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INTEGER SOLUTIONS OF THE EIGHTH DEGREE EQUATION WITH SEVEN VARIABLES $(x^2-y^2)(4x^2+4y^2-6xy)=2(w+P)(T+S)z^6$

S. VIDHYALAKSHMI[1], A. KAVITHA[2], M. A. GOPALAN[*3]

[1,2,3]Department of Mathematics,
Shrimati Indira Gandhi College, Trichy-620002, India.

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ABSTRACT

The non-homogeneous octic equation with five unknowns represented by the Diophantine equation $(x^2-y^2)(4x^2+4y^2-6xy)=2(w+P)(T+S)z^6$ is analyzed for its patterns of non-zero distinct integral solutions and four patterns of integral solutions are illustrated. Various interesting relations between the solutions and special numbers, namely, Pyramidal numbers, Pronic numbers, polygonal numbers, fourth, fifth and sixth dimensional figurate numbers are exhibited.

Keywords: *Octic non-homogeneous equation, Pyramidal numbers, Pronic numbers, Fourth, fifth and sixth dimensional figurate numbers.*

M. Sc 2000 Mathematics subject classification: *11D41.*

NOTATIONS

$t_{m,n}$: Polygonal number of rank *n* with size *m*
SO_n: Stella octangular number of rank *n*
Pr_n : Pronic number of rank *n*
G_n: Gnomonic number of rank *n*
CP_n^m :Centered Pyramidal number of rank n
j_n : Jacobsthal-Lucas number of rank n
$F_{4,n,6}$: Four dimensional hexagonal figurate number of rank n

INTRODUCTION

The theory of diophantine equations offers a rich variety of fascinating problems. In particular, homogeneous and non-homogeneous equations of higher degree have aroused the interest of numerous Mathematicians since antiquity [1, 2, 7, 9]. In [3-6, 8] heptic equations with three, four and five unknowns are analyzed. This communication analyses a non homogeneous octic equation with five unknowns given by $(x-y)(x^3+y^3)=4(w^2-p^2)T^6$ for determining its infinitely many non-zero integer quintuples (*x*, *y*, *w*, p, *T*) satisfying the above equation are obtained. Various interesting properties among the values of *x*, *y*, *p*, *w* and *T* are presented

1. METHOD OF ANALYSIS

The non-homogeneous octic equation with seven variables to be solved for its distinct non-zero integral solution is

$$(x^2-y^2)(4x^2+4y^2-6xy)=2(w+P)(T+S)z^6 \tag{1}$$

Introduction of the linear transformations

$$x=u+v,\quad y=u-v,\quad w=u+1,\quad P=u-1,\quad T=v+1,\quad S=v-1 \tag{2}$$

In (1) leads to

$$u^2+7v^2=z^6 \tag{3}$$

*Corresponding Author: M. A. Gopalan[*3], [1,2,3]Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002, India.*

Different methods of obtaining the patterns of integer solutions to (1) are illustrated below:

Pattern: 1
Let

$$z = (a^2 + 7b^2) \tag{4}$$

Using (4) in (3) and applying the method of factorization, define

$$u + i\sqrt{7}v = \alpha + i\sqrt{7}\beta \tag{5}$$

where $\left(\alpha + i\sqrt{7}\beta\right) = (a + i\sqrt{7}b)^6$

from which we have

$$\left.\begin{aligned}\alpha &= a^6 - 105a^4b^2 + 735a^2b^40 - 343b^6\\ \beta &= 6a^5b - 140a^3b^3 + 294ab^5\end{aligned}\right\} \tag{6}$$

Equating real and imaginary parts in (5), we get

$$\left.\begin{aligned}u &= \alpha\\ v &= \beta\end{aligned}\right\} \tag{7}$$

Using (7) and (2) the values of x, y, z, w, p, T and s are given by

$$\left.\begin{aligned}x &= \alpha + \beta\\ y &= \alpha - \beta\\ w &= \alpha + 1\\ P &= \alpha - 1\\ T &= \beta + 1\\ S &= \beta - 1\end{aligned}\right\} \tag{8}$$

Thus, (4) and (8) represented the non-zero distinct integral solutions to (1)

Properties:
(i) $x(n,1) + y(n,1) + 686 = 2t_{4,n}[6F_{4,n,6} - 2CP_n^9 - Pr_n - 106t_{4,n} + 735]$
$(ii) w(n,1) + P(n,1) + 686 = 2t_{4,n}[12F_{4,n,4} + 4CP_n^6 - 111t_{4,n} - Gn_n + 734]$
$(iii) T(n,1) + S(n,1) = 2[-6t_{4,n}(CP_n^6) + 140CP_n^6 + 583t_{3,n} - 294t_{4,n}]$

