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ON THE FUNCTION $\Delta_r(x,n)$

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ABSTRACT

Defining the function $\Delta_r(x,n)$ related to the r-totatives of n we study certain properties of it.

Key words: *r-totatives, r-analogues of Mobius and Euler Functions.*

1. INTRODUCTION

Throughout this paper r denotes a fixed positive integer. For positive integers a and b, their greatest rth power common divisor is denoted by $(a,b)_r$. It is clear that $(a,b)_1$ is the greatest common divisor *(a, b)* of *a* and *b*; and that $(a,b)_r = 1$ if and only if *(a, b)* is *r-free* (we recall that a positive integer is r-free if it is not divisible by the rth power of any prime).

For a positive integer *n*, a number τ with $(\tau,n)_r = 1$ will be called a *r-totative* of *n*. Note that 1-totatives of *n* are referred as totatives of *n* by J.J.Sylvester (see [7], p.124). V.L. Klee [4] has defined the function $\phi_r(n)$ as the number of integers *m* with $1 \le m \le n$ and $(m,n)_r = 1$. Note that $\phi_1(n) = \phi(n)$, the well-known Euler function; and that $\phi_r(n)$ is the number of r-totatives of *n* in $[0,n)$. Denote the number of r-totatives *m* of *n* with $m \le x$ by $\phi_r(x,n)$.

Here we define the function

(1.1) $$\Delta_r(x,n) = \sum_{\substack{m \le xn \\ (m,n)_r = 1}} 1 - x\phi_r(n) = \phi_r(xn,n) - x\phi_r(n)$$

Note that $\Delta(x,n) := \Delta_1(x,n)$ was studied by Codeca and Nair [1]. In this paper we present some proerties of (1.1) and the results involving this function in seciton 3.

2. PRELIMINARIES

The r-analogue of the Mobius function, $\mu_r(n)$, is defined (see [4]) by

(2.1) $$\mu_r(n) = \begin{cases} 1 & if \;\; n = 1 \\ (-1)^t & if \;\; n = p_1^r p_2^r \dots p_t^r \;\; where \;\; p_i\text{'s are distinct primes} \\ 0 & otherwise \end{cases}$$

and showed that it is multiplicative. V.L.Klee [4] has proved that

(2.2) $$\phi_r(n) = \sum_{d|n} \mu_r(d)\frac{n}{d} = \sum_{\delta|n} \mu_r\left(\frac{n}{\delta}\right)\delta$$

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Since $\phi_r(x,n)$ is the number of r-totatives *m* of *n* with $m \le x$, it is easy to show that

(2.3) $$\phi_r(x,n) = \sum_{d|n} \mu_r(d)\left[\frac{x}{d}\right] = \sum_{\delta|n} \mu_r\left(\frac{n}{\delta}\right)\left[\frac{x\delta}{n}\right]$$

where [y] is the greatest integer not exceeding y.

(2.4) Suppose for a given *n* let $N_r = N_r(n)$ is the r[th] power of the maximal square free divisor of n. Then note that $(a,n)_r = 1 \Leftrightarrow (a, N_r)_r = 1$. Hence we may assume, without loss to generality, that *n* itself is an r[th] power of a squarefree number *m,* say $n = m^r$. In all that follows *n* is always of this form.

Note that

(2.5) $\Delta_r(x,n)$ is periodic in *x* with period 1.

(2.6) Let $1 = a_1 < a_2 < ... < a_{\phi_r(n)} = n-1$ be the $\phi_r(n)$ r-totatives of *n* in the interval $[1,n)$. We write $a_o = 0$ and $a_{\phi_r(n)+1} = n$. Then $n - a_i = a_{\phi_r(n)-i+1}$ and $\frac{a_i}{n} \in [0,1]$ for $0 \le i \le a_{\phi_r(n)+1}$. If a_i's are defined as in (2.6) we observe that

(2.7) $$\Delta_r\left(\frac{a_i}{n}, n\right) = i - a_i\frac{\phi_r(n)}{n} \quad \text{for} \quad 0 \le i \le \phi_r(n)$$

(2.8) $\Delta_r(x,n) = \Delta_r\left(\frac{a_i}{n}, n\right) - \left(x - \frac{a_i}{n}\right)\phi_r(n)$ which imply that $\Delta_r(x,n)$ is a piecewise linear function of *x* with each line segment in $\left[\frac{a_i}{n}, \frac{a_{i+1}}{n}\right)$ having the gradient $-\phi_r(n)$.

