

A CLASSIFICATION OF NEARLY LORENTZIAN SASAKIAN MANIFOLD

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(Received On: 30-09-15; Revised & Accepted On: 25-10-15)

ABSTRACT

In 2011, R. Nivas and A. Bajpai [4] studied on generalized Lorentzian Para-Sasakian manifolds. Hayden [1] introduced the idea of metric connection with torsion tensor in a Riemannian manifold. In 1970, K.Yano [6] studied on semi-symmetric metric connections and their curvature tensors. In 1980, R. S. Mishra and S. N. Pandey [2] discussed on quarter-symmetric metric F -connection. In 1992, Nirmala S. Agashe and Mangala R. Chafle [3] studied semi-symmetric non-metric connection in a Riemannian manifold. In this paper, generalized nearly Sasakian and generalized nearly special Sasakian manifolds have been discussed and some of their properties have been established with generalized Co-symplectic manifolds. Semi-symmetric metric F -connection in a generalized special Sasakian manifold has also been studied.

Keywords: Generalized nearly Lorentzian Sasakian manifold, generalized nearly Lorentzian Special Sasakian manifold, generalized Lorentzian Co-symplectic manifolds and semi-symmetric metric F -connection.

1. INTRODUCTION

An $n (=2m+1)$ dimensional differentiable manifold M_n , on which there are defined covariant vector fields A_i , where $i = 3,4,5, \dots (n-1)$, the associated contravariant vector fields T_i , where $i = 3,4,5, \dots (n-1)$, a tensor field F of type $(1, 1)$ and a Lorentzian metric g , satisfying

$$(1.1) F^2 = -I_n - \sum_{i=3}^{n-1} A_i \otimes T_i, \quad FT_i = 0, \quad A_i(T_i) = -1, \quad A_i(FX) = 0, \quad \text{Rank } F = n - i$$
$$(1.2) g(FX, FY) = g(X, Y) + \sum_{i=3}^{n-1} A_i(X) A_i(Y), \quad \text{where } A_i(X) = g(X, T_i), \quad i = 3,4,5, \dots (n-1),$$
$${}^\flat F(X, Y) \stackrel{\text{def}}{=} g(FX, Y) = -{}^\flat F(Y, X),$$

Then M_n is called a generalized Lorentzian contact manifold (a generalized L-contact manifold) and the structure (F, T_i, A_i, g) is known as generalized L-contact structure [5]

Let D be a Riemannian connection on M_n , then we have

$$(1.3) (a) (D_X {}^\flat F)(FY, Z) - (D_X {}^\flat F)(Y, FZ) + \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(Z) + \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(Y) = 0$$
$$(b) (D_X {}^\flat F)(FY, F^2Z) = (D_X {}^\flat F)(F^2Y, FZ)$$

A generalized L-contact manifold is called a generalized Lorentzian Sasakian manifold, if

$$(1.4) (a) i(D_X F)(Y) - F^2X \sum_{i=3}^{n-1} A_i(Y) - g(FX, FY) \sum_{i=3}^{n-1} T_i = 0 \Leftrightarrow$$
$$(b) i(D_X {}^\flat F)(Y, Z) + g(FX, FZ) \sum_{i=3}^{n-1} A_i(Y) - g(FX, FY) \sum_{i=3}^{n-1} A_i(Z) = 0 \Leftrightarrow$$
$$(c) iD_X T_i = FX + T_i - \sum_{i=3}^{n-1} T_i,$$

From which, we get

$$(1.5) (a) i(D_X A_i)(FY) = g(FX, FY) \Leftrightarrow$$
$$(b) i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = {}^\flat F(X, Y)$$

