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A Common Fixed Point Theorem in Complex Valued b-Metric Spaces for Four Mappings

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ABSTRACT

In this paper we prove a common fixed point theorem for four self-mappings in a complete complex valued b-metric space.

Key Words: *Complex valued b-metric space, weakly compatible mappings.*

AMS Subject Classification (2010): *47H10, 54H25.*

1. INTRODUCTION

In 1989, Bakhtin [3] introduced the concept of b-metric space as a generalization of metric spaces. The concept of complex valued b-metric spaces was introduced in 2013 by Rao *et al.* [10], which was more general than the well-known complex valued metric spaces that were introduced in 2011 by Azam *et al.* [2]. The main purpose of this paper is to present common fixed point results of four self-mappings satisfying a rational inequality on complex valued b-metric spaces. The results presented in this paper are generalization of work done by Sanjib Kumar Dutta and Sultan Ali in [6].

Definition 1 (see [1]): Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d: X \times X \to \mathbb{C}$ is called a b-metric if for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x,y) = 0$ if and only if $x = y$
(ii) $d(x,y) = d(y,x)$
(iii) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X,d).

Example 2 (see [11]): Let (X,d) be a metric space and $\rho(x,y) = \left(d(x,y)\right)^p$, where $p > 1$ is a real number. Then (X,ρ) is a b-metric space with $s = 2^{p-1}$.

Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the followings holds:

(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$ and
(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e., one of (2), (3) and (4) is satisfied and we will write $z_1 \prec z_2$ if only (4) is satisfied.

Remark 1: We can easily check the followings:

(i) $a, b \in \mathbb{R}, a \leq b \Rightarrow az \precsim bz \ \forall\, z \in \mathbb{C}$.
(ii) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$.
$z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

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Definition 3 (see [2]): Let X be a nonempty set. A function $d: X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$
(ii) $d(x,y) = d(y,x)$
(iii) $d(x,y) \precsim d(x,z) + d(z,y)$

The pair (X,d) is called a complex valued metric space.

Example 4 (see [5]): Let $X = \mathbb{C}$. Define the mapping $d: X \times X \to \mathbb{C}$ by

$$d(x,y) = i|x - y|, \text{ for all } x, y \in X.$$

Then (X,d) is a complex valued metric space.

Definition 5 (see[10]): Let X be a nonempty set and let $s \geq 1$ be given real number. A function $d: X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$
(ii) $d(x,y) = d(y,x)$
(iii) $d(x,y) \precsim s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a complex valued b-metric space.

Example 6 (see [10]): Let $X = [0,1]$. Define the mapping $d: X\, x\, X \to \mathbb{C}$ by

$$d(x,y) = |x - y|^2 + i\,|x - y|^2, \text{ for all } x, y \in X.$$

Then (X,d) is a complex valued b-metric space with $s = 2$.

Definition 7(see[10]): Let (X,d) be a complex valued b-metric space. Consider the following .

(i) A point $x \in X$ is called an interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x,r) = \{y \in X : d(x,y) \prec r\} \subseteq A$.
(ii) A point $x \in X$ is called a limit point of a set A whenever, for every $0 \prec r \in \mathbb{C}$, $B(x,r) \cap (A - \{x\}) \neq \phi$.
(iii) A subset A of X is called open whenever each point of A is an interior point of A.
(iv) A subset A of X is called closed whenever each limit point of A belongs to A.
(v) A subbasis for a Hausdorff topology τ on X is a family

$$F = \{B(x,r) : x \in X \text{ } and \text{ } 0 \prec r\}.$$

Definition 8 (see [10]): Let (X,d) be a complex valued b-metric space and $\{x_n\}$ a sequence in X and $x \in X$. Consider the following.

(i) If for every c , with $0 \prec c$, there is $N \in \mathbb{N}$ such that, for all $n > N$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, if $\{x_n\}$ converges to x, and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(ii) If for every $\in \mathbb{C}$, with $0 \prec c$, there is $N \in \mathbb{N}$ such that, for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \epsilon \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.
(iii) If every Cauchy sequance in X is convergent, then (X,d) is said to be a complete complex valued b-metric space.

Definition 9 (see [7]): Let (X,d) be a complex valued metric space. The self-maps S and T are said to be commuting if $STx = TSx$ for all $x \in X$.

Definition 10 (see [8]): Let (X,d) be a complex valued metric space. The self-maps S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n\, , TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 11 (see [9]): Let (X,d) be a complex valued metric space. The self-maps S and T are said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$, i.e., they commute at their coincidence points.