3. MAIN RESULTS

3.1 *Lemma:* $\Delta_r(x,n) = -\mu_r(n)\sum_{d|n}\mu_r(d)\{xd\}$

Proof: By (1.1), (2.3) and (2.2) we get

$$\Delta_r(x,n) = \sum_{d|n}\mu_r(d)\left(\left[\frac{xn}{d}\right] - \frac{xn}{d}\right)$$
$$= -\sum_{d|n}\mu_r(d)\left\{\frac{xn}{d}\right\} = -\sum_{d|n}\mu_r\left(\frac{n}{d}\right)\{xd\}$$

where {*y*} denotes the fractional part of *y*. Since the contribution of divisiors *d* of *n* to the sum on the right is non-zero if and only if *d* is the r[th] power of square free integer, so that

$$\Delta_r(x,n) = \sum_{d|n}\frac{\mu_r(n)}{\mu_r(d)}\{xd\} = -\mu_r(n)\sum_{d|n}\mu_r(d)\{xd\},$$

proving the Lemma.

As a consequence of Lemma 3.1, we have the identity:

(3.2) If $p \nmid$ n, $\Delta_r(x, np^r) = \Delta_r(xp^r, n) - \Delta_r(x,n)$

It is easy to see that

(3.3) $$\Delta_r(x,n) = -\mu_r(n)\sum_{d|n}\mu_r(d)\left(\{xd\} - \frac{1}{2}\right)$$

Theorem A: If $(\ell,n)_r=1$ then $\sum_{n=0}^{\ell-1}\Delta_r\left(\frac{u+n}{\ell},n\right)=\Delta_r(u,n)$

Proof: By (3.3) we have

$$\sum_{n=0}^{\ell-1}\left(\frac{u+n}{\ell},n\right)=-\mu_r(n)\sum_{d|n}\mu_r(d)\sum_{n=0}^{\ell-1}\left(\left\{\frac{ud}{\ell}+\frac{n}{\ell}\right\}-\frac{1}{2}\right)$$

$$=-\mu_r(n)\sum_{d|n}\mu_r(d)\left(\{ud\}-\frac{1}{2}\right),$$

Since, $\sum_{n=0}^{\ell-1}\left(\left\{\frac{ud}{\ell}+\frac{n}{\ell}\right\}-\frac{1}{2}\right)=\{ud\}-\frac{1}{2}$, by a result of Landau ([5], p.170), we get

$$\sum_{n=0}^{\ell-1}\Delta_r\left(\frac{u+n}{\ell},n\right)=\Delta_r(u,n),$$

proving the theorem.

Theorem B: $\int_0^1\Delta_r^2(x,n)\,dx=\frac{1}{12}2^{\omega(n)}\frac{\phi_r(n)}{n}$

Proof: By (3.3) we have

$$\int_0^1\Delta_r^2(x,n)\,dx=\sum_{\substack{d_1|n\\ d_2|n}}\mu_r(d_1)\mu_r(d_2)\int_0^1\left(\{xd_1\}-\frac{1}{2}\right)\left(\{xd_2\}-\frac{1}{2}\right)dx$$

Now using the result of Franel [3], namely

$$\int_0^1\left(\{xd_1\}-\frac{1}{2}\right)\left(\{xd_2\}-\frac{1}{2}\right)dx=\frac{1}{12}\frac{(d_1,d_2)^2}{d_1d_2}$$

it follows that

(3.4) $$\int_0^1\Delta_r^2(x,n)\,dx=\frac{1}{12}\sum_{\substack{d_1|n\\ d_2|n}}\mu_r(d_1)\mu_r(d_2)\frac{(d_1,d_2)^2}{d_1d_2}$$

