CONSTRUCTION OF 24 RUN DESIGNS WITHIN A SPLIT-PLOT STRUCTURE

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ABSTRACT

In industries, often the size of an experiment needs to be kept small due to time and/or cost constraints. Hadamard matrices of order 24 have been used in this paper for the construction of some new 24 run designs within a split-plot structure. The D-values of the designs for different variance ratios are also computed.

Keywords: Hadamard Matrices; Hard-to-change factors; Easy-to-change factors; Split-plot designs; D-values.

AMS Subject Classification: Primary 62K20.

1. INTRODUCTION

In industrial response surface experiments, we often come across situations when there are two types of factors: hard-to-change (HTC) factors, whose levels are difficult to change, and easy-to-change (ETC) factors, whose levels are easy-to-change. In such situations complete randomization of the design is not appropriate and so split-plot structure is used. HTC factors are randomly applied to large sections of the units, called whole-plots. These whole-plots are then split into smaller units, called sub-plots and ETC factors are randomly applied to these plots. This leads to two error terms, one for the whole-plot treatments and second for the sub-plot treatments and the interaction between whole-plot and sub-plot treatments.

In literature, great deal of attention has been given to split-plot designs. They are extensively used in agricultural and industrial experiments. The concept of split-plot design has been discussed and analyzed by Box and Jones (1992), Bisgaard and Kulahci (2001), Goos and Vandebroek (2001), Kowalski (2002), Kowalski *et al.* (2007), Parker *et al.* (2008) and Jones and Nachtsheim (2009).

When the number of factors is large the regular designs have fairly large number of runs. In some circumstances, it is desirable to reduce the number of runs as much as possible while maintaining the ability to estimate all of the terms in the model. In literature, Hadamard matrices are available also which can be used to construct the designs with comparatively less number of runs for same number of factors.

Hadamard matrices are square matrices of +1's and -1's whose columns are orthogonal. These matrices are named after the French mathematician Jacques Hadamard (1865-1963). Spence (1995) gave an algorithm for the construction of the Hadamard matrices and studied their properties. N.J.A. Sloane gave an encyclopedia of Hadamard matrices which can be obtained from the website,www.research.att.com/~njas.

Here, we have used the columns of the Hadamard matrices of order 24 to construct the designs under split-plot structure. Since there is less number of runs involved to estimate the terms in the model, these designs are more economical. We have constructed the designs having four, five and six number of factors with two HTC factors and up to four ETC factors. We have then obtained D-values of the designs constructed.

The organization of this paper is as follows. In Section 2, we give the model and design selection criteria. We describe the method of the construction of the designs in Section 3.We illustrate the method with the help of examples. In Section 4, we conclude this paper with some discussion. The D-values and column combinations for some of the designs with two HTC and up to four ETC factors for different variance ratios are given in the Annexure. The complete list of D-values and all column combinations is available with Authors.

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2. MODEL AND DESIGN SELECTION CRITERIA

Usually, in industrial experiments the goal of an experimenter is to estimate all the main effects and also all two factor interactions: among HTC factors, between HTC and ETC factors and among ETC factors, while keeping the number of runs as small as possible. So the purpose of the designs constructed here is to achieve all goals.

Consider a split-plot design involving n HTC factors denoted by z_i ($i = 1, 2, \ldots, n$) and m ETC factors denoted by x_j ($j = 1, 2, \ldots, m$). The Model, considered here, for the uth ($u = 1, 2, \ldots, N$) run is

$$y_u = \mu + \sum_{i=1}^{n} a_i z_{iu} + \sum_{j=1}^{m} b_j x_{ju} + \sum_{\substack{i \neq i'=1 \\ i < i'}}^{n} a_{ii'} z_{iu} z_{i'u} + \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij} z_{iu} x_{ju} + \sum_{\substack{j \neq j'=1 \\ j < j'}}^{m} b_{jj'} x_{ju} x_{j'u} + \delta_u + \varepsilon_u \qquad (1)$$

where, y_u is the response, μ is the general mean effect, a's, b's and c's are regression coefficients, δ_u is whole-plots error and ε_u is sub-plots error with σ_δ^2 as whole-plots error variance and σ_ε^2 as sub-plots error variance. Here, Var(y_u) = $\sigma_\delta^2 + \sigma_\varepsilon^2$, for all u.

$$\text{Cov}(y_u, y_{u'}) = \begin{cases} \sigma_\delta^2, & \text{for } u \neq u' \\ 0, & \text{otherwise} \end{cases}$$

Various optimality criteria for the study of split-plot designs are available, of which D-optimality criterion is most frequently used. In this paper, we have obtained the D-values of the designs constructed. The formula for D-value is

$$D = \frac{1}{N}(X'V^{-1}X)^{1/p} \qquad (2)$$

where, X is the model matrix, V is the variance-covariance matrix, which is the block diagonal matrix with blocks of $\sigma_\varepsilon^2 I + \sigma_\delta^2 J = \sigma_\varepsilon^2 (I + \eta J)$ where η is variance ratio ($= \sigma_\delta^2/\sigma_\varepsilon^2$), I is the identity matrix and J is the matrix of ones, N is the total number of runs in the design and p is the number of parameters involved in the model.