Lemma 12 (see [10]): Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 13 (see [10]): Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ s a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

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Theorem 14 (see [6]): Let (X, d) be a complete complex valued metric space and let S, T, f and g be self-maps of X such that

(i) The pairs $\{S, f\}$ and $\{T, g\}$ are weakly compatible,
(ii) $TX \subseteq fX$ and $SX \subseteq gX$,
(iii) fX or gX is a complete subspace of X and
(iv) $d(Sx, Ty) \precsim \lambda d(fx, gy) + \frac{\mu d(fx,Sx)d(gy,Ty)}{1+d(fx,gy)}, \forall\, x, y \in X,$
where λ, μ are non-negative reals with $\lambda + \mu < 1$.

Then S, T, f and g have a unique common fixed point.

2. MAIN RESULT

My theorem is a generalization of Theorem 14 in complex valued b-metric spaces.

Theorem: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$. Let S, T, f and g be self-mappings of X such that

(i) The pairs $\{ S, f \}$ and $\{ T, g \}$ are weakly compatible,
(ii) $TX \subseteq fX$ and $SX \subseteq gX$,
(iii) fX or gX is a complete subspace of X and
(iv) $d(Sx, Ty) \precsim \lambda d(fx, gy) + \frac{\mu d(fx,Sx)d(gy,Ty)}{1+d(fx,gy)}, \forall\, x, y \in X,$ where λ, μ are non-negative reals with $s\lambda + \mu < 1$.

Then S, T, f and g have a unique common fixed point.

Proof: Let $x_0 \in X$ be arbitrary. Using the condition (ii), we defined a sequence $\{y_n\}$ in X as

$$y_{2k+1} = gx_{2k+1} = Sx_{2k}$$
$$y_{2k+2} = fx_{2k+2} = Tx_{2k+1},\ k = 0, 1, 2, \ldots\ldots\ldots.$$

Then

$$\begin{aligned} d(y_{2k+1}, y_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \\ &\precsim \lambda d(fx_{2k}, gx_{2k+1}) + \frac{\mu d(fx_{2k},Sx_{2k})d(gx_{2k+1},Tx_{2k+1})}{1+d(fx_{2k},gx_{2k+1})} \\ &= \lambda d(y_{2k}, y_{2k+1}) + \frac{\mu d(y_{2k},y_{2k+1})d(y_{2k+1},y_{2k+2})}{1+d(y_{2k},y_{2k+1})} \\ &\precsim \lambda\, d(y_{2k}, y_{2k+1}) + \mu d(y_{2k+1}, y_{2k+2}) \end{aligned}$$

Thus $d(y_{2k+1}, y_{2k+2}) \precsim \frac{\lambda}{1-\mu} d(y_{2k+1}, y_{2k+2})$ (1)

Similarly

$$\begin{aligned} d(y_{2k+2}, y_{2k+3}) &= d(Sx_{2k+2}, Tx_{2k+1}) \\ &\precsim \lambda d(fx_{2k+2}, gx_{2k+1}) + \frac{\mu d(fx_{2k+2},Sx_{2k+2})d(gx_{2k+1},Tx_{2k+1})}{1+d(fx_{2k+2},gx_{2k+1})} \\ &= \lambda d(y_{2k+2}, y_{2k+1}) + \frac{\mu d(y_{2k+2},)y_{2k+3}d(y_{2k+1},y_{2k+2})}{1+d(y_{2k+2},y_{2k+1})} \\ &\precsim \lambda\, d(y_{2k+2}, y_{2k+1}) + \mu d(y_{2k+2}, y_{2k+3}) \end{aligned}$$

Thus $d(y_{2k+2}, y_{2k+3}) \precsim \frac{\lambda}{1-\mu} d(y_{2k+2}, y_{2k+1})$ (2)

Now put $h = \frac{\lambda}{1-\mu}$

Since $0 \leq s\lambda + \mu < 1, s \geq 1,\ \lambda + \mu < 1$ and hence $0 \leq h < 1$.

