



UNIQUE FIXED POINT THEOREMS ON HAUSDORFF SPACES

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(Received on: 11-06-11; Accepted on: 30-06-11)

ABSTRACT

In present paper, some fixed point theorems in Hausdorff spaces have been proved, which generalize the theorem of YEH, C.C. [2].

$$d(Tx, Ty) < d(x, y); \quad \forall x, y \in X .$$

Keywords and Phrases: Result on Hausdorff spaces, fixed point theorem.

MSC (2000): 54H25, 28A78.

INTRODUCTION:

In 1962, M. Edelstein [1] proved the following result:

Theorem 1: Let (X, d) be a metric space, T be a contractive self-mapping of X . If for some $x \in X$, the sequence of iterates $\{T^n x\}$ has a convergent subsequence $\{T^{n_i} x\}$ converging to a point $x_0 \in X$, then $x_0 = \lim \{T^n x\}$ is unique fixed point.

Singh, S.P. and Zorzitto [3] generalized the result of M. Edelstein [1].

Here, we prove the following theorems, which generalized the result of YEH, C.C. [2]:

Theorem 2: Let T be a continuous mapping of a Hausdorff space X into itself and let f be a continuous mapping of $X \times X$ into non-negative reals such that

$$f(x, x) = 0; \quad \forall x \in X \quad \text{and} \quad f(x, y) \neq 0, \quad \forall x, y \in X, \quad x \neq y \tag{2.1}$$

$$f(x, z) \leq f(x, y) + f(y, z), \quad \text{for all } x, y, z \text{ in } X \tag{2.2}$$

$$f(Tx, Ty) \leq q \max \left\{ f(x, y), f(x, Tx), f(y, Ty), f(y, Tx), \frac{1}{2} f(x, Ty), \frac{f(x, Tx)f(y, Ty)}{f(x, y)}, \right. \\ \left. \frac{f(x, Ty)f(y, Tx)}{f(x, y)}, \frac{f(x, y)f(x, Ty)}{f(x, y) + f(Tx, Ty)}, \frac{f(x, Ty)f(Tx, Ty)}{f(x, y) + f(Tx, Ty)}, \frac{f(x, Tx)f(x, Ty)}{f(x, y) + f(Tx, Ty)}, \frac{f(y, Ty)f(x, Ty)}{f(x, y) + f(Tx, Ty)}, \right. \\ \left. \frac{[f(x, y)f(Tx, Ty) + f(x, Ty)f(y, Ty)]}{2f(x, y) + f(Tx, Ty)} \right\} \tag{2.3}$$

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hold for all $x, y \in X$ where $q \in [0,1)$. Then T has a unique fixed point in X.

Proof: Let x_0 be an arbitrary point in X and a sequence $\{x_n\}$ defined by $Tx_n = x_{n+1}$, n a positive integer. If for some n, $Tx_n = x_n$ then x_n is a fixed point. If not $Tx_n \neq x_n$ then by (2.3), we have.

$$\begin{aligned} f(x_1, x_2) &= f(Tx_0, Tx_1) \\ &\leq q \max \left\{ f(x_0, x_1), f(x_0, x_1), f(x_1, x_2), f(x_1, x_1) \frac{1}{2} f(x_0, x_2), \frac{f(x_0, x_1)f(x_1, x_2)}{f(x_0, x_1)}, \frac{f(x_0, x_2)f(x_1, x_1)}{f(x_0, x_1)}, \right. \\ &\quad \frac{f(x_0, x_1)f(x_0, x_2)}{f(x_0, x_1) + f(x_1, x_2)}, \frac{f(x_0, x_2)f(x_1, x_2)}{f(x_0, x_1) + f(x_1, x_2)}, \frac{f(x_0, x_1)f(x_0, x_2)}{f(x_0, x_1) + f(x_1, x_2)}, \frac{f(x_1, x_2)f(x_0, x_2)}{f(x_0, x_1) + f(x_1, x_2)}, \\ &\quad \left. \frac{f(x_0, x_1)f(x_1, x_2) + f(x_0, x_2)f(x_1, x_2)}{2f(x_0, x_1) + f(x_1, x_2)} \right\} \\ &\leq q \max \left\{ f(x_0, x_1), f(x_0, x_1)f(x_1, x_2), 0, \frac{1}{2}[f(x_0, x_1) + f(x_1, x_2)], f(x_1, x_2), 0, f(x_0, x_1), \right. \\ &\quad \left. f(x_1, x_2), f(x_0, x_1), f(x_1, x_2), f(x_1, x_2) \right\} \\ \text{i.e. } &\leq q \max \left\{ f(x_0, x_1), f(x_1, x_2) \frac{1}{2}[f(x_0, x_1) + f(x_1, x_2)] \right\} \end{aligned}$$