A generalized L-contact manifold is called a generalized Lorentzian Special Sasakian manifold (a generalized LS-Sasakian manifold), if

$$(1.6) (a) i(D_X F)(Y) + FX \sum_{i=3}^{n-1} A_i(Y) - {}^\flat F(X, Y) \sum_{i=3}^{n-1} T_i = 0 \Leftrightarrow$$
$$(b) i(D_X {}^\flat F)(Y, Z) + {}^\flat F(X, Z) \sum_{i=3}^{n-1} A_i(Y) - {}^\flat F(X, Y) \sum_{i=3}^{n-1} A_i(Z) = 0 \Leftrightarrow$$
$$(c) iD_X T_i = F^2X + T_i - \sum_{i=3}^{n-1} T_i$$

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From which, we get

$$(1.7) \text{ (a) } i(D_X A_i)(FY) = \text{\textbackslash} F(X, Y) \Leftrightarrow$$

$$\text{ (b) } i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = -g(FX, FY)$$

A generalized L-contact manifold is called a generalized Lorentzian Co-symplectic manifold, if

$$(1.8) \text{ (a) } (D_X F)Y - \sum_{i=3}^{n-1} A_i(Y)FD_X T_i - \sum_{i=3}^{n-1} (D_X A_i)(FY)T_i = 0 \Leftrightarrow$$

$$\text{ (b) } (D_X \text{\textbackslash} F)(Y, Z) + \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(FZ) - \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(FY) = 0$$

Therefore, a generalized Lorentzian Co-symplectic manifold will be a generalized Lorentzian Sasakian manifold, if

$$(1.9) \text{ (a) } i(D_X A_i)(FY) = g(FX, FY) \Leftrightarrow \text{ (b) } i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = \text{\textbackslash} F(X, Y) \Leftrightarrow$$

$$\text{ (c) } iD_X T_i = FX + T_i - \sum_{i=3}^{n-1} T_i$$

And a generalized Lorentzian Co-symplectic manifold will be a generalized LS-Sasakian manifold, if

$$(1.10) \text{ (a) } i(D_X A_i)(FY) = \text{\textbackslash} F(X, Y) \Leftrightarrow$$

$$\text{ (b) } i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = -g(FX, FY) \Leftrightarrow \text{ (c) } iD_X T_i = F^2 X + T_i - \sum_{i=3}^{n-1} T_i$$

Nijenhuis tensor in a generalized almost contact metric manifold is given by

$$(1.11) \text{\textbackslash} N(X, Y, Z) = (D_{FX} \text{\textbackslash} F)(Y, Z) - (D_{FY} \text{\textbackslash} F)(X, Z) + (D_X \text{\textbackslash} F)(Y, FZ) - (D_Y \text{\textbackslash} F)(X, FZ)$$

Where $\text{\textbackslash} N(X, Y, Z) \stackrel{\text{def}}{=} g(N(X, Y), Z)$

2. GENERALIZED NEARLY LORENTZIAN SASAKIAN MANIFOLD

A generalized L-contact manifold is called a generalized nearly Lorentzian Sasakian manifold, if

$$(2.1) \text{ (a) } i(D_X \text{\textbackslash} F)(Y, Z) + \sum_{i=3}^{n-1} A_i(Y)g(FZ, FX) - \sum_{i=3}^{n-1} A_i(Z)g(FX, FY)$$

$$= i(D_Y \text{\textbackslash} F)(Z, X) + \sum_{i=3}^{n-1} A_i(Z)g(FX, FY) - \sum_{i=3}^{n-1} A_i(X)g(FY, FZ)$$

$$= i(D_Z \text{\textbackslash} F)(X, Y) + \sum_{i=3}^{n-1} A_i(X)g(FY, FZ) - \sum_{i=3}^{n-1} A_i(Y)g(FZ, FX)$$

