

**SOLUTION OF LINEAR HIGHER, NONLINEAR HIGHER ORDER BOUNDARY VALUE  
PROBLEMS WITH FRACTIONAL ORDER FUNCTION BY USING  
DIFFERENTIAL TRANSFORM METHOD**

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**ABSTRACT**

*In this paper we solve some numerical examples of linear and nonlinear higher order with fractional order function by using Differential Transform Method, it gives the convergent series form effectively and the result obtains by Differential Transform Method shows the approximate solution of the Differential equations.*

**Keywords:** Series solutions, Differential transform Method, boundary value problem, Numerical solution.

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**INTRODUCTION**

A variety of methods, exact, approximate and purely numerical are available for the solution of differential equations. Most of these methods are computationally intensive because they are trial-and error in nature, or need complicated symbolic computations. The differential transformation technique is one of the numerical methods for ordinary differential equations. The concept of differential transformation was first proposed by Zhou in 1986 [15] and Arikhologlu[1] and Ozkol, Ayaz [3], Chen [6-7] and Ho, 1996, 1999; Hassan[9] and Abdel-Halim[2] 2008, Duan [8], Khaled Batiha[10], it was applied to solve linear and nonlinear initial value problems in electric circuit analysis, Kho, Mantri [11-12], Bert [4]. This method constructs a semi analytical, numerical technique that uses the Taylor series for the solution of differential equations in the form of a polynomial. It is different from the high-order Taylor series method which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally time-consuming, especially for higher order equations [5]. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. The Differential transformation method is very effective and powerful for solving various kinds of differential equation.

In this paper, we solve some numerical examples of linear, nonlinear higher order and fractional order boundary value problem Shahid [13], V. Ananathaswamy [14] by using Differential Transform Method the numerical results are compared with their exact solutions. The main advantage of the DTM is that, it can be applied directly to linear and nonlinear ordinary differential equations without linearization; it reduces the size of computational work and providing the series solution with convergence.

**BASIC IDEAS OF DIFFERENTIAL TRANSFORM METHOD**

The transformation of the  $k^{th}$  Derivative of a function with one variable follows:

$$U(k) = \frac{1}{k!} \left( \frac{d^k u(x)}{dx^k} \right)_{x=x_0} \quad (1)$$

Where  $u(x)$  is the original function and  $U(k)$  is the transformed function and the differential inverse transformation  $U(k)$  is defined by,

$$u(x) = \sum_{k=0}^{\infty} U(k)(x - x_0)^k \quad (2)$$

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When  $x_0 = 0$ , the function  $u(x)$  Defined in (2) is expressed as

$$u(x) = \sum_{k=0}^{k=\infty} U(k)x^k \tag{3}$$

Equation (3) implies that the concept of one dimensional differential transform is almost is same as the one dimensional Taylors series expansion. We use following fundamental theorems on differential transform method

**Theorem 1:** If  $u(x) = \alpha g(x) \pm \beta h(x)$  then  $U(k) = \alpha G(k) \pm \beta H(k)$

**Theorem 2:** If  $u(x) = x^m$  then  $U(K) = \delta(k - m)$  where  $\delta(k - m) = \begin{cases} 1, & \text{if } k = m \\ 0, & \text{if } k \neq m \end{cases}$

**Theorem 3:** If  $u(x) = e^x$  then  $U(k) = \frac{1}{k!}$

**Theorem 4:** If  $u(x) = g(x)h(x)$  then  $U(k) = \sum_{l=0}^k G(l)H(k-l)$

**Theorem 5:** If  $y(x) = y_1(x)y_2(x)$  then  $Y(k) = \sum_{k_1=0}^k Y_1(k_1)Y_2(k-k_1)$

**Theorem 6:** If  $y(x) = \frac{d^n y_1(x)}{dx^n}$ , then  $Y(k) = \frac{(k+n)!}{k!} Y_1(k+n)$

**Theorem 7:** If  $y(x) = e^{\lambda x}$  then  $Y(k) = \frac{\lambda^k}{k!}$ ,  $\lambda$  is constant

**Theorem 8:** If  $y(x) = \sin(wx + \alpha)$  then  $Y(k) = \frac{w^k}{k!} \sin(\frac{k\pi}{2} + \alpha)$ , where  $\alpha, w$  are constant.

**Theorem 9:** If  $y(x) = \cos(wx + \alpha)$  then  $Y(k) = \frac{w^k}{k!} \cos(\frac{k\pi}{2} + \alpha)$ , where  $\alpha, w$  are constant.