Then we have either $f(x_1, x_2) < qf(x_1, x_2)$ which is a contradiction as $q < 1$

or $f(x_1, x_2) < qf(x_0, x_1) < f(x_0, x_1)$ as $q < 1$

and $f(x_1, x_2) \leq \frac{q}{2-q} f(x_0, x_1) \leq qf(x_0, x_1) < f(x_0, x_1)$ as $q < 1$ Proceeding in the same manner we have

$$f(x_0, x_1) > f(x_1, x_2) > f(x_2, x_3) > \dots$$

Thus we have a monotone sequence of positive reals which converges with all its subsequences to some $x \in X$, x being real.

Again $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ in X which converges to some $x \in X$. From the continuity of T,

we have

$$\begin{aligned} Tx &= T\{\lim x_n\} = \lim Tx_n \\ &= \lim(x_{n+1}) \end{aligned}$$

$$\begin{aligned} T^2(x) &= T(Tx) = T\{\lim x_{n_k+1}\} \\ &= \lim\{Tx_{n_k+1}\} = x_{n_k+2} \end{aligned}$$

Now we are to prove that x is a fixed point for T.

$$\begin{aligned} f(x, Tx) &= f(\lim x_n, \lim x_{n+1}) \\ &= \lim f(x_n, x_{n+1}) \\ &= \lim f(x_{n+1}, x_{n+2}) \\ &= \lim f(Tx, T^2x) \end{aligned}$$

Again by (2.3) we have

$$\begin{aligned}
 f(Tx, T^2x) &\leq q \max \left\{ f(x, Tx), f(x, Tx), f(Tx, T^2x), f(Tx, Tx), \frac{1}{2} f(x, T^2x), \frac{f(x, Tx)f(Tx, T^2x)}{f(x, Tx)}, \right. \\
 &\quad \left. \frac{f(x, T^2x)f(Tx, Tx)}{f(x, Tx)}, \frac{f(x, Tx)f(x, T^2x)}{f(x, Tx) + f(Tx, T^2x)}, \frac{f(x, T^2x)f(Tx, T^2x)}{f(x, Tx) + f(Tx, T^2x)}, \right. \\
 &\quad \left. \frac{f(Tx, T^2x)f(x, T^2x)}{f(x, Tx) + f(Tx, T^2x)}, \frac{f(x, Tx)f(Tx, T^2x) + f(x, T^2x)f(Tx, T^2x)}{2f(x, Tx) + f(Tx, T^2x)} \right\} \\
 &\leq q \max \left\{ f(x, Tx), f(x, Tx), f(Tx, T^2x), 0, \frac{1}{2} \{f(x, Tx) + f(Tx, T^2x)\}, \right. \\
 &\quad \left. f(Tx, T^2x), 0, f(x, Tx), f(Tx, T^2x), f(x, Tx), f(Tx, T^2x), f(Tx, T^2x) \right\} \\
 &\leq q \max \left\{ f(x, Tx), f(Tx, T^2x), \frac{1}{2} [f(x, Tx) + f(Tx, T^2x)] \right\}
 \end{aligned}$$

which gives

$$f(Tx, T^2x) \leq q f(x, Tx) \leq f(x, Tx) \quad \text{as } q < 1$$

So we arrive at a contradiction hence $x = Tx$ i.e. x is a fixed point for T .

