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SUPRA FUZZY TOPOLOGICAL SPACE AND HAUSDORFFNESS

MANJARI SRIVASTAVA, PARIJAT SINHA*

V. S. S. D. College, Kanpur, India.

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ABSTRACT

Here we have introduced and studied a new definition of Hausdorffness in a fuzzy supra topological space. A complete comparison of this concept with other existing definitions has been established and we have proved the appropriateness of our concept by proving several interesting results.

***Key words**: Fuzzy supra topology, Hausdorff fuzzy supra topological space, Fuzzy supra continuity, goodness of extension.*

INTRODUCTION

Fuzzy set theory is a useful tool to describe a situation in which the data is imprecise or vague or there is no clear cut boundary. Fuzzy set handle such situation by attributing a degree of membership to which a certain object belongs to the set. In 1965, Fuzzy set was introduced by Zadeh as follows:

Let X be a set, a fuzzy set A in X is characterised by a membership function $\mu_A : X \rightarrow [0, 1]$

Later, in 1968 Chang introduced fuzzy topology as a family τ of fuzzy sets in X which satisfies the following conditions

(i) $\emptyset, X \in \tau$
(ii) If $A, B \in \tau$, then $A \cap B \in \tau$
(iii) If $A_i \in \tau$ for each $i \in \wedge$ then $\cup A_i \in \tau$

In 1976, Lowen modified this definition as all constant functions should belong to τ otherwise constant functions will not be continuous.

In 1983, Mashhour *et al*. introduced the concepts of supra topological spaces, supra open sets and supra closed sets. Later on in 1987, Monsef et al. introduced the concept of fuzzy supra topological spaces as a natural generalization of the notion of supra topological spaces. They defined a fuzzy supra topology on X as a family $\tau \subseteq I^X$ which is closed under arbitrary union and contains $\emptyset$ and X. Here in this paper we have modified the definition of a fuzzy supra topology on X. We have called a family $\tau \subseteq I^X$, a fuzzy supra topology on X if it is closed under arbitrary union and contains all constant fuzzy sets in X.

We concentrate here mainly on Hausdorffness in a fuzzy supra topological space. Earlier this concept has been introduced and studied by S. Dang et al. [3] and A. Kandil [4]. Here we have given another definition of Hausdorffness in an fsts which is on parallel lines as in [13] and which is a natural generalization of the corresponding concept in case of topological spaces. We also note that by replacing 'fuzzy singleton' by 'fuzzy point' in A. Kandil's definition, we get a different definition of Hausdorffness in an fsts. A complete comparison of our definition with the remaining three has been given. It turns out that all the four definitions satisfy good extension property.

We have proved the appropriateness of our definition by proving several interesting relevant results eg. It is equivalent to Δ_X being fuzzy s-closed' and that it is productive and hereditary.

Corresponding Author: Parijat Sinha, V. S. S. D. College, Kanpur, India.*

PRELIMINARIES

Here we shall follow Lowen's definition of fuzzy topology. I denote the unit interval [0, 1], a constant fuzzy set taking value $\alpha \epsilon [0,1]$ will be denoted by $\underline{\alpha}$ and A′ will denote the complement of a fuzzy set A in X. If A $\subseteq$ X, then we shall identify the characteristic function χ_A with A itself. Also α_A will denote the fuzzy set in X, which takes the constant value α on A and zero otherwise. As in [6], a fuzzy point 'x_r' is a fuzzy set in X, taking value r ϵ (0, 1) at x and zero otherwise. x and r are respectively called the support and value the fuzzy point x_r. x_r is said to belong to a fuzzy set A in X iff r <A(x).

The following definitions are from [6].

A fuzzy singleton 'x_r' in X is a fuzzy set in X taking value x ϵ $(0, 1]$ at x and zero elsewhere. A fuzzy singleton x_r is said to be quasi-coincident with a fuzzy set A (notation: x_r q A) iff r+A(x) > 1, if x_r is not quasi-coincident with A, we write $x_r\, \bar{q}$ A. Two fuzzy sets A and B in X are said to be quasi-coincident if $\exists$ x $\in$X such that A(x)+ B(x) > 1. If A and B are not quasi-coincident, then we write A $\bar{q}$ B. It can be checked easily that A $\bar{q}$ B $\Leftrightarrow$ A $\subseteq$ B′.

