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# ON THE FUNCTION $\Delta_r(x,n)$

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***ABSTRACT***

***D**efining the function $\Delta_r(x,n)$ related to the r-totatives of n we study certain properties of it.*

***Key words:*** *r-totatives, r-analogues of Mobius and Euler Functions.*

## 1. INTRODUCTION

Throughout this paper r denotes a fixed positive integer. For positive integers a and b, their greatest r$^{th}$ power common divisor is denoted by $(a,b)_r$. It is clear that $(a,b)_1$ is the greatest common divisor *(a, b)* of *a* and *b*; and that $(a,b)_r=1$ if and only if *(a, b)* is *r-free* (we recall that a positive integer is r-free if it is not divisible by the r$^{th}$ power of any prime).

For a positive integer *n*, a number $\tau$ with $(\tau,n)_r=1$ will be called a *r-totative* of *n*. Note that 1-totatives of *n* are referred as totatives of *n* by J.J.Sylvester (see [7], p.124). V.L. Klee [4] has defined the function $\phi_r(n)$ as the number of integers *m* with $1\le m\le n$ and $(m,n)_r=1$. Note that $\phi_1(n)=\phi(n)$, the well-known Euler function; and that $\phi_r(n)$ is the number of r-totatives of *n* in $[0,n)$. Denote the number of r-totatives *m* of *n* with $m\le x$ by $\phi_r(x,n)$.

Here we define the function

**(1.1)** $$\Delta_r(x,n)=\sum_{\substack{m\le xn\\(m,n)_r=1}}1-x\phi_r(n)=\phi_r(xn,n)-x\phi_r(n)$$

Note that $\Delta(x,n):=\Delta_1(x,n)$ was studied by Codeca and Nair [1]. In this paper we present some proerties of (1.1) and the results involving this function in seciton 3.

## 2. PRELIMINARIES

The r-analogue of the Mobius function, $\mu_r(n)$, is defined (see [4]) by

**(2.1)** $$\mu_r(n)=\begin{cases}1 & if\ \ n=1\\(-1)^t & if\ \ n=p_1^r p_2^r\ldots p_t^r\ \ where\ \ p_i\text{'s are distinct primes}\\0 & otherwise\end{cases}$$

and showed that it is multiplicative. V.L.Klee [4] has proved that

**(2.2)** $$\phi_r(n)=\sum_{d|n}\mu_r(d)\frac{n}{d}=\sum_{\delta|n}\mu_r\left(\frac{n}{\delta}\right)\delta$$

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Since $\phi_r(x,n)$ is the number of r-totatives *m* of *n* with $m \le x$, it is easy to show that

**(2.3)** $\phi_r(x,n) = \sum_{d|n} \mu_r(d)\left[\frac{x}{d}\right] = \sum_{\delta|n} \mu_r\left(\frac{n}{\delta}\right)\left[\frac{x\delta}{n}\right]$

where [y] is the greatest integer not exceeding y.

**(2.4)** Suppose for a given *n* let $N_r = N_r(n)$ is the r^th power of the maximal square free divisor of n. Then note that $(a,n)_r = 1 \Leftrightarrow (a,N_r)_r = 1$. Hence we may assume, without loss to generality, that *n* itself is an r^th power of a squarefree number *m*, say $n = m^r$. In all that follows *n* is always of this form.

Note that

**(2.5)** $\Delta_r(x,n)$ is periodic in *x* with period 1.

**(2.6)** Let $1 = a_1 < a_2 < \ldots < a_{\phi_r(n)} = n-1$ be the $\phi_r(n)$ r-totatives of *n* in the interval $[1,n)$. We write $a_o = 0$ and $a_{\phi_r(n)+1} = n$. Then $n - a_i = a_{\phi_r(n)-i+1}$ and $\frac{a_i}{n} \in [0,1]$ for $0 \le i \le a_{\phi_r(n)+1}$. If $a_i$'s are defined as in (2.6) we observe that

**(2.7)** $\Delta_r\left(\frac{a_i}{n}, n\right) = i - a_i\frac{\phi_r(n)}{n}$ for $0 \le i \le \phi_r(n)$

**(2.8)** $\Delta_r(x,n) = \Delta_r\left(\frac{a_i}{n}, n\right) - \left(x - \frac{a_i}{n}\right)\phi_r(n)$ which imply that $\Delta_r(x,n)$ is a piecewise linear function of *x* with each line segment in $\left[\frac{a_i}{n}, \frac{a_{i+1}}{n}\right)$ having the gradient $-\phi_r(n)$.