3. CONSTRUCTION OF THE DESIGNS

In this paper, we have constructed designs based on the model discussed in Section 2. We have considered three cases that use combinations of two HTC factors and two, three and four ETC factors. The procedure for constructing the designs having two HTC and m ETC factors ($m = 2, 3$ and 4) is as follows:

First, we take H.24-1Hadamard matrix from the list of sixty Hadamard matrices given by N.J.A. Sloane. Next, we take first combination of ($2 + m$) columns out of $^{23}C_{2+m}$ column combinations. We have not considered the first column of all ones in constructing the design. In the ($2 + m$) columns combination selected, the first two columns correspond to HTC factors and the next m columns correspond to the ETC factors. The allocation of HTC and ETC factors to the ($2 + m$) columns of the Hadamard matrices can also be interchanged. This forms the design for two HTC and m ETC factors. The design is then sorted in ascending (or descending) order, first on the basis of first HTC factor z_1 and then on the basis of second HTC factor z_2. This forms four whole-plots with six sub-plots in each whole-plot. We now generate the model matrix X as per equation (1). The D-values, using equation (2), for different values of the variance ratio(η) are then computed. The above procedure is repeated for all $^{23}C_{2+m}$ column combinations of the H.24-1Hadamard matrix. The design having the highest D-value is selected.

The above procedure is used for constructing the designs with the remaining fifty-nine Hadamard matrices of order 24.

The above procedure, for two HTC and two ETC, two HTC and three ETC and two HTC and four ETC factors, is explained with the help of the following examples. In all the examples we have taken same Hadamard matrix.

Example 3.1: Consider the H.24-1Hadamard matrix. Let us consider the column combination with column numbers: {1, 2, 4, 6}. The HTC factors z_1 and z_2 are allocated to column 1 and 2 and the ETC factors x_1 and x_2 are assigned to column 4 and 6 respectively. The design is given in Table1.

Table-1: Design with two HTC and two ETC factors

z_1	z_2	x_1	x_2
1	1	1	1
1	1	1	1
1	1	1	-1
1	-1	-1	1
1	1	-1	1
1	1	-1	-1
1	-1	1	1
1	1	-1	-1
1	-1	-1	-1
1	-1	1	-1
1	-1	-1	1
1	-1	1	-1
-1	1	1	1
-1	1	1	1
-1	1	1	-1
-1	-1	-1	1
-1	1	-1	1
-1	1	-1	-1
-1	-1	1	1
-1	1	-1	-1
-1	-1	-1	-1
-1	-1	1	-1
-1	-1	-1	1
-1	-1	1	-1

This design is first sorted on the basis of HTC factor z_1 and then on the basis of HTC factor z_2 generating four whole-plots of size six each. This is shown in Table 2.

Table-2: Design under Split-plot Structure with two HTC and two ETC factors

z_1	z_2	x_1	x_2
-1	-1	-1	1
-1	-1	1	1
-1	-1	-1	-1
-1	-1	1	-1
-1	-1	-1	1
-1	-1	1	-1
-1	1	1	1
-1	1	1	1
-1	1	1	-1
-1	1	-1	1
-1	1	-1	-1
-1	1	-1	-1
1	-1	-1	1
1	-1	1	1
1	-1	-1	-1
1	-1	1	-1
1	-1	-1	1
1	-1	1	-1

$$
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 \\
1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1
\end{array}
$$

Using the above design, model matrix X is generated according to equation (1), given in Table 3. Here *k* is column of all ones representing general mean term.