Thus using (1) and (2) for $n \in \mathbb{N}$, we get that

$$d(y_n, y_{n+1}) \precsim hd(y_{n-1}, y_n) \precsim h^2\, d(y_{n-2}, y_{n-1}) \precsim \ldots\ldots\ldots\ldots\ldots.. \precsim h^{n-1}\, d(y_1, y_2).$$

So for $m, n \in \mathbb{N}$,

$$\begin{aligned} d(y_n, y_{m+n}) &\precsim s[d(y_n, y_{n+1}) + d(y_{n+1}, y_{m+n})] \\ &\precsim sd(y_n, y_{n+1}) + s^2\,[\,d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{m+n})] \\ &\precsim sd(y_n, y_{n+1}) + s^2\, d(y_{n+1}, y_{n+2}) + s^3\, d(y_{n+2}, y_{n+3}) + \cdots \ldots + s^{m-1}\, d(y_{n+m-2}, y_{n+m-1}) \\ &\qquad + s^{m-1}\, d(y_{n+m-1}, y_{n+m}) \\ &\precsim sd(y_n, y_{n+1}) + s^2\, d(y_{n+1}, y_{n+2}) + s^3\, d(y_{n+2}, y_{n+3}) + \ldots\ldots + s^{m-1}\, d(y_{n+m-2}, y_{n+m-1}) \\ &\qquad + s^m\, d(y_{n+m-1}, y_{n+m}) \\ &\precsim sh^{n-1}d(y_1, y_2) + s^2h^n d(y_1, y_2) + \ldots\ldots\ldots\ldots + s^{m-1}h^{n+m-3}d(y_1, y_2) + s^m\, h^{n+m-2}d(y_1, y_2) \\ &= sh^{n-1}[\,1 + sh + s^2h^2 + \ldots\ldots\ldots\ldots\ldots + s^{m-1}h^{m-1}]d(y_1, y_2) \\ &\precsim \frac{sh^{n-1}}{1-sh} d(y_1, y_2) \end{aligned}$$

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Thus $|d(y_n, y_{m+n})| \leq \frac{sh^{n-1}}{1-sh}\ |d(y_1, y_2)|\ \to 0$ as $n\ \to\ \infty$, where $m \in \mathbb{N}$.

Hence $\{y_n\}$ is a Cauchy sequence in X.

Since X is complete, there exists $z \in X$ such that $y_n \to\ z$ as $n\ \to\ \infty$.

Thus $\lim_{n \to \infty} Sx_{2n} = \lim_{n \to \infty} gx_{2n+1} = \lim_{n \to \infty} Tx_{2n+1} = \lim_{n \to \infty} fx_{2n+2} = z$ (3)

Now if fX is a complete subspace of X, there exists $u \in X$ such that $fu\ =\ z$

From the condition (iv), we have

$$
\begin{aligned}
d(Su, z) &\precsim sd(Su, Tx_{2n+1}) + sd(Tx_{2n+1}, z) \\
&\precsim s\left[\lambda d(fu, gx_{2n+1}) + \frac{\mu d(fu,Su) d(gx_{2n+1},Tx_{2n+1})}{1+d(fu,gx_{2n+1})}\right] + sd(Tx_{2n+1}, z) \\
&= s\left[\lambda d(fu, y_{2n+1}) + \frac{\mu d(fu,Su) d(y_{2n+1},y_{2n+2})}{1+d(fu,y_{2n+1})}\right] + sd(y_{2n+2}, z)
\end{aligned}
$$

Therefore $|d(Su, z)| \leq s\left[\lambda |d(fu, y_{2n+1})| + \frac{\mu |d(fu,Su)||d(y_{2n+1},y_{2n+2})|}{1+|d(fu,y_{2n+1})|}\right] + s|d(y_{2n+2}, z)|$

Letting $n \to \infty$ and using (3) and Lemma 12, we get that $|d(Su, z)| \leq\ 0$.

Thus $|d(Su, z)|\ =\ 0$. i.e. $d(Su, z)\ =\ 0$ and hence $Su\ =\ z$.

Since $SX\ \subseteq gX$, there exists $v \in X$ such that $gv = z$.

Again from condition (iv), we have

$$
\begin{aligned}
d(z, Tv)\ &=\ d(Su, Tv) \\
&\precsim \lambda d(fu, gv) + \frac{\mu d(fu,Su) d(gv,Tv)}{1+d(fu,gv)} \\
&= 0
\end{aligned}
$$

Thus $d(z, Tv)\ =\ 0$ and hence $Tv\ =\ z$.

Thus $fu\ =\ Su\ =\ z\ =\ gv\ =\ Tv$.

Since f and S are weakly compatible,

$$fz\ =\ fSu\ =\ Sfu\ = Sz.$$

Now we will show that $Sz\ =\ z$.

From condition (iv),

$$
\begin{aligned}
d(Sz, z)\ &=\ d(Sz, Tv) \\
&\precsim \lambda d(fz, gv) + \frac{\mu d(fz,Sz) d(gv,Tv)}{1+d(fz,gv)} \\
&= \lambda d(Sz, z)
\end{aligned}
$$

Thus $(1 - \lambda)|d(Sz, z)|\ \leq 0$

Thus $d(Sz, z) = 0$ and hence $Sz\ =\ z$.