Uniqueness: Let $y (\neq x)$ be another fixed point. Then by (2.3) we have

$$\begin{aligned}
 f(x, y) &\leq q \max \left\{ f(x, y), f(x, x), f(y, y), f(y, x), \frac{1}{2} f(x, y), \frac{f(x, x)f(y, y)}{f(x, y)}, \frac{f(x, y)f(y, x)}{f(x, y)}, \right. \\
 &\quad \left. \frac{f(x, y)f(x, y)}{f(x, y) + f(x, y)}, \frac{f(x, y), f(x, y)}{f(x, y)f(x, y)}, \frac{f(x, y)f(x, y)}{f(x, y) + f(x, y)}, \frac{f(y, y)f(x, y)}{f(x, y) + f(x, y)}, \frac{f(x, y)f(x, y) + f(x, y)f(y, y)}{2f(x, y) + f(x, y)} \right\}
 \end{aligned}$$

$$f(x, y) \leq q f(x, y) < f(x, y) \quad \text{which is a contradiction as } q < 1$$

Hence $x = y$ i.e. x is a unique fixed point of T .

This completes the proof of the theorem.

Theorem 3: Let T_1 and T_2 be two commuting continuous mapping of a Hausdorff space X into it self and let f be a symmetric continuous mapping of $X \times X$ into non-negative reals such that

$$f(x, x) = 0 \quad \forall x \in X \quad \text{and} \quad f(x, y) \neq 0 \quad \forall x, y \in X, x \neq y \tag{3.1}$$

$$f(x, z) \leq f(x, y) + f(y, z) \tag{3.2}$$

$$\begin{aligned}
 f(T_1x, T_2y) &\leq q \max \left\{ f(x, y), f(y, T_1x), \frac{1}{2} f(x, T_2y), f(y, T_2y), f(y, T_1x), \frac{f(x, T_1x)f(y, T_2y)}{f(x, y)}, \right. \\
 &\quad \frac{f(y, T_2y)f(x, T_2y)}{f(x, y)}, \frac{f(x, y)f(x, T_2y)}{f(x, y) + f(T_1x, T_2y)}, \frac{f(T_1x, T_2y)f(x, T_2y)}{f(x, y) + f(T_1x, T_2y)}, \frac{f(x, T_1x)f(y, T_2y)}{f(x, y) + f(T_1x, T_2y)}, \\
 &\quad \left. \frac{f(x, T_1x)f(y, T_2y)}{f(x, y) + f(T_1x, T_2y)}, \frac{f(x, y)f(T_1x, T_2y) + f(x, T_2y)f(y, T_2y)}{2f(x, y) + f(T_1x, T_2y)} \right\} \tag{3.3}
 \end{aligned}$$

hold for all $x, y \in X$ where $q \in [0,1)$. If for some $x_0 \in X$, the sequence $\{x_n\}$ where $T_1 x_{2n} = x_{2n+1}$ and $T_2 x_{2n+1} = x_{2n+2}$ for $n = 0, 1, 2, 3, \dots$ has a convergent. Subsequence of the type $\{x_{(2p+1)n}\}$ where $p \in N$, is fixed and $n \in N$, then T_1 and T_2 have a unique fixed point.

Proof: Using (3.3) we have

$$f(x_1, x_2) = f(T_1 x_0, T_2 x_1) \leq q \max \left\{ f(x_0, x_1), f(x_1, x_2), f(x_1, x_2), f(x_1, x_1), \frac{1}{2} f(x_0, x_2), \right. \\ \left. \frac{f(x_0, x_1)f(x_1, x_2)}{f(x_0, x_1)}, \frac{f(x_1, x_1)f(x_0, x_2)}{f(x_0, x_1)}, \frac{f(x_0, x_1)f(x_0, x_2)}{f(x_0, x_1) + f(x_1, x_2)}, \right. \\ \left. \frac{f(x_1, x_2)f(x_0, x_2)}{f(x_0, x_1) + f(x_1, x_2)}, \frac{f(x_0, x_1)f(x_0, x_2)}{f(x_0, x_1) + f(x_1, x_2)}, \frac{f(x_1, x_2)f(x_0, x_2)}{f(x_0, x_1) + f(x_1, x_2)}, \frac{f(x_0, x_1)f(x_1, x_2) + f(x_0, x_2)f(x_1, x_2)}{2f(x_0, x_1) + f(x_1, x_2)} \right\} \\ \leq q \max \left\{ f(x_0, x_1), f(x_1, x_2), 0, \frac{1}{2} \{f(x_0, x_1) + f(x_1, x_2)\} \right\}$$

Then we have either $f(x_1, x_2) \leq q f(x_1, x_2)$ which is a contradiction as $q < 1$.

or $f(x_1, x_2) \leq q f(x_0, x_1)$ as $q < 1$. and $f(x_1, x_2) \leq \frac{q}{2-q} f(x_0, x_1)$ as $q < 1$.