Table-3: Model Matrix X under Split-plot Structure with two HTC and two ETC factors

k	z_1	z_2	x_1	x_2	z_1z_2	z_1x_1	z_1x_2	z_2x_1	z_2x_2	x_1x_2
1	-1	-1	-1	1	1	1	-1	1	-1	-1
1	-1	-1	1	1	1	-1	-1	-1	-1	1
1	-1	-1	-1	-1	1	1	1	1	1	1
1	-1	-1	1	-1	1	-1	1	-1	1	-1
1	-1	-1	-1	1	1	1	-1	1	-1	-1
1	-1	-1	1	-1	1	-1	1	-1	1	-1
1	-1	1	1	1	-1	-1	-1	1	1	1
1	-1	1	1	1	-1	-1	-1	1	1	1
1	-1	1	1	-1	-1	-1	1	1	-1	-1
1	-1	1	-1	1	-1	1	-1	-1	1	-1
1	-1	1	-1	-1	-1	1	1	-1	-1	1
1	-1	1	-1	-1	-1	1	1	-1	-1	1
1	1	-1	-1	1	-1	-1	1	1	-1	-1
1	1	-1	1	1	-1	1	1	-1	-1	1
1	1	-1	-1	-1	-1	-1	-1	1	1	1
1	1	-1	1	-1	-1	1	-1	-1	1	-1
1	1	-1	-1	1	-1	-1	1	1	-1	-1
1	1	-1	1	-1	-1	1	-1	-1	1	-1
1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	-1	1	1	-1	1	-1	-1
1	1	1	-1	1	1	-1	1	-1	1	-1
1	1	1	-1	-1	1	-1	-1	-1	-1	1
1	1	1	-1	-1	1	-1	-1	-1	-1	1

The D-values of the design should be in the following form:

Variance Ratio (η):	0.2	0.4	0.6	0.8	1.0
D-values :	0.7270	0.6206	0.5560	0.5110	0.4772

The above column combination gives the highest D-values of the design. There are other column combinations with the same D-values.

Using the same Hadamard matrix H. 24-1, we now obtain the D-values of the design with two HTC and three ETC factors for different variance ratios.

Example 3.2: Consider the column combination (1, 2, 4, 6, 9) of H.24-1Hadamard matrix for two HTC and three ETC factors. The design obtained is given in Table 4.

Table-4: Design with two HTC and three ETC factors

z_1	z_2	x_1	x_2	x_3
1	1	1	1	1
1	1	1	1	1
1	1	1	-1	-1
1	-1	-1	1	1
1	1	-1	1	-1
1	1	-1	-1	-1
1	-1	1	1	-1
1	1	-1	-1	1
1	-1	-1	-1	1
1	-1	1	-1	1
1	-1	-1	1	-1
1	-1	1	-1	-1
-1	1	1	1	-1
-1	1	1	1	-1
-1	1	1	-1	1
-1	-1	-1	1	-1
-1	1	-1	1	1
-1	1	-1	-1	1
-1	-1	1	1	1
-1	1	-1	-1	-1
-1	-1	-1	-1	-1
-1	-1	1	-1	-1
-1	-1	-1	1	1
-1	-1	1	-1	1

Sorting the design first on the basis of HTC factor z_1 and then on the basis of HTC factor z_2 in ascending (or descending) order generates split-plot design with four whole-plots each of size six. The model matrix X is generated as per equation (1).

We then obtained D-values of the design for different values of variance ratio (η) using equation (2) and are given below:

Variance Ratio (η):	0.2	0.4	0.6	0.8	1.0
D-values :	0.7408	0.6644	0.6160	0.5814	0.5547

This column combination gives the highest D-values of the design. Besides the above given column combination, there are other column combinations with the same D-values.

Similarly, we use the Hadamard matrix H. 24-1 to obtain the D-values of the design for two HTC and four ETC factors for different variance ratios.

Example 3.3: Consider the column combination (1, 2, 4, 6, 9, 12) of H.24-1 Hadamard matrix for two HTC and four ETC factors. The design obtained is given in Table 5.

Table-5: Design with two HTC and four ETC factors

z_1	z_2	x_1	x_2	x_3	x_4
1	1	1	1	1	1
1	1	1	1	1	-1
1	1	1	-1	-1	1
1	-1	-1	1	1	1
1	1	-1	1	-1	1
1	1	-1	-1	-1	-1
1	-1	1	1	-1	-1
1	1	-1	-1	1	-1
1	-1	-1	-1	1	1
1	-1	1	-1	1	-1
1	-1	-1	1	-1	-1
1	-1	1	-1	-1	1
-1	1	1	1	-1	1
-1	1	1	1	-1	-1
-1	1	1	-1	1	1
-1	-1	-1	1	-1	1
-1	1	-1	1	1	1
-1	1	-1	-1	1	-1
-1	-1	1	1	1	-1
-1	1	-1	-1	-1	-1
-1	-1	-1	-1	-1	1
-1	-1	1	-1	-1	-1
-1	-1	-1	1	1	-1
-1	-1	1	-1	1	1

The split-plot design with four whole-plots of size six each is obtained by sorting the design first on the basis of HTC factor z_1 and then on the basis of HTC factor z_2 in ascending (or descending) order. The model matrix X is then generated as per equation (1).

