs are available, of which D-optimality criteria is most frequently used. In this paper, we have obtained D-values of the designs constructed. The formula for D-value is

$$D = \frac{1}{N} \left| X' V^{-1} X \right|^{1/P} \tag{2.2}$$

where, X is the model matrix, V is the variance-covariance matrix, which is the block diagonal matrix with blocks of $\sigma_\varepsilon^2 I + \sigma_\delta^2 J = \sigma_\varepsilon^2 (I + \eta J)$ where η is variance ratio ($=\sigma_\delta^2/\sigma_\varepsilon^2$), I is the identity matrix and J is the matrix of ones, N is the total number of runs in the design and p is the number of parameters involved in the model.

3. METHOD OF CONSTRUCTION

In this paper, we have constructed designs based on the model discussed in Section 2. We have considered three cases: designs with one HTC and six ETC factors, two HTC and five ETC factors and three HTC and four ETC factors. We have also fixed all HTC factors at two levels and all ETC factors at three levels. The procedure for constructing the designs having n HTC and m ETC factors (where, n = 1, 2, 3 and m = 4, 5, 6) is as follows:

Consider mixed-level orthogonal array OA(36, $2^{11} \times 3^{12}$, 2) given by Zhang *et al.* (2001). Consider the design with n HTC and m ETC factors. Next, we take first combination of n columns out of ${}^{11}C_n$ column combinations at two levels and first combination of m columns out of ${}^{12}C_m$ column combinations at three levels. The combination of these (n + m) columns forms the design in which first n columns are allocated to HTC factors and next m columns are allocated to ETC factors. The design is then sorted in ascending (or descending) order on the basis of HTC factors. This forms split-plot design with whole-plots and sub-plots. We now generate the model matrix X as per equation (2.1).The D-values, using equation (2.2), for different values of the variance ratio (η) are then computed. The above procedure is repeated for all possible ${}^{11}C_n$ and ${}^{12}C_m$ column combinations of the orthogonal array. The design having the highest D-value is selected.

The above procedure is explained with the help of the following example.

Example 3.1: Consider the case of two HTC and five ETC factors. Let us consider the column combination with column numbers: (6, 7, 14, 17, 19, 20, 23). The HTC factor z_1 and z_2 are allocated to columns 6 and 7 respectively and the ETC factors x_1, x_2, x_3, x_4 and x_5 are assigned to column14, 17, 19, 20 and 23 respectively. The design is given in Table 1.

Table-1: Design with two HTC and five ETC factors

z_1	z_2	x_1	x_2	x_3	x_4	x_5
-1	-1	-1	1	-1	1	-1
-1	1	-1	-1	1	-1	1
1	1	1	1	-1	-1	1
-1	1	1	-1	1	1	-1
-1	1	1	0	0	0	1
-1	-1	1	-1	-1	1	0
1	1	-1	0	0	1	0
1	-1	-1	-1	-1	0	1
1	-1	-1	1	0	-1	-1
1	-1	-1	0	1	0	0
-1	-1	1	1	0	0	0
1	1	1	0	1	-1	-1
-1	-1	0	-1	0	-1	0
-1	1	0	0	-1	0	-1
1	1	-1	-1	0	0	-1
-1	1	-1	0	-1	-1	0
-1	1	-1	1	1	1	-1
-1	-1	-1	0	0	-1	1
1	1	0	1	1	-1	1
1	-1	0	0	0	1	-1
1	-1	0	-1	1	0	0
1	-1	0	1	-1	1	1
-1	-1	-1	-1	1	1	1
1	1	-1	1	-1	0	0
-1	-1	1	0	1	0	1
-1	1	1	1	0	1	0
1	1	0	0	1	1	0
-1	1	0	1	0	0	1
-1	1	0	-1	-1	-1	0
-1	-1	0	1	1	0	-1
1	1	1	-1	-1	0	-1
1	-1	1	1	1	-1	0
1	-1	1	0	-1	1	1
1	-1	1	-1	0	-1	-1
-1	-1	0	0	-1	-1	-1
1	1	0	-1	0	1	1

This design is first sorted on the basis of HTC factor z_1 and then on the basis of z_2 in ascending (or descending) order. This generates four whole-plots of size nine each. This is shown in Table 2.

Table-2: Design under Split-plot Structure with two HTC and five ETC factors

z_1	z_2	x_1	x_2	x_3	x_4	x_5
-1	-1	-1	1	-1	1	-1
-1	-1	1	-1	-1	1	0
-1	-1	1	1	0	0	0
-1	-1	0	-1	0	-1	0
-1	-1	-1	0	0	-1	1
-1	-1	-1	-1	1	1	1
-1	-1	1	0	1	0	1
-1	-1	0	1	1	0	-1
-1	-1	0	0	-1	-1	-1
-1	1	-1	-1	1	-1	1
-1	1	1	-1	1	1	-1
-1	1	1	0	0	0	1
-1	1	0	0	-1	0	-1
-1	1	-1	0	-1	-1	0
-1	1	-1	1	1	1	-1
-1	1	1	1	0	1	0
-1	1	0	1	0	0	1
-1	1	0	-1	-1	-1	0
1	-1	-1	-1	-1	0	1
1	-1	-1	1	0	-1	-1
1	-1	-1	0	1	0	0
1	-1	0	0	0	1	-1
1	-1	0	-1	1	0	0
1	-1	0	1	-1	1	1
1	-1	1	1	1	-1	0
1	-1	1	0	-1	1	1
1	-1	1	-1	0	-1	-1
1	1	1	1	-1	-1	1
1	1	-1	0	0	1	0
1	1	1	0	1	-1	-1
1	1	-1	-1	0	0	-1
1	1	0	1	1	-1	1
1	1	-1	1	-1	0	0
1	1	0	0	1	1	0
1	1	1	-1	-1	0	-1
1	1	0	-1	0	1	1

Using the above design, model matrix X is generated according to equation (2.1). The D-values of the design for different values of variance ratio (η) are then obtained using equation (2.2) and are:

Variance Ratio (η)	0.1	0.2	0.3	0.4	0.5
D-values	0.4290	0.4066	0.3913	0.3797	0.3705

The above column combinations give the highest D-values of the design. There are other column combinations with the same D-values.

Same procedure is applied for constructing the designs with one HTC and six ETC factors and three HTC and four ETC factors. The highest D-values and the column combinations for these designs for different variance ratios (η) are given in Annexure I.

4. CONCLUSIONS

In most of the situations, when the number of factors is large, due to time and/or cost constraints, it is required to use a design with less number of runs to estimate the parameters in the model. Also there are cases where HTC and ETC factors are at different levels. In this paper we have constructed the designs involving less number of runs and in which factors are at different levels.We have considered up to three HTC factors and up to six ETC factors. We have used mixed-level orthogonal array to construct the designs under split-plot structure. We have also computed the D-values of these designs for different variance ratios (η). Some of the designs having highest D-values for different cases are given in the Annexure I.

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ANNEXURE- I

Table-3
Highest D-values of the Design having one HTC and six ETC Factors with Column Combination (3, 12, 14, 16, 19, 21, 23)

Variance Ratio (η)	0.1	0.2	0.3	0.4	0.5
D-values	0.3803	0.3675	0.3592	0.3531	0.3483

Table-4
Highest D-values of the Design having three HTC and four ETC Factors with Column Combination (2, 5, 7, 15, 16, 17, 21)

Variance Ratio (η)	0.1	0.2	0.3	0.4	0.5
D-values	0.4780	0.4450	0.4210	0.4020	0.3870

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