### ANALYSIS OF METHOD

Consider an nth order boundary value problem of the form

$$y^n(x) = f(x, y, y', y'', \dots, y^{n-1}), \quad 0 \leq x \leq 1 \tag{4}$$

with the initial conditions

$$B\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^{n-1}y}{dx^{n-1}}\right) \tag{5}$$

The differential transform of (4) is

$$Y(k+n) = \frac{F(k)}{(k+n)!} \tag{6}$$

where  $F(k)$  is the differential transform of  $f(x, y, y', y'', \dots, y^{n-1})$ .

The transformed boundary condition can be written as

$$Y(k) = A, Y(m) = \sum_{k=0}^N \prod_{i=1}^{m-1} (k-i)Y(k) = B_m, (m < n), \tag{7}$$

where  $m$  is the order of the derivative in the boundary conditions and  $A, B_m$  are real constants.

Using (6) and (7), the value of  $Y(i), i = 1, 2, 3, \dots$  can be determined and by using inverse differential transformation, the following approximate solution up  $O(x^{N+1})$  can be determined

$$y(x) = \sum_{k=0}^N Y(k)x^k \tag{8}$$

**ANALYSIS OF METHOD FOR NONLINEAR HIGHER FRACTIONAL ORDER BOUNDARY VALUE PROBLEMS**

Consider an nth order boundary value problem of the form

$$y^n(x) = e^{-x}y^{-\frac{1}{m}}(x)$$

is given by

$$Y(k+n) = \frac{k!}{(k+n)!} \left[ \left( \frac{1}{\left(\frac{1}{m}-1\right)^n} \right) \left( \frac{(-1)^K}{K!} \right) K! Y(k) \right], k = 0,2,4 \dots$$

$$Y(k+n) = \frac{k!}{(k+n)!} \left[ \left( \frac{1}{\left(1-\frac{1}{m}\right)^n} \right) \left( \frac{(-1)^K}{K!} \right) K! Y(k) \right], k = 1,3,5, \dots \tag{9}$$

**Example 1:** Consider a linear sixth order boundary value problem

$$y^6(x) = y(x) - 6e^x, 0 < x < 1, \tag{10}$$

with the boundary conditions

$$y(0) = 1, y(1) = 0, y'(0) = 0, y'(1) = -e, y''(0) = -1, y''(1) = -2e \tag{11}$$

By applying theorems (6), (7) on the equation (10) we get,

$$Y(k+6) = \frac{1}{\prod_{n=1}^6 (k+n)} \left[ Y(k) - \frac{6}{k!} \right] \tag{12}$$

Differential Transformation of the boundary condition equation (11) is

$$Y(0) = 1, Y(1) = 0, Y(2) = -\frac{1}{2}, \sum_{k=0}^n Y(k) = 0$$

$$\sum_{k=0}^n kY(k) = -e, \sum_{k=0}^n k(k-1)Y(k) = -2e \tag{13}$$

Put  $n = 3, k = 0$  in (6) we get  $Y(3) = -\frac{1}{3}$ , put  $n = 4, k = 0$  in (6) we get  $Y(4) = -\frac{1}{8}$

Put  $n = 5, k = 0$  in (6) we get  $Y(5) = -\frac{1}{30}$ , put  $n = 6, k = 0$  in (6) we get  $Y(6) = -\frac{1}{144}$

Put  $n = 7, k = 0$  in (6) we get  $Y(7) = -\frac{1}{840}$ , put  $n = 8, k = 0$  in (6) we get  $Y(k) = -\frac{1}{5760}$

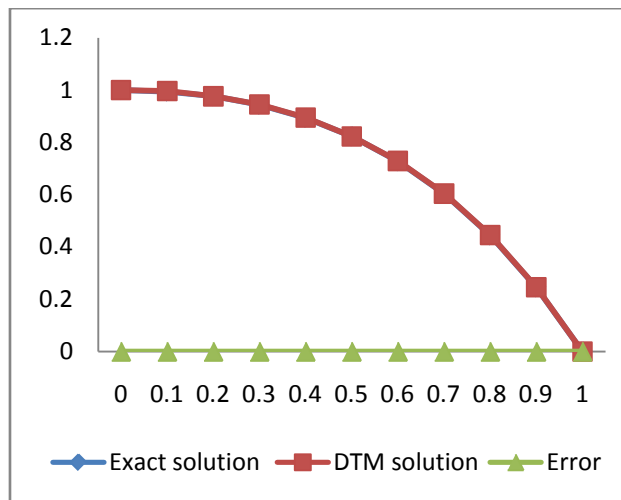
By using inverse differential transformation equation (3) and using boundary conditions we obtain the following series solution up to  $O(x^8)$ ,

$$y(x) = \sum_{k=0}^{k=8} Y(k)x^k = 1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{8} - \frac{x^5}{30} - \frac{x^6}{144} - \frac{x^7}{840} - \frac{x^8}{5760}$$