The D-values of the design for different values of variance ratio (η) are then obtained and are:

Variance Ratio (η):	0.2	0.4	0.6	0.8	1.0
D-values	: 0.6623	0.6119	0.5791	0.5552	0.5366

This column combination gives the highest D-values of the design. Besides the above given column combination, there are other column combinations with the same D-values.

Same procedure is applied for constructing the designs with remaining fifty-nine Hadamard matrices. The highest D-values and the column combinations for some of the designs with two HTC and m (=2, 3, 4) ETC factors for different variance ratios (η) for different Hadamard matrices are given in Annexure.

4. CONCLUSIONS

In most situations, when the number of factors is large, due to time and/or cost constraints, it is required to use a design with less number of runs to estimate the parameters in the model. Here we have considered two HTC factors and up to four ETC factors. We have used Hadamard matrices of order 24 to construct the designs under split-plot structure.

Even though for four and five factors (two HTC and two ETC and two HTC and three ETC respectively), less number of runs are required to estimate the parameters in the given model (1) but here we have constructed the designs having better D-values using Hadamard matrices of order 24. Sometimes the experimenter would prefer a design with better D-values. For six factors (two HTC and four ETC), these designs are more economical since less number of runs is involved. We have also computed the D-values of these designs for different variance ratios (η). Some of the designs having highest D-values for different cases are given in the Annexure.

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ANNEXURE

Table-6: Highest D-values and column combinations for different combinations of HTC and ETC Factors (H.24-2 Hadamard Matrix)

Var. ratio (η)	2 HTC & 2 ETC {1, 2, 4, 10}	2 HTC & 3 ETC {2, 4, 7, 8, 14}	2 HTC & 4 ETC {1, 2, 4, 6, 11, 15}
	D-value	D-value	D-value
0.2	0.7270	0.7603	0.6650
0.4	0.6206	0.6819	0.6144
0.6	0.5560	0.6323	0.5815
0.8	0.5110	0.5967	0.5575
1.0	0.4772	0.5693	0.5388

Table-7: Highest D-values and column combinations for different combinations of HTC and ETC Factors (H.24-5 Hadamard Matrix)

Var. ratio (η)	2 HTC & 2 ETC {1, 2, 5, 7}	2 HTC & 3 ETC {2, 4, 7, 8, 13}	2 HTC & 4 ETC {1, 2, 4, 7, 8, 15}
	D-value	D-value	D-value
0.2	0.7270	0.7603	0.6650
0.4	0.6206	0.6819	0.6144
0.6	0.5560	0.6323	0.5815
0.8	0.5110	0.5967	0.5575
1.0	0.4772	0.5693	0.5388

Table-8: Highest D-values and column combinations for different combinations of HTC and ETC Factors (H.24-20 HadamardMatrix)

Var. ratio (η)	2 HTC & 2 ETC {1, 2, 5, 7}	2 HTC & 3 ETC {1, 2, 6, 12, 17}	2 HTC & 4 ETC {1, 2, 5, 6, 13, 16}
	D-value	D-value	D-value
0.2	0.7270	0.7710	0.6867
0.4	0.6206	0.6915	0.6345
0.6	0.5560	0.6412	0.6006
0.8	0.5110	0.6051	0.5758
1.0	0.4772	0.5773	0.5564

Table-9: Highest D-values and column combinations for different combinations of HTC and ETC Factors
(H.24-44 Hadamard Matrix)

Var. ratio (η)	2 HTC & 2 ETC {1, 2, 3, 15}	2 HTC & 3 ETC {1, 2, 3, 18, 23}	2 HTC & 4 ETC {1, 2, 3, 4, 16, 20}
	D-value	D-value	D-value
0.2	0.7270	0.7710	0.6867
0.4	0.6206	0.6915	0.6345
0.6	0.5560	0.6412	0.6006
0.8	0.5110	0.6051	0.5758
1.0	0.4772	0.5773	0.5564

Table-10: Highest D-values and column combinations for different combinations of HTC and ETC Factors
(H.24-60 HadamardMatrix)

Var. ratio (η)	2 HTC & 2 ETC {1, 2, 3, 4}	2 HTC & 3 ETC {1, 2, 3, 5, 22}	2 HTC & 4 ETC {1, 2, 3, 4, 6, 10}
	D-value	D-value	D-value
0.2	0.7270	0.7603	0.6579
0.4	0.6206	0.6819	0.6078
0.6	0.5560	0.6323	0.5753
0.8	0.5110	0.5967	0.5516
1.0	0.4772	0.5693	0.5330

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