From which, we get

$$(2.2) \text{ (a) } i(D_X F)Y + i(D_Y F)X - \sum_{i=3}^{n-1} A_i(Y)F^2 X - \sum_{i=3}^{n-1} A_i(X)F^2 Y - 2 \sum_{i=3}^{n-1} T_i g(FX, FY) = 0 \Leftrightarrow$$

$$\text{ (b) } i(D_X \text{\textbackslash} F)(Y, Z) + i(D_Y \text{\textbackslash} F)(X, Z) + \sum_{i=3}^{n-1} A_i(Y)g(FX, FZ) + \sum_{i=3}^{n-1} A_i(X)g(FY, FZ) -$$

$$2 \sum_{i=3}^{n-1} A_i(Z)g(FX, FY) = 0$$

From which, we get

$$(2.3) \text{ (a) } i(D_X F)FY + i(D_{FY} F)X + \sum_{i=3}^{n-1} A_i(X)FY + 2 \sum_{i=3}^{n-1} T_i \text{\textbackslash} F(X, Y) = 0 \Leftrightarrow$$

$$\text{ (b) } i(D_X \text{\textbackslash} F)(FY, Z) - i(D_{FY} \text{\textbackslash} F)(Z, X) + \sum_{i=3}^{n-1} A_i(X) \text{\textbackslash} F(Y, Z) + 2 \sum_{i=3}^{n-1} A_i(Z) \text{\textbackslash} F(X, Y) = 0$$

$$(2.4) \text{ (a) } i(D_X F)F^2 Y + i(D_{F^2 Y} F)X + \sum_{i=3}^{n-1} A_i(X)F^2 Y + 2 \sum_{i=3}^{n-1} T_i g(FX, FY) = 0 \Leftrightarrow$$

$$\text{ (b) } i(D_X \text{\textbackslash} F)(F^2 Y, Z) - i(D_{F^2 Y} \text{\textbackslash} F)(Z, X) - \sum_{i=3}^{n-1} A_i(X)g(FY, FZ) + 2 \sum_{i=3}^{n-1} A_i(Z)g(FX, FY) = 0$$

$$(2.5) \text{ (a) } (D_X F)Y + (D_Y F)X - \sum_{i=3}^{n-1} A_i(Y)\{FD_X T_i - (D_{T_i} F)X\} - \sum_{i=3}^{n-1} A_i(X)\{FD_Y T_i - (D_{T_i} F)Y\}$$

$$- \sum_{i=3}^{n-1} T_i\{(D_X A_i)(FY) + (D_Y A_i)(FX)\} = 0 \Leftrightarrow$$

$$\text{ (b) } (D_X \text{\textbackslash} F)(Y, Z) + (D_Y \text{\textbackslash} F)(X, Z) + \sum_{i=3}^{n-1} A_i(Y)\{(D_X A_i)(FZ) - (D_{T_i} \text{\textbackslash} F)(Z, X)\}$$

$$+ \sum_{i=3}^{n-1} A_i(X)\{(D_Y A_i)(FZ) + (D_{T_i} \text{\textbackslash} F)(Y, Z)\} - \sum_{i=3}^{n-1} A_i(Z)\{(D_X A_i)(FY) + (D_Y A_i)(FX)\} = 0$$

Pre-multiplying X, Y, Z by F in (1.11) and using equations (2.1), (1.3) (b), we see that a generalized nearly Lorentzian Sasakian manifold will be completely integrable, if

$$(2.6) (D_{\overline{X}} \text{\textbackslash} F)(\overline{Y}, \overline{Z}) = (D_{\overline{Y}} \text{\textbackslash} F)(\overline{X}, \overline{Z})$$

3. GENERALIZED NEARLY LORENTZIAN SPECIAL SASAKIAN MANIFOLD

A generalized L-contact manifold is called a generalized nearly Lorentzian Special Sasakian manifold (a generalized nearly LS-Sasakian manifold), if

$$(3.1) \text{ (a) } i(D_X \text{\textbackslash} F)(Y, Z) - \sum_{i=3}^{n-1} A_i(Y) \text{\textbackslash} F(Z, X) - \sum_{i=3}^{n-1} A_i(Z) \text{\textbackslash} F(X, Y)$$

$$= i(D_Y \text{\textbackslash} F)(Z, X) - \sum_{i=3}^{n-1} A_i(Z) \text{\textbackslash} F(X, Y) - \sum_{i=3}^{n-1} A_i(X) \text{\textbackslash} F(Y, Z)$$

$$= i(D_Z \text{\textbackslash} F)(X, Y) - \sum_{i=3}^{n-1} A_i(X) \text{\textbackslash} F(Y, Z) - \sum_{i=3}^{n-1} A_i(Y) \text{\textbackslash} F(Z, X)$$