S. Dang et al. [3] and A. Kandil *et al.* [4] have defined a fuzzy supra topology on X as a subfamily $\tau \subseteq I^x$ which is closed under arbitrary union and contains X, $\emptyset$. We make a modification here and take the following definition:

Definition 2.1: A subfamily $\tau \subseteq I^x$ is called a fuzzy supra topology on X if it contains all constant fuzzy sets and is closed under arbitrary union.

If τ is a fuzzy supra topology on X then (X, τ) is called a fuzzy supra topological space, in short, an fsts. Members of τ are called fuzzy supra open sets (in short fuzzy s-open sets) and their complements are called fuzzy supra closed sets (in short fuzzy s-closed sets) in X.

Definition 2.2 [3]: A fuzzy set A in an fsts is called a fuzzy supra neighbourhood of a fuzzy singleton x_r if $\exists$ B$\in$τ such that $x_r \subseteq$ B $\subseteq$ A.

Definition 2.3 [3]: Let (X, τ) be an fsts. A subfamily ß of τ is called a base for τ if each U$\in$ τ can be expressed as a union of members of ß.

Definition 2.4 [3]: A mapping f: (X, τ_1)→(Y, τ_2) between two fsts is called fuzzy supra continuous (fuzzy s-continuous, in short) if $f^{-1}(V) \in \tau_1$ for every V$\in$ τ_2.

Proposition 2.1: A fuzzy set U in an fsts (X, τ) is fuzzy S-open iff it is fuzzy supra neighbourhood of each of its fuzzy points.

Proof: Let U be fuzzy S-open in (X, τ). Take any fuzzy point $x_r \in$ U. Clearly U is a fuzzy supra neighbourhood of x_r. Conversely, let U be a fuzzy supra neighbourhood of each of its fuzzy points. Then for any fuzzy point x_r in U, $\exists$ a fuzzy S-open set say Vx_r such that $x_r \subseteq Vx_r \subseteq$ U, therefore, $\bigcup_{xr \in U}$ xr $\subseteq$ $\bigcup_{xr \in U}$ Vxr $\subseteq$ U which implies that $\bigcup$ Vxr = U (Since $\bigcup_{xr \in U}$ xr = U).

Proposition 2.2: A fuzzy point $x_r \in \cup A_i$ iff $x_r \in A_i$ for some i.

Proposition 2.3: A fuzzy set U in an fsts (X, τ) is fuzzy S-open iff for every fuzzy point x_r in X, $\exists$ a basic fuzzy S-open set B such that x_r $\in$ B $\subseteq$U.

Proof: Let ß a base of (X, τ), U be a fuzzy S-open set in X and x_r be a fuzzy point belonging to U, let U=$\cup${B_i : $B_i \in$ ß*$\subseteq$ ß}. Then $x_r \in \cup B_i$, implying that $x_r \in B_i$ for some i (using Proposition 2.2). Thus $x_r \in B_i \subseteq$U. Conversely, let $\forall$ x_r $\in$ U, $\exists$ a basic fuzzy S-open set say Bx_r such that $x_r \in Bx_r \subseteq$ U. Thus, $\bigcup_{xr \in U}$ xr $\subseteq$ $\bigcup_{xr \in U}$ Bxr $\subseteq$ U Implying that U= $\bigcup_{xr \in U}$ Bxr and hence U is a fuzzy S-open.

Definition 2.5 [4]: Let (X, T) be supra topological space. Then, w (T) = {μ $\in$ I^x: μ^{-1} (α, 1) $\in$ T, ∀α $\in$ (0, 1)} is the induced fuzzy supra topology of the supra topology T.

Definition 2.6 [4]: Let (X, δ) be an fsts. The α-level of the fuzzy supra topology δ is $\iota_\alpha(\delta)$ ={μ^{-1} (α, 1] :μ $\in$ δ}. It can be checked that $\iota_\alpha(\delta)$ is a supra topology on X. The initial supra topology on X is $\iota(\delta) = Sup_{\alpha \in [0,1)}\, \iota_{(\delta)}$

Definition 2.7[3]: Let (X, τ) be an fsts and Y $\subseteq$ X, then τ_Y={Y$\cap A : A \in \tau$} is called the fuzzy supra subspace topology on Y and (Y, τ_Y) is called a fuzzy supra subspace of (X, τ).

