

## ON TWO INTERESTING TRIPLE INTEGER SEQUENCES

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### ABSTRACT

*We search for three non-zero distinct integers such that each of the triples  $(x^2, z^2, y^2)$  and  $(y^2 z^2, x^2 y^2, z^2 x^2)$  forms Harmonic progression. A few interesting properties among the solutions are also presented.*

**Keywords:** Harmonic progression, Integer solution.

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## 1. INTRODUCTION

Number theory, called the Queen of Mathematics, is a broad and diverse part of Mathematics that developed from the study of the integers. The foundations for Number theory as a discipline were laid by the Greek mathematician Pythagoras and his disciples (known as Pythagoreans). One of the oldest branches of mathematics itself, is the Diophantine equations since its origins can be found in texts of the ancient Babylonians, Chinese, Egyptians, Greeks and so on [7-8]. Diophantine problems were first introduced by Diophantus of Alexandria who studied this topic in the third century AD and he was one of the first Mathematicians to introduce symbolism to Algebra. The theory of Diophantine equations is a treasure house in which the search for many hidden relation and properties among numbers form a treasure hunt. In fact, Diophantine problems dominated most of the celebrated unsolved mathematical problems. Certain Diophantine problems come from physical problems or from immediate Mathematical generalizations and others come from geometry in a variety of ways. Certain Diophantine problems are neither trivial nor difficult to analyze [1-6].

This communication consists of two sections A and B

**In section A,** we search for three non-zero distinct integers such that the triple  $(x^2, z^2, y^2)$  form a harmonic progression

**In section B,** we search for three non-zero distinct integers such that the triple  $(y^2 z^2, x^2 y^2, z^2 x^2)$  form a harmonic progression

## 2. METHOD OF ANALYSIS

**SECTION A:** Let  $x, y, z$  be three non-zero distinct integers such that  $(x^2, z^2, y^2)$  forms a harmonic progression (H.P)

By the definition of H.P, the above problem is equivalent to solving the Diophantine equation

$$2x^2 y^2 = z^2 (x^2 + y^2) \tag{1}$$

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$$\text{which is written as } x^2 + y^2 = 2t^2 \quad (2)$$

$$\text{where } t = \frac{xy}{z} \quad (3)$$

As x, y are integers, the value of t on the RHS of (2) also represents an integer. This means that z divides xy as can be seen from (3).

$$\text{Let } d = \gcd(x, y, t) \text{ so that } x = ad, y = bd, t = cd \quad (4)$$

where  $\gcd(a, b, c) = 1$

$$\text{Substituting (4) in (3), we have } z = \frac{abd}{c} \quad (5)$$

For z to be an integer, c should divide d as  $\gcd(a, c) = 1$  &  $\gcd(b, c) = 1$

$$\text{Let } d = kc \quad (6)$$

Substituting (6) in (4) and (5), we have

$$x = kac, y = kbc, t = kc^2, z = kab \quad (7)$$

Again, substituting the above values of x, y, t in (2), we have  $a^2 + b^2 = 2c^2$  which is satisfied by

$$a = \alpha^2 - \beta^2 - 2\alpha\beta,$$

$$b = \alpha^2 - \beta^2 + 2\alpha\beta,$$

$$c = \alpha^2 + \beta^2$$

Thus, in view of (7), the non-zero distinct integer values of x, y, z satisfying (1) are given by

$$x = k(\alpha^2 + \beta^2)(\alpha^2 - \beta^2 - 2\alpha\beta),$$

$$y = k(\alpha^2 + \beta^2)(\alpha^2 - \beta^2 + 2\alpha\beta),$$

$$z = k(\alpha^2 - \beta^2 - 2\alpha\beta)(\alpha^2 - \beta^2 + 2\alpha\beta)$$

A few numerical examples are presented below:

k	$\alpha$	$\beta$	a	b	c	x	y	z	t
1	2	1	-1	7	5	-5	35	-7	25
2	3	2	-7	17	13	-182	442	-238	338
2	4	1	7	23	17	238	782	322	578
1	3	1	2	14	10	20	140	28	100