From which, we obtain

$$(3.2) \text{ (a) } i(D_X F)Y + i(D_Y F)X + \sum_{i=3}^{n-1} A_i(Y)FX + \sum_{i=3}^{n-1} A_i(X)FY = 0 \Leftrightarrow$$

$$\text{ (b) } i(D_X \text{\textbackslash} F)(Y, Z) + i(D_Y \text{\textbackslash} F)(X, Z) - \sum_{i=3}^{n-1} A_i(Y) \text{\textbackslash} F(Z, X) + \sum_{i=3}^{n-1} A_i(X) \text{\textbackslash} F(Y, Z) = 0$$

This gives

$$\begin{aligned}
 (3.3) \quad (a) \quad & i(D_X F)FY + i(D_{FY} F)X + \sum_{i=3}^{n-1} A_i(X)F^2Y = 0 \Leftrightarrow \\
 & (b) \quad i(D_X \backslash F)(FY, Z) - i(D_{FY} \backslash F)(Z, X) - \sum_{i=3}^{n-1} A_i(X)g(FY, FZ) = 0 \\
 (3.4) \quad (a) \quad & i(D_X F)F^2Y + i(D_{F^2Y} F)X - \sum_{i=3}^{n-1} A_i(X)FY = 0 \Leftrightarrow \\
 & (b) \quad i(D_X \backslash F)(F^2Y, Z) - i(D_{F^2Y} \backslash F)(Z, X) - \sum_{i=3}^{n-1} A_i(X) \backslash F(Y, Z) = 0 \\
 (3.5) \quad (a) \quad & (D_X F)Y + (D_Y F)X - \sum_{i=3}^{n-1} A_i(Y)\{FD_X T_i - (D_{T_i} F)X\} - \sum_{i=3}^{n-1} A_i(X)\{FD_Y T_i - (D_{T_i} F)Y\} = 0 \Leftrightarrow \\
 & (b) \quad (D_X \backslash F)(Y, Z) + (D_Y \backslash F)(X, Z) + \sum_{i=3}^{n-1} A_i(Y)\{(D_X A_i)(FZ) - (D_{T_i} \backslash F)(Z, X)\} \\
 & \quad + \sum_{i=3}^{n-1} A_i(X)\{(D_Y A_i)(FZ) + (D_{T_i} \backslash F)(Y, Z)\} = 0
 \end{aligned}$$

Pre-multiplying X, Y, Z by F in (1.11) and using equations (3.1), (1.3) (b), we see that a generalized nearly LS-Sasakian manifold will be completely integrable, if

$$(3.6) \quad (D_{\bar{X}} \backslash F) \left(\bar{Y}, \bar{Z} \right) + (D_{\bar{Y}} \backslash F) \left(\bar{Z}, \bar{X} \right)$$

4. GENERALIZED NEARLY LORENTZIAN CO-SYMPLECTIC MANIFOLD

A generalized L-contact manifold is called a generalized nearly Lorentzian Co-symplectic manifold, if

$$\begin{aligned}
 (4.1) \quad & (D_X \backslash F)(Y, Z) + \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(FZ) - \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(FY) \\
 & = (D_Y \backslash F)(Z, X) + \sum_{i=3}^{n-1} A_i(Z)(D_Y A_i)(FX) - \sum_{i=3}^{n-1} A_i(X)(D_Y A_i)(FZ) \\
 & = (D_Z \backslash F)(X, Y) + \sum_{i=3}^{n-1} A_i(X)(D_Z A_i)(FY) - \sum_{i=3}^{n-1} A_i(Y)(D_Z A_i)(FX)
 \end{aligned}$$

