International Journal of Mathematical Archive-6(10), 2015, 1-4
IJMA Available online through www.ijma.info ISSN 2229 – 5046

# ON TWO INTERESTING TRIPLE INTEGER SEQUENCES

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*(Received On: 14-09-15; Revised & Accepted On: 08-10-15)*

***ABSTRACT***

*We search for three non-zero distinct integers such that each of the triples* $(x^2, z^2, y^2)$ *and* $(y^2z^2, x^2y^2, z^2x^2)$ *forms Harmonic progression. A few interesting properties among the solutions are also presented.*

***Keywords:*** *Harmonic progression, Integer solution.*

***2010 Mathematics subject classification:*** *11D09, 11D25.*

## 1. INTRODUCTION

Number theory, called the Queen of Mathematics, is a broad and diverse part of Mathematics that developed from the study of the integers. The foundations for Number theory as a discipline were laid by the Greek mathematician Pythagoras and his disciples (known as Pythagoreans)**.** One of the oldest branches of mathematics itself, is the Diophantine equations since its origins can be found in texts of the ancient Babylonians, Chinese, Egyptians, Greeks and so on[7-8]. Diophantine problems were first introduced by Diophantus of Alexandria who studied this topic in the third century AD and he was one of the first Mathematicians to introduce symbolism to Algebra. The theory of Diophantine equations is a treasure house in which the search for many hidden relation and properties among numbers form a treasure hunt. In fact, Diophantine problems dominated most of the celebrated unsolved mathematical problems. Certain Diophantine problems come from physical problems or from immediate Mathematical generalizations and others come from geometry in a variety of ways. Certain Diophantine problems are neither trivial nor difficult to analyze [1-6].

This communication consists of two sections A and B

**In section A,** we search for three non-zero distinct integers such that the triple $(x^2, z^2, y^2)$ form a harmonic progression

**In section B,** we search for three non-zero distinct integers such that the triple $(y^2z^2, x^2y^2, z^2x^2)$ form a harmonic progression

## 2. METHOD OF ANALYSIS

**SECTION A:** Let x, y, z be three non-zero distinct integers such that $(x^2, z^2, y^2)$ forms a harmonic progression (H.P**)**

By the definition of H.P, the above problem is equivalent to solving the Diophantine equation

$$2x^2y^2 = z^2(x^2 + y^2) \tag{1}$$

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which is written as $x^2 + y^2 = 2t^2$ (2)

where $t = \frac{xy}{z}$ (3)

As x, y are integers, the value of t on the RHS of (2) also represents an integer. This means that z divides xy as can be seen from (3).

Let $d = \gcd(x, y, t)$ so that $x = ad, y = bd, t = cd$ (4)
where $\gcd(a, b, c) = 1$

Substituting (4) in (3), we have $z = \frac{abd}{c}$ (5)

For z to be an integer, c should divide d as $\gcd(a, c) = 1 \,\&\, \gcd(b, c) = 1$

Let $d = kc$ (6)

Substituting (6) in (4) and (5), we have
$x = kac, y = kbc, t = kc^2, z = kab$ (7)

Again, substituting the above values of x, y, t in (2), we have $a^2 + b^2 = 2c^2$ which is satisfied by

$$a = \alpha^2 - \beta^2 - 2\alpha\beta,$$
$$b = \alpha^2 - \beta^2 + 2\alpha\beta,$$
$$c = \alpha^2 + \beta^2$$

Thus, in view of (7), the non-zero distinct integer values of x, y, z satisfying (1) are given by

$$x = k(\alpha^2 + \beta^2)(\alpha^2 - \beta^2 - 2\alpha\beta),$$
$$y = k(\alpha^2 + \beta^2)(\alpha^2 - \beta^2 + 2\alpha\beta),$$
$$z = k(\alpha^2 - \beta^2 - 2\alpha\beta)(\alpha^2 - \beta^2 + 2\alpha\beta)$$