Repeating this argument we have

$$f(x_0, x_1) > f(x_1, x_2) > f(x_2, x_3) > \dots$$

Thus, the sequence $\{f(x_n, x_{n+1})\}$ is converging to some z . Again sequence $\{x_n\}$ has a subsequence $\{x_{(2p+1)2n}\}$.

From the continuity of T_1 and T_2 , we have $T_1 x = T_1 \lim x_{(2p+1)2n} = \lim x_{(2p+1)2n+1}$

$$T_2 T_1 x = T_2 \lim x_{(2p+1)2n+1} = \lim x_{(2p+1)2n+2}$$

Now, we have to show that x is a fixed point for T_1 and T_2 , we get

$$f(x, T_1 x) = f(\lim x_{(2p+1)2n}, x_{(2p+1)2n+1}) \\ = f(\lim x_{(2p+1)2n}, x_{(2p+1)2n+1}) \\ = f(\lim x_{(2p+1)2n+1}, x_{(2p+1)2n+2}) \\ = f(T_1 x, T_2 T_1 x)$$

Suppose $x \neq T_1 x$ then by (3.1), we have

$$f(T_1 x, T_2 T_1 x) \leq q \max \left\{ f(x, T_1 x), f(x, T_1 x), f(T_1 x, T_2 T_1 x) f(T_1 x, T_1 x), \frac{1}{2} f(x, T_2 T_1 x), \right. \\ \left. \frac{f(x, T_1 x)f(T_1 x, T_2 T_1 x)}{f(x, T_1 x)}, \frac{f(x, T_2 T_1 x)f(T_1 x, T_1 x)}{f(x, T_1 x)}, \frac{f(T_1 x, T_2 T_1 x)f(x, T_2 T_1 x)}{f(x, T_1 x) + f(T_1 x, T_2 T_1 x)}, \right. \\ \left. \frac{f(x, T_1 x)f(x, T_2 T_1 x)}{f(x, T_1 x) + f(T_1 x, T_2 T_1 x)}, \frac{f(x, T_1 x)f(x, T_2 T_1 x)}{f(x, T_1 x) + f(T_1 x, T_2 T_1 x)}, \frac{f(x, T_2 T_1 x)f(T_1 x, T_2 T_1 x)}{f(x, T_1 x) + f(T_1 x, T_2 T_1 x)}, \right. \\ \left. \frac{f(x, T_1 x)f(T_1 x, T_2 T_1 x) + f(x, T_2 T_1 x)f(T_1 x, T_2 T_1 x)}{2f(x, T_1 x) + f(T_1 x, T_2 T_1 x)} \right\}$$

Implies

$$f(T_1 x, T_2 T_1 x) \leq q f(x, T_1 x) < f(x, T_1 x) \text{ a contradiction as } q < 1.$$

Thus $x = T_1x$, similarly if $\{x_{(2p+1)2n+1}\}$ be subsequence of $\{x_{(2p+1)n}\}$ then we can get easily $x = T_2x$. Therefore x is a fixed point for T_1 and T_2 .

Uniqueness: Let $y (\neq x)$ be another fixed point of T_1 and T_2 . By applying (3.3) we can prove that $f(T_1x, T_2y) \leq f(x, y)$, a contradiction as $q < 1$.

Hence x is a unique fixed point of T_1 and T_2 .

This completes the proof of the theorem.