## 3. MAIN RESULTS

**3.1 *Lemma:*** $\Delta_r(x,n) = -\mu_r(n)\sum_{d|n}\mu_r(d)\{xd\}$

***Proof:*** By (1.1), (2.3) and (2.2) we get

$$\Delta_r(x,n) = \sum_{d|n}\mu_r(d)\left(\left[\frac{xn}{d}\right] - \frac{xn}{d}\right)$$
$$= -\sum_{d|n}\mu_r(d)\left\{\frac{xn}{d}\right\} = -\sum_{d|n}\mu_r\left(\frac{n}{d}\right)\{xd\}$$

where {*y*} denotes the fractional part of *y*. Since the contribution of divisiors *d* of *n* to the sum on the right is non-zero if and only if *d* is the r^th power of square free integer, so that

$$\Delta_r(x,n) = \sum_{d|n}\frac{\mu_r(n)}{\mu_r(d)}\{xd\} = -\mu_r(n)\sum_{d|n}\mu_r(d)\{xd\},$$

proving the Lemma.

As a consequence of Lemma 3.1, we have the identity:

**(3.2)** If $p \nmid$ n, $\Delta_r(x, np^r) = \Delta_r(xp^r, n) - \Delta_r(x,n)$

It is easy to see that

**(3.3)** $\Delta_r(x,n) = -\mu_r(n)\sum_{d|n}\mu_r(d)\left(\{xd\} - \frac{1}{2}\right)$

**Theorem A:** If $(\ell,n)_r = 1$ then $\sum_{n=0}^{\ell-1} \Delta_r\left(\frac{u+n}{\ell}, n\right) = \Delta_r(u,n)$

**Proof:** By (3.3) we have

$$\sum_{n=0}^{\ell-1}\left(\frac{u+n}{\ell}, n\right) = -\mu_r(n)\sum_{d|n}\mu_r(d)\sum_{n=0}^{\ell-1}\left(\left\{\frac{ud}{\ell}+\frac{n}{\ell}\right\}-\frac{1}{2}\right)$$

$$= -\mu_r(n)\sum_{d|n}\mu_r(d)\left(\{ud\}-\frac{1}{2}\right),$$

Since, $\sum_{n=0}^{\ell-1}\left(\left\{\frac{ud}{\ell}+\frac{n}{\ell}\right\}-\frac{1}{2}\right) = \{ud\}-\frac{1}{2}$, by a result of Landau ([5], p.170), we get

$$\sum_{n=0}^{\ell-1}\Delta_r\left(\frac{u+n}{\ell}, n\right) = \Delta_r(u,n),$$

proving the theorem.

**Theorem B:** $\int_0^1 \Delta_r^2(x,n)\,dx = \frac{1}{12}2^{\omega(n)}\frac{\phi_r(n)}{n}$

**Proof:** By (3.3) we have

$$\int_0^1 \Delta_r^2(x,n)\,dx = \sum_{\substack{d_1|n\\ d_2|n}}\mu_r(d_1)\mu_r(d_2)\int_0^1\left(\{xd_1\}-\frac{1}{2}\right)\left(\{xd_2\}-\frac{1}{2}\right)dx$$

Now using the result of Franel [3], namely

$$\int_0^1\left(\{xd_1\}-\frac{1}{2}\right)\left(\{xd_2\}-\frac{1}{2}\right)dx = \frac{1}{12}\frac{(d_1, d_2)^2}{d_1 d_2}$$

it follows that

**(3.4)** $$\int_0^1 \Delta_r^2(x,n)\,dx = \frac{1}{12}\sum_{\substack{d_1|n\\ d_2|n}}\mu_r(d_1)\mu_r(d_2)\frac{(d_1,d_2)^2}{d_1 d_2}$$

