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TWO POINT BOUNDARY VALUE PROBLEM WITH UNKNOWN INTERVAL

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ABSTRACT

Two point boundary value problems where the interval is unknown cannot be handled by standard numerical methods. By using transformation the problem is brought to usual boundary value problem with interval length occurring in the ordinary differential equation. Two methods are used to obtain the solution of this differential equation while finding the interval length an extra condition is used. First method is by the application of Newton's method to find the interval length and second method is Quasilinearization technique where method of finding interval length uses criteria worth noting for such problems.

Keywords: *Boundary Value Problem, Finite difference method, Newton's method, Quasilinearization technique.*

INTRODUCTION

This paper is concerned with the development of numerical methods to solve two point boundary value problem defined over [a, T] where T is unknown. If the differential equation is of n^{th} order we need n conditions defined at both ends of the interval and also T is unknown.

Consider,

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right) \tag{1.1}$$

with conditions $x(0) = a,\ x(T) = b\ \ \& \frac{dx}{dt}(T) = 0$. As T is unknown so usual methods for solving two point boundary value problems are not directly applicable. We have applied the well known Newton's method to find T and its solution. We have also applied the well known Bellman's quasilinearization technique [2] to solve this problem. We have presented following sections for choosing the method for solving any practical problem.

SECTION-2: STANDARD BOUNDARY VALUE PROBLEM

With the help of transformation

$$\theta = 1 - \frac{t}{T}$$

the problem (1.1) takes the form

$$\begin{aligned}\frac{d^2x}{d\theta^2} &= T^2 f\left((1-\theta)T, x, -\frac{1}{T}\frac{dx}{d\theta}\right)\\ &= S\ F\left(\theta, x, \frac{dx}{d\theta}\right) \qquad where\ (S = T^2)\end{aligned} \tag{2.2}$$

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And the boundary conditions are

$$x(0)=b,\quad x(1)=a \quad \& \quad \frac{dx}{d\theta}(0)=0$$

Now differentiating equation (2.2) with respect to S, we get an initial value problem, with T being unknown. Assuming an initial approximation for S, and next approximation is obtained by Newton's method .

$$S^{(1)}=S^{(0)}-\frac{x\left(1,S^{(0)}\right)}{\frac{dx}{ds}\left(1,S^{(0)}\right)}$$

We apply this technique to specific problem presented in section 3.

SECTION-3: PROBLEM

A problem treated in ([3],[4]) describes the motion of the mass acted upon by a force, $-xe^{-x}$ with x as the displacement at time t and which initially at origin. The problem is to find the duration of the mass to reach $x=x_0$ to be at rest. Mathematically the problem is

$$\frac{d^2x}{dt^2}=-xe^{-x} \tag{3.1}$$

$$x(0)=0,\ x(T)=x_0 \ \ \& \ \ \frac{dx}{dt}(T)=0$$

The author ([3], [4]) assumed an initial approximation x as $x^{(0)}$ and developed an iterative method defined by $x^{(n+1)}=y+Sz$ with $S=\frac{dx^{(n)}}{dt}$. He then obtain two separate equations for y & z and built a sequence $x^{(n)}$.

We have found that although the method is logical the solution is found to be not correct. We apply our technique to this problem (3.1)

Differentiating (3.1) with respect to the parameter S & denoting $X=\frac{dx}{dS}$, $X'=\frac{dx'}{dS}$, $X''=\frac{dx''}{dS}$

$$\frac{d^2X}{d\theta^2}=-xe^{-x}-S[-xe^{-x}X+Xe^{-x}] \tag{3.2}$$

$$X(0)=0,\quad X'(0)=0$$

Assuming $S=S^{(0)}$

Solve (3.1) up to $\theta=1$ with $x(1,S^{(0)})$ known. Knowing $x(S,\theta)$ solve (3.2) for X to obtain $X(1)=\frac{dx}{dS}(1)$.

Next approximation to S is $S^{(1)}=S^{(0)}-\frac{x(1,S^{(0)})}{X(1,S^{(0)})}$.

Starting with $S=3$, the iteration converged up to four significant digits $S=3.7675$ and hence $T=\sqrt{S}=1.9410$ as against $T=0.45$ given by [4].

SECTION-4: QUASILINEARIZATION TECHNIQUE

Above problem is also solved by quasilinearization technique for two reasons. First finding the starting approximation for the quasilinearization technique satisfying all the boundary conditions is not a simple exercise. Secondly, the methodology involves for finding the interval length T is different and may help in tackling in some other problem.

The starting approximation for the problem (3.1) is found to be

$$x^{(0)} = x_0\left[e^{-\theta^2} - \frac{\theta^2}{e}\right]$$

satisfies all the three boundary conditions. The linearization problem by ([1], [2]) is

$$\frac{d^2x}{d\theta^2} = S\left[-x^{(0)^2} e^{-x^{(0)}} + e^{-x^{(0)}} x(x^{(0)} - 1)\right] \tag{4.1}$$

$$x(0) = x_0\,,\ \frac{dx}{d\theta}(0) = 0\,,\ x(1) = 0$$

From the linearized problem we obtain tridiagonal system of equations

$$x_{j-1} - x_j\left[2 + Sh^2(x_j^{(0)} - 1)\, e^{-x_j^{(0)}}\right] + x_{j+1} = -Sh^2 x_j^{(0)^2} e^{-x_j^{(0)}} \tag{4.2}$$

where $j = 1,2,3,\dots,N-1$ $(Here\ x_0 = 0.5\ ,\ x_N = 0)$ and S is unknown. The third condition $\frac{dx}{d\theta}(0) = 0$ through its discrete equivalent (by forward difference approximation with second order)

$$F = 4x_1 - x_2 - 3x_0 = 0 \tag{4.3}$$

is used to find S.

Algorithm to find S

1. Assume S_L and S_F as two values of S between which the true value lies by the criteria (4.2).
2. Solve the discrete analogue of the equation by using Thomas algorithm.
3. Obtain F_L and F_F using (4.3). By the method of Bisection we find the value of S satisfying all the conditions of the problem and obtain the next approximation for quasilinearization technique.
4. Go back to step 1 until convergence occur.

The unknown duration T is found to be T=1.9406 when (x_0 =0.5).

A summary of the numerical solutions by using above algorithm is given in following **Table 1.** Sample solution of Equation (3.1) $(x_0 = 0.5)$.

Table: 1

T	0.0000	0.0970	0.1941	0.2911	0.3881	0.4851	0.5822	0.6792
x_0	0.5000	0.4986	0.4943	0.4872	0.4772	0.4645	0.4491	0.4309
T (Conti…)	0.7762	0 .8733	0.9703	1.0673	1.1644	1.2614	1.3584	1.4554
x_0	0.4101	0.3867	0.3609	0.3327	0.3022	0.2697	0.2352	0.1989
T (Conti…)	1.5525	1.6495	1.7465	1.8436	1.9406			
x_0	0.1611	0.1221	0.0820	0.0412	0.0000			

(Where T- time duration, x_0 -position of the mass)

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