If x_r is a fuzzy point in $Y \subseteq X$ then we will identify x_r with the fuzzy point in X which take the value r at x and zero otherwise.

Hausdorff Fuzzy supra topological spaces-In [3], S dang et al. defined a Hausdorff fsts as follows:

Definition 3.1: An fsts (X,τ) is called Hausdorff if for each x, y $\in$ X, x$\neq$y, there exist U,V$\in \tau$ such that U(x)=1=V(y) and $U \subseteq V'$. K and il *et. al* [4] defined a Hausdorff fsts as:

Definition 3.2: An fsts (X, τ) is called Hausdorff if for any pair of fuzzy singletons x_t, y_r in X, such that $x_t \bar{q} y_r$, there exist U, V $\in \tau$ such that $x_t \subseteq U$, $y_r \subseteq V$ and $U \bar{q} V$.

Remark 3.1: In an fsts considered here (which includes all constant fuzzy sets in X), for $x_t \bar{q} y_r$ having x=y, the above condition for Housdorffness in definition 3.2 is automatically satisfied with $U=\underline{t}$, $V=\underline{r}$.

If we replace 'fuzzy singleton' by fuzzy point in definition 3.2, we get another definition of Housdorffness in an fsts as follows:

Definition 3.3: An fsts (X, τ) is called Housdorff if for any pair of fuzzy points x_t, y_r in X, such that $x_t \bar{q} y_r$, there exists U, V $\in \tau$ such that $x_t \subseteq U$, $y_r \subseteq V$ and $U \bar{q} V$.

Now we give another definition of Housdorffness in an fsts which is on parallel lines as in [9].

Definition 3.4: An fsts (X, τ) is called Housdorff if for pair of distinct fuzzy points x_r and y_s in X, there exist U, V $\in \tau$ such that $x_r \in U$, $y_s \in V$ and $U \cap V = \emptyset$. In short, Housdorffness in the sense of definitions 3.1, 3.2, 3.3 and 3.4 will be respectively denoted by ST_2 (i), ST_2 (ii), ST_2 (iii) and ST_2 (iv). It can be easily seen that definitions 3.1 and 3.2 are equivalent.

Now we compare definition 3.4 with the other three definitions, in the following theorem

Theorem 3.1: Let (X, τ) be an fsts, then
(a) ST_2 (iv) and ST_2 (i) are independent
(b) ST_2 (iv) and ST_2 (ii) are independent
(c) ST_2 (iv) $\Rightarrow$ ST_2 (iii) but ST_2 (iii) $\nRightarrow$ ST_2 (iv)

Proof: The following two counter examples show that ST_2 (iv) and ST_2 (i) are independent.

Counter example 3.1: ST_2 (i) $\nRightarrow$ ST_2 (iv)

Let X be an infinite set and τ be the fuzzy topology on X generated by
$\{\alpha : \alpha \in [0, 1]\} \cup \{X\text{-}\{x\} : x \in X\} \cup \{A_{xy}, B_{xy} : x, y \in X, x \neq y\}$
Where A_{xy} and B_{xy} are defined as:
$A_{xy}(x) = 0$, $A_{xy}(y) = 1$, $A_{xy}(z) = \frac{1}{2}$ for $z \neq x, y$
And $B_{xy}(x) = 1$, $B_{xy}(y) = 0$, $B_{xy}(z) = \frac{1}{3}$ for $z \neq x, y$

Then (X, τ) is an fsts which is ST_2 (i) but not and ST_2 (iv), since there is no fuzzy supra open set in X, which takes zero except at finite number of points of X .

For ST_2 (iv) $\nRightarrow$ ST_2 (i), the counter example given on [5] will work here also.

Follows in view of the fact that ST_2 (i) and ST_2 (ii) are equivalent.

For ST_2 (iii) $\nRightarrow$ ST_2 (iv), the counter example in [5] will again work.