A few numerical examples are presented below:

| $k$ | $\alpha$ | $\beta$ | a | b | c | x | y | z | t |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | -1 | 7 | 5 | -5 | 35 | -7 | 25 |
| 2 | 3 | 2 | -7 | 17 | 13 | -182 | 442 | -238 | 338 |
| 2 | 4 | 1 | 7 | 23 | 17 | 238 | 782 | 322 | 578 |
| 1 | 3 | 1 | 2 | 14 | 10 | 20 | 140 | 28 | 100 |

Some interesting properties are as follows:

(*) $6k\left(\frac{xy}{z}\right)$ is a Nasty number

(*) $8k^2(x^2 + y^2)$ is a biquadratic integer.

(*) $\left(\frac{(y-x)^2}{(x^2+y^2)}\right) \equiv 0 \pmod{8}$

(*) $\left(\frac{y-x}{y+x}\right) \equiv 0 \pmod{2}$

**SECTION B:** Let x, y, z be three non-zero distinct integers such that $(y^2z^2, x^2y^2, z^2x^2)$ form a harmonic progression(H.P**),** By the definition of H.P, the above problem is equivalent to solving the Diophantine equation
$x^2 + y^2 = 2z^2$ (8)

 

which is satisfied by $x = a^2 - b^2 - 2ab, y = a^2 - b^2 + 2ab, z = a^2 + b^2$ representing the required values for x,y,z. Also, we have an another set of solutions to (8) which is obtained as follows.

Rewrite (8) as $x^2 + y^2 = 2z^2 * 1$ (9)

Assume $2 = (1+i)(1-i), 1 = \frac{(3+4i)(3-4i)}{25}, z = a^2 + b^2$ (10)

Substituting (10) in (9) and using the method of factorization, we get

$$x + iy = (1+i)(a^2 - b^2 + i2ab)\left(\frac{3+4i}{5}\right)$$

Equating the real and imaginary parts, we get

$$\left.\begin{aligned} x &= \frac{1}{5}(-a^2 + b^2 - 14ab) \\ y &= \frac{1}{5}(7a^2 - 7b^2 - 2ab) \end{aligned}\right\} \qquad (11)$$

Since our interest is on finding integer solutions, choosing
$a = 5A \,\&\, b = 5B$ in (10) and (11), we have

$$x = -5A^2 + 5B^2 - 70AB$$

$$y = 35A^2 - 35B^2 - 10AB$$

$$z = 25(A^2 + B^2)$$

A few numerical examples are as follows:

| A | B | x | y | z |
|---|---|---|---|---|
| 1 | 1 | -70 | -10 | 50 |
| 2 | 2 | -280 | -40 | 200 |
| 3 | 1 | -250 | 250 | 250 |
| 2 | 3 | -395 | -235 | 325 |

Each of the expressions forms an Arithmetic progression:

$$(1): (x^2z^2, z^4, y^2z^2)$$

$$(2): [(2x^2 - y^2)z, z^3, (2y^2 - x^2)z]$$

$$(3): (x^2 - x + y, z^2, y^2 - y + x)$$

$$(4): [(x^2 - x + y)z, z^3, (y^2 - y + x)z]$$

$$(5): [(x \pm y)^2 z, z^3, \mp 2xyz]$$

**Note:** In (10), the representation for 1 may be considered as

$$1 = \left(\frac{(m^2 - n^2 + i2mn)((m^2 - n^2 - i2mn)}{(m^2 + n^2)^2}\right) \text{ (or)}$$

$$1 = \left(\frac{(2mn + i(m^2 - n^2))(2mn - i(m^2 - n^2))}{(m^2 + n^2)^2}\right)$$

Employing the above representation and following the analysis presented above an infinite number of triples forming Harmonic Progression are obtained.

## 3. CONCLUSION

In this communication, we have exhibited two different triples each forming a Harmonic progression. To conclude, one may search for other choices of triples forming Harmonic progression along with their corresponding properties.

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**Source of support: Nil, Conflict of interest: None Declared**