Exact solution for equation (10) is  $y(x) = (1-x)e^x$ ,

In the following table we compare the exact solution and DTM solution,

$x$	Exact solution	DTM solution	Error
0	1	1	0
0.1	0.994350367	0.996320492	$1 \times 10^{-3}$
0.2	0.976526077	0.976309706	$2 \times 10^{-4}$
0.3	0.944036589	0.944901165	$8 \times 10^{-5}$
0.4	0.894002981	0.895094824	$1 \times 10^{-3}$
0.5	0.823103881	0.822436068	$1 \times 10^{-3}$
0.6	0.727514353	0.728847758	$1 \times 10^{-3}$
0.7	0.602836803	0.604126777	$1 \times 10^{-3}$
0.8	0.444022296	0.445111434	$1 \times 10^{-3}$
0.9	0.245255774	0.245969802	$1 \times 10^{-4}$
1	0	0.000002480	$1 \times 10^{-5}$



**Example 2:** Consider an eighth order boundary value problem having fractional order with nonlinear function

$$y^8(x) = e^{-x} \sqrt{y(x)} \tag{14}$$

with the boundary conditions

$$y(0) = 1, y(1) = -2, y'(0) = 1, y'(1) = e, y''(0) = 1,$$

$$y''(1) = 2, y'''(0) = 1, y'''(1) = -\frac{1}{6} \tag{15}$$

By applying theorems (6), (7) on the equation (14) and using (9) and the transformation of (15) becomes,

if  $n = 1, m = \frac{1}{2}$

$$Y(k + 1) = \frac{k!}{(k + 1)!} \left[ \left( \frac{1}{\left(\frac{1}{m} - 1\right)} \right) \left( \frac{(-1)^k}{K!} \right) K! Y(k) \right], k = 0, 2, 4 \dots$$

$$Y(k + 1) = \frac{k!}{(k + 1)!} \left[ \left( \frac{1}{\left(1 - \frac{1}{m}\right)} \right) \left( \frac{(-1)^k}{K!} \right) K! Y(k) \right], k = 1, 3, 5, \dots$$

$Y(0) = 1$

$$Y(k + 8) = \frac{k!}{(k + 8)!} \left[ \left( \frac{1}{\left(\frac{1}{m} - 1\right)^8} \right) \left( \frac{(-1)^k}{K!} \right) K! Y(k) \right], k = 0, 2, 4 \dots$$

$$Y(k + 8) = \frac{k!}{(k + 1)!} \left[ \left( \frac{1}{\left(1 - \frac{1}{m}\right)^8} \right) \left( \frac{(-1)^k}{K!} \right) K! Y(k) \right], k = 1, 3, 5, \dots$$

$$\begin{aligned}
 & \text{if } k = 0, Y(1) = -2, \text{ if } k = 1, Y(2) = 2, \text{ if } k = 2, Y(3) = -1.333333333, \\
 & \text{if } k = 3, Y(4) = 0.111111111, \text{ if } k = 4, Y(5) = -1.851851852 \times 10^{-3}, \\
 & \text{if } k = 5, Y(6) = 5.144032922 \times 10^{-6}, \text{ if } k = 6, Y(7) = -2.041282905 \times 10^{-9}, \\
 & \text{if } k = 7, Y(8) = 1.012541124 \times 10^{-13}, \text{ if } k = 8, Y(9) = -5.580583795 \times 10^{-19}, \\
 & \text{if } k = 9, Y(10) = 3.075718582 \times 10^{-25}, \text{ if } k = 10, Y(11) = -1.695171176 \times 10^{-31}
 \end{aligned}$$

By using inverse differential transformation equation and using boundary conditions we obtain the following series solution up to  $O(x^{11})$ ,

$$\begin{aligned}
 y(x) &= \sum_{k=0}^{k=9} Y(k)x^k \\
 &= 1 - 2x + 2x^2 - 1.333333333x^3 + 0.111111111x^4 - 1.851851852 \times 10^{-3}x^5 \\
 &\quad + 5.144032922 \times 10^{-6}x^6 - 2.041282905 \times 10^{-9}x^7 + 1.012541124 \times 10^{-13}x^8 \\
 &\quad - 5.580583795 \times 10^{-19}x^9 + 3.075718582 \times 10^{-25}x^{10} - 1.695171176 \times 10^{-31}x^{11}
 \end{aligned}$$

## CONCLUSION

In this paper, the differential transformation method has been applied to obtain the numerical solution of linear and nonlinear higher order boundary value problems. The present method has been applied in a direct way without using linearization, discretization, or perturbation. By increasing the order of approximation more accuracy can be obtained. Comparison of the numerical results with the existing technique shows that the present method is more accurate.

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