Similarly since g and T are weakly compatible,

$$gz\ =\ gTv\ =\ Tgv\ =\ Tz.$$

Also

$$
\begin{aligned}
d(z, Tz) &=\ d(Sz, Tz) \\
&\precsim \lambda d(fz, gz) + \frac{\mu d(fz,Sz) d(gz,Tz)}{1+d(fz,gz)} \\
&= \lambda d(z, Tz)
\end{aligned}
$$

Thus $d(z, Tz)\ =\ 0$ and hence $Tz\ =\ z$.

Thus $Sz\ =\ fz\ =\ gz\ =\ Tz\ =\ z$.

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i.e. z is a common fixed point of four mappings S, T, f and g.

Now we show that z is the unique common fixed point.

Let $z^* \in X$ such that $fz^* = Sz^* = gz^* = Tz^* = z^*$.

Then we have,

$$\begin{aligned} d(z, z^*) &= d(Sz, Tz^*) \\ &\precsim \lambda d(fz, gz^*) + \frac{\mu d(fz, Sz) d(gz^*, Tz^*)}{1 + d(fz, gz^*)} \\ &= \lambda d(z, z^*) \end{aligned}$$

Thus $d(z, z^*) = 0$ and so $z = z^*$. Thus z is the unique common fixed point of S, T, f and g.

If gX is complete, we can similarly prove the theorem.

Corollary 1: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$. Let S, T be self-mappings of X such that

$$d(Sx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Sx) d(y, Ty)}{1 + d(x, y)}, \forall\, x, y \in X,$$

where λ, μ are non-negative reals with $s\lambda + \mu < 1$.

Then S, T have a unique common fixed point.

Proof: Taking $f(x) = x$ and $g(x) = x$, $\forall\, x \in X$ in the above theorem we get the result.

Corollary 2: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$. Let T, f and g be self-mappings of X such that

(i) The pairs $\{T, f\}$ and $\{T, g\}$ are weakly compatible
(ii) $TX \subseteq fX$ and $TX \subseteq gX$
(iii) fX or gX is a complete subspace of X and
(iv) $d(Tx, Ty) \precsim \lambda d(fx, gy) + \frac{\mu d(fx, Tx) d(gy, Ty)}{1 + d(fx, gy)}, \forall\, x, y \in X$, where λ, μ are non-negative reals with $s\lambda + \mu < 1$.

Then T, f and g have a unique common fixed point.

Proof: Taking $S = T$ in the above theorem, we get the result.

Corollary 3: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$. Let T, f and g be self-mappings of X and n is a possitive integers, satisfying the following conditions

(i) The pairs $\{T^n, f\}, \{T^n, g\}, \{T, f\}$ and $\{T, g\}$ are weakly compatible
(ii) $T^n X \subseteq fX$ and $T^n X \subseteq gX$
(iii) fX or gX is a complete subspace of X and
(iv) $d(T^n x, T^n y) \precsim \lambda d(fx, gy) + \frac{\mu d(fx, T^n x) d(gy, T^n y)}{1 + d(fx, gy)}, \forall\, x, y \in X$, where λ, μ are non-negative reals wit $s\lambda + \mu < 1$.

Then T, f and g have a unique common fixed point.

Proof: Applying corollary 2, we get a unique common fixed point z of T^n, f and g.

Therefore $T^n z = fz = gz = z$.

Now we note that $T^n Tz = TT^n z = Tz$.

Also since the pairs $\{T, f\}$ and $\{T, g\}$ are weakly compatible,

$$fTz = Tfz = Tz \text{ and } gTz = Tgz = Tz.$$

Thus we see that Tz is also a common fixed point of T^n, f and g.

Thus by uniqueness of z, we have $Tz = z$.

Hence z is a common fixed point of T, f and g.

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Since any common fixed point of T, f and g is also a common fixed point of T^n, f and g, the common fixed point z of T, f and g is unique.

This complete the proof.

Corollary 4: Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$. Let T be a self-mappings of X and n is a possitive integers , such that

$$d(T^n x, T^n y) \precsim \lambda d(x, y) + \frac{\mu d(x, T^n x) d(y, T^n y)}{1 + d(x, y)}, \forall\, x, y \in X,$$

where λ, μ are non-negative reals with $s\lambda + \mu < 1$.

Then T has a unique common fixed point.

Proof: In corollary 3, if we take $f(x) = x$ and $g(x) = x$, for all $x \in X$, then the required result follows.

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