Some interesting properties are as follows:

$$(*) 6k \left( \frac{xy}{z} \right) \text{ is a Nasty number}$$

$$(*) 8k^2(x^2 + y^2) \text{ is a biquadratic integer.}$$

$$(*) \left( \frac{(y-x)^2}{(x^2 + y^2)} \right) \equiv 0 \pmod{8}$$

$$(*) \left( \frac{y-x}{y+x} \right) \equiv 0 \pmod{2}$$

**SECTION B:** Let x, y, z be three non-zero distinct integers such that  $(y^2z^2, x^2y^2, z^2x^2)$  form a harmonic progression(H.P). By the definition of H.P, the above problem is equivalent to solving the Diophantine equation

$$x^2 + y^2 = 2z^2 \quad (8)$$

which is satisfied by  $x = a^2 - b^2 - 2ab$ ,  $y = a^2 - b^2 + 2ab$ ,  $z = a^2 + b^2$  representing the required values for  $x, y, z$ . Also, we have another set of solutions to (8) which is obtained as follows.

$$\text{Rewrite (8) as } x^2 + y^2 = 2z^2 \quad (9)$$

$$\text{Assume } 2 = (1+i)(1-i), 1 = \frac{(3+4i)(3-4i)}{25}, z = a^2 + b^2 \quad (10)$$

Substituting (10) in (9) and using the method of factorization, we get

$$x + iy = (1+i)(a^2 - b^2 + i2ab) \left( \frac{3+4i}{5} \right)$$

Equating the real and imaginary parts, we get

$$\left. \begin{aligned} x &= \frac{1}{5}(-a^2 + b^2 - 14ab) \\ y &= \frac{1}{5}(7a^2 - 7b^2 - 2ab) \end{aligned} \right\} \quad (11)$$

Since our interest is on finding integer solutions, choosing

$a = 5A$  &  $b = 5B$  in (10) and (11), we have

$$x = -5A^2 + 5B^2 - 70AB$$

$$y = 35A^2 - 35B^2 - 10AB$$

$$z = 25(A^2 + B^2)$$

A few numerical examples are as follows:

A	B	x	y	z
1	1	-70	-10	50
2	2	-280	-40	200
3	1	-250	250	250
2	3	-395	-235	325

Each of the expressions forms an Arithmetic progression:

$$(1) : (x^2 z^2, z^4, y^2 z^2)$$

$$(2) : [(2x^2 - y^2)z, z^3, (2y^2 - x^2)z]$$

$$(3) : (x^2 - x + y, z^2, y^2 - y + x)$$

$$(4) : [(x^2 - x + y)z, z^3, (y^2 - y + x)z]$$

$$(5) : [(x \pm y)^2 z, z^3, \mp 2xyz]$$

**Note:** In (10), the representation for 1 may be considered as

$$1 = \left( \frac{(m^2 - n^2 + i2mn)(m^2 - n^2 - i2mn)}{(m^2 + n^2)^2} \right) \quad (\text{or})$$

$$1 = \left( \frac{(2mn + i(m^2 - n^2))(2mn - i(m^2 - n^2))}{(m^2 + n^2)^2} \right)$$

Employing the above representation and following the analysis presented above an infinite number of triples forming Harmonic Progression are obtained.

### 3. CONCLUSION

In this communication, we have exhibited two different triples each forming a Harmonic progression. To conclude, one may search for other choices of triples forming Harmonic progression along with their corresponding properties.

### 4. REFERENCES

1. Andre weil, Number Theory: An Approach through History, From Hammurapito to Legendre, Birkahsuser, Boston, 1987.
2. Bibhotibhusan Batta and Avadhesh Narayanan Singh, History of Hindu Mathematics, Asia Publishing House, 1938.
3. Boyer.C.B., A History of mathematics, John Wiley& sons Inc., New York, 1968.
4. Dickson L. E., History of Theory of Numbers, Vol. 11, Chelsea Publishing Company, New York (1952).
5. Davenport, Harold (1999), The higher Arithmetic: An Introduction to the Theory of Numbers (7<sup>th</sup>ed.) Cambridge University Press.
6. John Stilwell, Mathematics and its History, Springer Verlag, New York, 2004.
7. James Matteson, M.D,"A Collection of Diophantine problems with solutions", Washington", Artemas Martin, 1888.
8. Titu andreescu, Dorin Andrica, "An Introduction to Diophantine equations" GIL Publicating House, 2002.

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