Pattern: 2
Consider (3) as

$$u^2 + 7v^2 = z^6 \ *1 \tag{9}$$

Write 1 as

$$1 = \frac{(3+i\sqrt{7})(3-i\sqrt{7})}{16} \tag{10}$$

Substituting (4) and (10) in (9) and employing the factorization method, define

$$\left(u + i\sqrt{7}v\right) = \frac{1}{4}(3 + i\sqrt{7})(a + i\sqrt{7}b)^6$$

$$\left(u + i\sqrt{7}v\right) = \frac{1}{4}(3 + i\sqrt{7})(\alpha + i\sqrt{7}\beta)$$

Equating real and imaginary parts, we have

$$\left.\begin{aligned}u &= \frac{1}{4}(3\alpha - 7\beta)\\ &\frac{1}{4}(\alpha + 3\beta)\end{aligned}\right\} \tag{11}$$

As our interest is on finding integer solutions, we choose a and b suitably so that u and v are integers. Replace a by 4a and b by 4b in (6). Substituting the corresponding values of α and β in (11) and employing (2) non-zero integral solutions to (1) are found to be

$$\left.\begin{aligned}x(a,b) &= 4^5(4\alpha - 4\beta)\\ y(a,b) &= 4^5(2\alpha - 10\beta)\\ w(a,b) &= 4^5(3\alpha - 7\beta) + 1\\ P(a,b) &= 4^5(3\alpha - 7\beta) - 1\\ T(a,b) &= 4^5(\alpha + 3\beta) + 1\\ S(a,b) &= 4^5(\alpha + 3\beta) - 1\\ z &= 4^2(a^2 + 7b^2)\end{aligned}\right\} \tag{12}$$

Thus, (12) represent the non-zero distinct integer solutions to (1)

Properties:
(i) $x(n,1) - 2y(n,1) = 4^7[-CP_n^6(6t_{4,n} - 140) + 588t_{3,n} - 294t_{4,n}]$
$(ii)3S(n,1) - 4w(n,1) + 4 = 4^7[-6CP_n^6(t_{4,n}) - 14(3CP_n^{16} + SO_n - 30t_{3,n} + 15t_{4,n})]$
(iii)$z(n,1) - 96 = 16j_{2n}$

Pattern: 3
In addition to (10), 1 can be written as

$$1 = \frac{(1+i3\sqrt{7})(1-i3\sqrt{7})}{64} \tag{13}$$

Following the procedure as presented in pattern.2, the corresponding non-zero integral solutions to (1) are given by
$x(a,b) = 8^5(4\alpha - 20\beta)$
$y(a,b) = 8^5(-2\alpha - 22\beta)$
$w(a,b) = 8^5(\alpha - 21\beta) + 1$
$P(a,b) = 8^5(\alpha - 21\beta) - 1$
$T(a,b) = 8^5(3\alpha + \beta) + 1$
$S(a,b) = 8^5(3\alpha + \beta) - 1$
$z(a,b) = 8^2(a^2 + 7b^2)$

Pattern.4
(3) can be written as

$$u^2 + 7v^2 = (z^3)^2 \tag{14}$$

Then (14) is satisfied by

$$\left.\begin{aligned} v &= 2rs \\ u &= 7r^2 - s^2 \end{aligned}\right\} \tag{15}$$

and

$$z^3 = 7r^2 + s^2 \tag{16}$$

To find z:

Substituting (4) in (16) and employing the method of factorization, define

$$s + i\sqrt{7}r = (a^3 - 21ab^3) + i\sqrt{7}(3a^2b - 7b^3) \tag{17}$$

Equating real and imaginary parts, we have

$$\left.\begin{aligned} s &= a^3 - 21ab^2 \\ r &= 3a^2b - 7b^3 \end{aligned}\right\} \tag{18}$$

Using (18), (15) and (2) we have

$$\left.\begin{aligned} u &= 7(3a^2b - 7b^3)^2 - (a^3 - 21ab^2)^2 \\ v &= 2(3a^2b - 7b^3)(a^3 - 21ab2) \end{aligned}\right\} \tag{19}$$

Substituting (19) in (2) the corresponding non-zero integer solutions to (1) are given by
$x(a,b) = 7(3a^2b - 7b^3)^2 - (a^3 - 21ab^2)^2 + 2(3a^2b - 7b^3)(a^3 - 21ab^2)$
$y(a,b) = 7(3a^2b - 7b^3)^2 - (a^3 - 21ab^2)^2 - 2(3a^2b - 7b^3)(a^3 - 21ab^2)$
$w(a,b) = 7(3a^2b - 7b^3)^2 - (a^3 - 21ab^2)^2 + 1$
$P(a,b) = 7(3a^2b - 7b^3)^2 - (a^3 - 21ab^2)^2 - 1$
$T(a,b) = 2(3a^2b - 7b^3)(a^3 - 21ab^2) + 1$
$S(a,b) = 2(3a^2b - 7b^3)(a^3 - 21ab^2) - 1$

CONCLUSION

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the non-homogeneous octic equation with five unknowns. As the octic equations are rich in variety, one may search for other forms of octic equation with variables greater than or equal to five and obtain their corresponding properties.

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