Let $D=(d_1,d_2)$ so that $d_1=D\delta_1$, $d_2=D\delta_2$ and $(\delta_1,\delta_2)=1$, Then (3.4) gives

$$\int_0^1\Delta_r^2(x,n)\,dx=\frac{1}{12}\sum_{D|n}\sum_{\delta_1\delta_2|\frac{N}{D}}\frac{\mu_r(\delta_1\delta_2)}{\delta_1\delta_2}=\frac{1}{12}\sum_{D|n}\sum_{\delta|\frac{N}{D}}\frac{\mu_r(\delta)\tau_r(\delta)}{\delta}$$

(3.5) $$\int_0^1\Delta_r^2(x,n)\,dx=\frac{1}{12}g(n),$$

where $g(n)=\sum_{D|n}f\left(\frac{n}{D}\right)$ in which $f(m)=\sum_{d|m}\frac{\mu_r(d)\tau_r(d)}{d}$,

clearly $f(m)$ is a multiplicative arithmetic function and $f(p^r)=1-\frac{1}{p^r}$.

Therefore $g(p^r)=f(p^r)+f(1)=2\left(1-\frac{1}{p^r}\right)=2\frac{\phi_r(p^r)}{p^r}$.

Again since $g(n)$ is multiplicative, it gives that $g(n)=2^{\omega(n)}\dfrac{\phi_r(n)}{n}$.

Hence $\displaystyle\int_0^1 \Delta_r^2(x,n)\,dx=\frac{1}{12}2^{\omega(n)}\frac{\phi_r(n)}{n}$,

proving the theorem.

We need the following Lemma proved in [1] (Corollary. p.347) for our next result:

3.6 Lemma: Let $\alpha_1<\alpha_2<\alpha_3<\ldots<\alpha_\ell$ be the points in $(0,1)$ such that they are symmetric about $\frac{1}{2}$ and if $s(x)=\sum\limits_{\substack{i\\ \alpha_i\le x}}1 - x\ell$ then $\displaystyle\frac{1}{\ell}\sum_{i=1}^{\ell}s^2(\alpha_i)=\int_0^1 s^2(x)\,dx+\frac{1}{6}$.

Theorem C: For $n>1$ and if $a_1<a_2<\ldots<a_{\phi_r(n)}$ are the r-totatives of *n* then

$$\frac{1}{\phi_r(n)}\sum_{i=1}^{\phi_r(n)}\Delta_r^2\left(\frac{a_i}{n},n\right)=\frac{1}{12}2^{\omega(n)}\frac{\phi_r(n)}{n}+\frac{1}{6}$$

Proof: Since $(\tau,n)_r=1\Leftrightarrow(n-\tau,n)_r=1$, the intervals $\left[0,\frac{n}{2}\right)$ and $\left[\frac{n}{2},n\right)$ have the same number of r-totatives, it follows that the numbers $\frac{a_i}{n}$ are symmetrically distributed about $\frac{1}{2}$ in $(0,1)$. Taking $\alpha_i=\frac{a_i}{n}$ for $1\le i\le\phi_r(n)$ in Lemma 3.6 and noting $s\left(\frac{a_i}{n}\right)=\Delta_r\left(\frac{a_i}{n},n\right)$, we get

$$\frac{1}{\phi_r(n)}\sum_{i=1}^{\phi_r(n)}\Delta_r^2\left(\frac{a_i}{n},n\right)=\int_0^1\Delta_r^2(x,n)\,dx+\frac{1}{6}.$$

Using Thereom B, we have

$$\frac{1}{\phi_r(n)}\sum_{i=1}^{\phi_r(n)}\Delta_r^2\left(\frac{a_i}{n},n\right)=\frac{1}{12}2^{\omega(n)}\frac{\phi_r(n)}{n}+\frac{1}{6}$$

proving the theorem.

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