Therefore, a generalized nearly Lorentzian Sasakian manifold will be a generalized nearly Lorentzian Co-symplectic manifold, if

$$\begin{aligned}
 (4.2) \quad (a) \quad & i(D_X A_i)(FY) = g(FX, FY) \Leftrightarrow \\
 & (b) \quad i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = \backslash F(X, Y) \Leftrightarrow \quad (c) \quad iD_X T_i = FX + T_i - \sum_{i=3}^{n-1} T_i
 \end{aligned}$$

And a generalized nearly LS-Sasakian manifold will be a generalized nearly Lorentzian Co-symplectic manifold, if

$$\begin{aligned}
 (4.3) \quad (a) \quad & i(D_X A_i)(FY) = \backslash F(X, Y) \Leftrightarrow \\
 & (b) \quad i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = -g(FX, FY) \Leftrightarrow \quad (c) \quad iD_X T_i = F^2X + T_i - \sum_{i=3}^{n-1} T_i
 \end{aligned}$$

5. SEMI-SYMMETRIC METRIC F-CONNECTION IN A GENERALIZED LS-SASAKIAN MANIFOLD

Let M_{2m-1} be submanifold of M_{2m+1} and let $c : M_{2m-1} \rightarrow M_{2m+1}$ be the inclusion map such that

$$d \in M_{2m-1} \rightarrow cd \in M_{2m+1},$$

Where c induces a Jacobian map (linear transformation) $J : T'_{2m-1} \rightarrow T'_{2m+1}$.

T'_{2m-1} is tangent space to M_{2m-1} at point d and T'_{2m+1} is tangent space to M_{2m+1} at point cd such that

$$\hat{X} \text{ in } M_{2m-1} \text{ at } d \rightarrow J\hat{X} \text{ in } M_{2m+1} \text{ at } cd$$

Let \tilde{g} be the induced Lorentzian metric in M_{2m-1} , then

$$(5.1) \quad \tilde{g}(\hat{X}, \hat{Y}) = ((g(J\hat{X}, J\hat{Y}))b$$

Semi-symmetric metric F-connection B in generalized Lorentzian special Sasakian manifold M_n is given by

$$(5.2) \quad iB_X Y = iD_X Y - \sum_{i=3}^{n-1} A_i(Y)X + \sum_{i=3}^{n-1} g(X, Y)T_i - 2 \sum_{i=3}^{n-1} A_i(X)Y$$

Where X and Y are arbitrary vector fields of M_{2m+1} . Let

$$(5.3) \quad T_i = Jt_i + \rho_i M + \sigma_i N, \text{ where } i = 3, 4, 5, \dots, (n-1).$$

Where $t_i, i = 3, 4, 5, \dots, (n-1)$, are C^∞ vector fields in M_{2m-1} . M, N are unit normal vectors to M_{2m-1} .

Denoting by \hat{D} the connection induced on the submanifold from D . Gauss equation is

$$(5.4) \quad D_{JX} J\hat{Y} = J(\hat{D}_X \hat{Y}) + p(\hat{X}, \hat{Y})M + q(\hat{X}, \hat{Y})N$$

Where p and q are symmetric bilinear functions in M_{2m-1} . Also

$$(5.5) \quad B_{JX} J\hat{Y} = J(\hat{B}_X \hat{Y}) + h(\hat{X}, \hat{Y})M + k(\hat{X}, \hat{Y})N,$$

Where \hat{B} is the connection induced on the submanifold from B and h, k are symmetric bilinear functions in M_{2m-1} .