Theorem 4: Let $T_1, T_2, T_3, \dots, T_n$ be continuous mappings of a Hausdorff space X into itself and let f be a continuous mapping of $X \times X$ into the non-negative reals such that

$$f(x, x) = 0 \quad \forall x \in X \quad \text{and} \quad f(x, y) \neq 0 \quad \forall x, y \in X, x \neq y \tag{4.1}$$

$$f(x, z) \leq f(x, y) + f(y, z) \tag{4.2}$$

$$f(T_r x, T_{r+1} y) \leq q \max \left\{ f(x, y), f(x, T_r x), f(x, T_{r+1} y), f(y, T_r x), \frac{1}{2} f(x, T_{r+1} y), \right. \\ \left. \frac{f(x, T_r x) f(y, T_{r+1} y)}{f(x, y)}, \frac{f(y, T_r x) f(x, T_{r+1} y)}{f(x, y)}, \frac{f(x, y) f(x, T_{r+1} y)}{f(x, y) + f(T_r x, T_{r+1} x)}, \right. \\ \left. \frac{f(x, T_r x) f(x, T_{r+1} y)}{f(x, y) + f(T_r x, T_{r+1} y)}, \frac{f(T_r x, T_{r+1} y) f(x, T_{r+1} y)}{f(x, y) + f(T_r x, T_{r+1} y)}, \frac{f(x, T_{r+1} y) f(y, T_{r+1} y)}{f(x, y) + f(T_r, T_{r+1} y)}, \right. \\ \left. \frac{f(x, y) f(T_r x, T_{r+1} y) + f(x, T_{r+1} y) f(y, T_{r+1} y)}{2 f(x, y) + f(T_r x, T_{r+1} y)} \right\} \tag{4.3}$$

hold for all distinct $x, y \in X$ where $q \in [0, 1)$ and $T_k x = x_{k+1}$.

If for some $x_0 \in X$, the sequence $\{x_n\}$ where

$$\begin{aligned} x_1 = T_1 x_0, \quad x_2 = T_2 x_1, \quad x_3 = T_3 x_2, \dots, \dots, \dots, \quad x_k = T_k x_{k-1} \\ \dots, \dots, \dots \\ x_{k+1} = T_1 x_k, \quad x_{k+2} = T_2 x_{k+1}, \dots, \dots, \dots, \quad x_{2k} = T_k x_{2k-1} \\ x_{n_k+1} = T_1 x_{n_k}, \quad x_{n_k+2} = T_2 x_{n_k+1}, \dots, \dots, \dots, \quad x_{(n_k+1)k} = T_k x_{(n_k+1)k-1} \end{aligned}$$

for $n = 0, 1, 2, 3, \dots$ has a convergent subsequence of the type $\{x_{mk+1}\}$ where $m \in N$ is fixed and $n \in N$. Then $T_1, T_2, T_3, \dots, T_k$ has a unique fixed point.

Proof: Using (4.3) we have

$$f(x_1, x_2) = f(T_1 x_0, T_2 x_1) \leq q \max \left\{ f(x_0, x_1), f(x_0, x_1), f(x_1, x_2), 0, \frac{1}{2} [f(x_0, x_1) + f(x_1, x_2)] \right. \\ \left. f(x_1, x_2), 0, f(x_0, x_1), f(x_1, x_2), f(x_0, x_1), f(x_1, x_2) f(x_1, x_2) \right\} \\ \leq q \max \left\{ f(x_0, x_1), f(x_1, x_2), \frac{1}{2} [f(x_0, x_1) + f(x_1, x_2)] \right\}$$

which implies

$$f(x_1, x_2) \leq q f(x_0, x_1) < f(x_0, x_1) \quad \text{as } q < 1$$

Analogously

$$f(x_2, x_3) < f(x_1, x_2) \dots\dots\dots f(x_{k-1}, x_k) < f(x_{k-2}, x_{k-1})$$

By (4.3) we have

$$f(x_k, x_{k+1}) = f(T_k x_{k-1}, T_k x_k) \leq q \max \left\{ f(x_{k-1}, x_k) f(x_k, x_{k+1}) \frac{1}{2} [f(x_{k-1}, x_k) + f(x_k, x_{k+1})] \right\}$$

implies

$$f(x_k, x_{k+1}) \leq q f(x_{k-1}, x_k) < f(x_{k-1}, x_k) \text{ as } q < 1$$

Hence

$$f(x_n, x_{n+1}) < f(x_{n-1}, x_n), n = 0, 1, 2, 3, \dots$$

The uniqueness of $T_1, T_2, T_3, \dots, T_k$ is similar as of theorem-3.

This completes the proof of theorem.

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