Now let us prove that ST_2(iv)$\Rightarrow$$ST_2$(iii)

Let x_r, y_s be two fuzzy points in X with $x_r \bar{q} y_s$ for x=y, the requirement for ST_2(ii) follows automatically in view of remark 3.1, Now suppose x$\neq$y. Then x_r and y_s will be two distinct fuzzy points in X and hence in view of ST_2 (iv),there exist supra fuzzy open sets U,V in X with $x_r \in U$,$y_s \in V$,$U \cap V = \emptyset$ but this implies that $x_r \subseteq U$,$y_s \subseteq V$ and $U \bar{q} V$, proving that (X, τ) is ST_2 (iii)

Next we show that all the four definitions 3.1 to 3.4 satisfy good extension property.

Theorem3.2: A supra topological space (X, τ) is Hausdorff iff (X, w(τ)) is ST_2(i).

Proof: Let (X, T) be Hausdorff. To show that (X, w(T)) is ST_2(i), take any two distinct points x, y ∈ X. Using that (X, T) is Hausdorff ∃ U, V ∈ T such that x ∈ U, y ∈ V and U ∩ V = ∅. Consider U, V ∈ w(T), then U(x) = 1, V(y) = 1 and U⊆ V′ which shows that (X, w(T)) is ST_2(i).

Conversely, let (X, w(T)) be ST_2(i). Then for x, y ∈ X, x ≠ y, ∃ U, V ∈ w(T) such that U(x) = 1, V(y) = 1, U⊆ V′. Consider now $U^{-1}(\frac{1}{2}, 1]$, $V^{-1}(\frac{1}{2}, 1]$which belong to T, then $x \in U^{-1}(\frac{1}{2}, 1]$, $y \in V^{-1}(\frac{1}{2}, 1]$ and $U^{-1}(\frac{1}{2}, 1] \cap V^{-1}(\frac{1}{2}, 1] = \emptyset$ for otherwise there will exist z ∈ X such that U(z) > $\frac{1}{2}$, V(z) > $\frac{1}{2}$ implying that U q V, a contradiction showing that (X, T) is Hausdorff.

Now good extension of ST_2(ii)follows in view of its equivalence of ST_2(i) and ST_2(ii) and the theorem 3.2.

Theorem 3.3: A supra topological space (X, T) is Hausdorff iff (X, w(T)) is ST_2(iii).

Proof: Let (X, T) be Hausdorff. Take any two fuzzy points x_r and y_s in X. In view of remark 3.1, it is sufficient to consider the case when x ≠ y. using Hausdorff of (X, T), ∃ U, V ∈ T such that x ∈ U, y ∈ V and U ∩ V = ∅. Consider U, V ∈ w(T). Then $x_r \subseteq U$, $y_s \subseteq V$ and U $\overline{q}$ V.

Conversely, let (X, w(T)) be Hausdorff. Let x, y ∈ X, take r =3/4, then $x_{3/4}$, $y_{3/4}$ are distinct fuzzy points in X, then ∃ U, V ∈ w(T) such that $x_{3/4} \subseteq U$, $y_{3/4} \subseteq V$ and U $\overline{q}$ V. Now consider $U^{-1}(\frac{1}{2}, 1)$ and $V^{-1}(\frac{1}{2}, 1)$ belonging to T. Then $x \in U^{-1}(\frac{1}{2}, 1)$, $y \in V^{-1}(\frac{1}{2}, 1)$ and $U^{-1}(\frac{1}{2}, 1) \cap V^{-1}(\frac{1}{2}, 1) = \emptyset$ (as in theorem 3.2) which proves that (X, T) is Hausdorff.

Theorem 3.4: A supra topological space (X, T) is Hausdorff iff (X, w(T)) is ST_2(iv).

Proof: Let (X, T) be Hausdorff. We have to show that (X, w(T)) is Hausdorff. For this let x_r, y_s be any two distinct fuzzy points in X since (X, T) is Hausdorff, ∃ U, V ∈ T such that x ∈ U, y ∈ V and U ∩ V = ∅. Consider U, V ∈ w(T). Then $x_r \in U$, $y_s \in V$ and U ∩ V = ∅ which shows that (X, w(T)) is Hausdorff.

Conversely, let (X, w(T)) be a Hausdorff fsts . Let x, y ∈ X, x ≠ y, choose r ∈ (0, 1). Consider the distinct fuzzy points x_r, y_r. Since (X, w(T)) is Hausdorff U, V ∈ w(T) such that $x_r \in U$, $y_r \in V$ and U ∩ V = ∅. Consider $U^{-1}(r, 1]$, $V^{-1}(r, 1]$ which are disjoint supra open sets in (X, T) and also $x \in U^{-1}(r, 1]$, $y \in V^{-1}(r, 1]$. Thus (X, T) is Hausdorff.