Inconsequence of (5.2), we have

$$(5.6) \quad iB_{JX} J\hat{Y} = iD_{JX} J\hat{Y} - \sum_{i=3}^{n-1} A_i(J\hat{Y})J\hat{X} + \sum_{i=3}^{n-1} g(J\hat{X}, J\hat{Y})T_i - 2 \sum_{i=3}^{n-1} A_i(J\hat{X})J\hat{Y}$$

Using (5.4), (5.5) and (5.6), we have

$$(5.7) \quad ij(\hat{B}_X \hat{Y}) + ih(\hat{X}, \hat{Y})M + ik(\hat{X}, \hat{Y})N = ij(\hat{D}_X \hat{Y}) + ip(\hat{X}, \hat{Y})M + iq(\hat{X}, \hat{Y})N - \sum_{i=3}^{n-1} A_i(J\hat{Y})J\hat{X} + \sum_{i=3}^{n-1} g(J\hat{X}, J\hat{Y})T_i - 2 \sum_{i=3}^{n-1} A_i(J\hat{X})J\hat{Y}$$

Using (5.3), we get

$$(5.8) \quad ij(\hat{B}_X \hat{Y}) + ih(\hat{X}, \hat{Y})M + ik(\hat{X}, \hat{Y})N = ij(\hat{D}_X \hat{Y}) + ip(\hat{X}, \hat{Y})M + iq(\hat{X}, \hat{Y})N - \sum_{i=3}^{n-1} a_i(\hat{Y})J\hat{X} + \sum_{i=3}^{n-1} (Jt_i + \rho_i M + \sigma_i N) \tilde{g}(\hat{X}, \hat{Y}) - 2 \sum_{i=3}^{n-1} a_i(\hat{X})J\hat{Y}$$

Where $\tilde{g}(\hat{Y}, t_i) \stackrel{\text{def}}{=} a_i(\hat{Y})$

This implies

$$(5.9) \quad i\hat{B}_X \hat{Y} = i\hat{D}_X \hat{Y} - \sum_{i=3}^{n-1} a_i(\hat{Y})\hat{X} + \sum_{i=3}^{n-1} \tilde{g}(\hat{X}, \hat{Y})t_i - 2 \sum_{i=3}^{n-1} a_i(\hat{X})\hat{Y}$$

Iff

$$(5.10) \quad (a) \quad ih(\hat{X}, \hat{Y}) = ip(\hat{X}, \hat{Y}) + \sum_{i=3}^{n-1} \rho_i \tilde{g}(\hat{X}, \hat{Y})$$

$$(b) \quad ik(\hat{X}, \hat{Y}) = iq(\hat{X}, \hat{Y}) + \sum_{i=3}^{n-1} \sigma_i \tilde{g}(\hat{X}, \hat{Y})$$

Thus, we have

Theorem 5.1: The connection induced on a submanifold of a generalized Lorentzian special Sasakian manifold with a Semi-symmetric metric F-connection with respect to unit normal vectors M and N is also Semi-symmetric metric F-connection iff (5.10) holds.

REFERENCES

1. Hayden, H. A., Subspaces of a space with torsion, Proc. London Math. Soc., 34, (1932), pp. 27-50.
2. Mishra, R. S. and Pandey, S. N., On quarter-symmetric metric F-connection, Tensor, N.S., 34, (1980), 1-7.
3. Nirmala S. Agashe and Mangala R. Chafle, A Semi-symmetric non-metric connection on a Riemmanian manifold, Indian J. pure appl. Math., 23(6), (1992), 399-409.
4. Nivas, R. and Bajpai, A., Study of Generalized Lorentzian Para-Sasakian Manifolds, Journal of international Academy of Physical Sciences, Vol. 15 No.4, (2011), 405-412.
5. Pandey, L.K., A note on generalized L-contact structure, International J. of mathematical archive, 6(8), (2015), pp. 197-200.
6. Yano, K., On semi-symmetric metric connection, Rev. Roum. Math. pures et appl." tome XV, No 9, Bucarest, (1970), 1579-1584.

Source of support: Nil, Conflict of interest: None Declared

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