Let $D = (d_1, d_2)$ so that $d_1 = D\delta_1$, $d_2 = D\delta_2$ and $(\delta_1,\delta_2) = 1$, Then (3.4) gives

$$\int_0^1 \Delta_r^2(x,n)\,dx = \frac{1}{12}\sum_{D|n}\sum_{\delta_1\delta_2|\frac{N}{D}}\frac{\mu_r(\delta_1\delta_2)}{\delta_1\delta_2} = \frac{1}{12}\sum_{D|n}\sum_{\delta|\frac{N}{D}}\frac{\mu_r(\delta)\tau_r(\delta)}{\delta}$$

**(3.5)** $$\int_0^1 \Delta_r^2(x,n)\,dx = \frac{1}{12}g(n),$$

where $g(n) = \sum_{D|n} f\left(\frac{n}{D}\right)$ in which $f(m) = \sum_{d|m}\frac{\mu_r(d)\tau_r(d)}{d}$,

clearly $f(m)$ is a multiplicative arithmetic function and $f(p^r) = 1-\frac{1}{p^r}$.

Therefore $g(p^r) = f(p^r) + f(1) = 2\left(1-\frac{1}{p^r}\right) = 2\frac{\phi_r(p^r)}{p^r}$.

 

Again since $g(n)$ is multiplicative, it gives that $g(n)=2^{\omega(n)}\dfrac{\phi_r(n)}{n}$.

Hence $\int_0^1 \Delta_r^2(x,n)\,dx=\dfrac{1}{12}2^{\omega(n)}\dfrac{\phi_r(n)}{n}$,

proving the theorem.

We need the following Lemma proved in [1] (Corollary. p.347) for our next result:

**3.6 Lemma:** Let $\alpha_1<\alpha_2<\alpha_3<\ldots<\alpha_\ell$ be the points in $(0,1)$ such that they are symmetric about $\dfrac{1}{2}$ and if $s(x)=\sum\limits_{\substack{i\\ \alpha_i\le x}}1 - x\ell$ then $\dfrac{1}{\ell}\sum\limits_{i=1}^{\ell}s^2(\alpha_i)=\int_0^1 s^2(x)\,dx+\dfrac{1}{6}$.

**Theorem C:** For $n>1$ and if $a_1<a_2<\ldots<a_{\phi_r(n)}$ are the r-totatives of *n* then

$$\frac{1}{\phi_r(n)}\sum_{i=1}^{\phi_r(n)}\Delta_r^2\left(\frac{a_i}{n},n\right)=\frac{1}{12}2^{\omega(n)}\frac{\phi_r(n)}{n}+\frac{1}{6}$$

**Proof:** Since $(\tau,n)_r=1\Leftrightarrow(n-\tau,n)_r=1$, the intervals $\left[0,\dfrac{n}{2}\right)$ and $\left[\dfrac{n}{2},n\right)$ have the same number of r-totatives, it follows that the numbers $\dfrac{a_i}{n}$ are symmetrically distributed about $\dfrac{1}{2}$ in $(0,1)$. Taking $\alpha_i=\dfrac{a_i}{n}$ for $1\le i\le\phi_r(n)$ in Lemma 3.6 and noting $s\left(\dfrac{a_i}{n}\right)=\Delta_r\left(\dfrac{a_i}{n},n\right)$, we get

$$\frac{1}{\phi_r(n)}\sum_{i=1}^{\phi_r(n)}\Delta_r^2\left(\frac{a_i}{n},n\right)=\int_0^1\Delta_r^2(x,n)\,dx+\frac{1}{6}.$$

Using Thereom B, we have

$$\frac{1}{\phi_r(n)}\sum_{i=1}^{\phi_r(n)}\Delta_r^2\left(\frac{a_i}{n},n\right)=\frac{1}{12}2^{\omega(n)}\frac{\phi_r(n)}{n}+\frac{1}{6}$$

proving the theorem.

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