From now onwards we shall take Hausdorff in an fsts in the sense of definition 3.4.

Theorem 3.5: If (X, δ) be a Hausdorff fsts. Then (X, ι_α(δ)) is Hausdorff.

Proof: Let x, y ∈ X, x ≠ y. Now x_α, y_α are two distinct fuzzy points in X, hence due to Hausdorffness of (X, δ), ∃ fuzzy supra open sets U, V ∈ δ such that $x_\alpha \in U$, $y_\alpha \in V$ and U ∩ V = ∅. Now consider $U^{-1}(\alpha, 1]$, $V^{-1}(\alpha, 1] \in \tau_\alpha(\delta)$. We have $x \in U^{-1}(\alpha, 1]$, $y \in V^{-1}(\alpha, 1]$ and $U^{-1}(\alpha, 1] \cap V^{-1}(\alpha, 1] = \emptyset$.

Thus (X, τ_α(δ)) is Hausdorff.

Similarly it can be shown that

Remark 3.2: The converse of the above theorem 3.5 is not true in case of fuzzy topological spaces, a counter example is given in [5]. Since any fuzzy topological space is also a fuzzy supra topological space, the same counter example will work here also.

Next we prove the following result:

Theorem 3.6: A fsts (X, τ) is Hausdorff iff the diagonal set Δ_x is fuzzy S-closed in (X x X, τ x τ)

Proof: Let us assume that fsts (X, τ) is Hausdorff then to show that diagonal set Δ_x is fuzzy S-closed in (X x X, τ x τ) i.e. X x X-Δ_x is fuzzy S-open in (X x X, τ x τ), let $(x, y)_r$ be a fuzzy point in X x X-Δ_x. Then x≠y and so x_r, y_r are two distinct fuzzy points in X. Now due to Hausdorffness of (X, τ), ∃ fuzzy S-open sets U, V in τ such that $x_r \in U$, $y_r \in V$ and U ∩ V = ∅. Now consider the basic fuzzy S-open set U x V in (X x X, τ x τ). Then (x, y)r ∈ U x V ⊆ X x X-Δ_x since U x V (x, x) = U ∩ V(x) = 0. Thus X x X-Δ_x is fuzzy S-open in (X x X, τ x τ).

Conversely, let X x X-Δ_x be fuzzy S-open in (X x X, τ x τ). To show that (X, τ) is Hausdorff, take any two distinct fuzzy points x_r, y_s in X. Let $s < r$, then consider $(x, y)_r$. Since X x X-Δ_x is fuzzy S-open in (X x X, τ x τ), $\exists$ a basic fuzzy S-open set in X x X, say U x V such that $(x, y)_r \in U \times V \subseteq X \times X - \Delta_x$. Now it can be seen that $x_r \in U$, $y_r \in V$ and $U \cap V = \emptyset$ proving that (X, τ) is Hausdorff.

Theorem 3.7: If $\{(X_i, \tau_i) : i \in A\}$ be a family of fsts. Then the product fsts ($\pi i\, X_i$, $\pi i\, \tau_i$) is Hausdorff iff each coordinate fsts is Hausdorff.

The proof of this theorem is on similar lines as in case of fuzzy topological spaces as given in [8].

Theorem 3.8: Hausdorefness in an fsts is hereditary.

Proof: Let us assume that (Y, τ_y) is an fuzzy subspace of a Hausdorff fsts (X, τ). Let $x_r, y_s \in Y \subseteq X$. Since (X, τ) is ST_2, for distinct fuzzy points x_r, y_s there exist two disjoint fuzzy supra open sets sets U and V that $x_r \in U$, $y_s \in U$

Take $U_y = U \cap Y$ and $V_y = V \cap Y$

Clearly $U_y, V_y \in \tau_y$ and $x_r \in U_y$, $y_s \in V_y$

Also
$$\begin{aligned} U_y \cap V_y &= (U \cap Y) \cap (V \cap Y) \\ &= (U \cap V) \cap Y \\ &= \emptyset \cap Y \\ &= \emptyset \end{aligned}$$

Hence (Y, τ_y) is Hausdorff.

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