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A NOTE ON FUNCTIONS DEFINED WITH RELATED TO (j,k)-SYMMETRIC POINTS

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ABSTRACT.

In this paper using the concept of (j,k)-symmetrical functions which generalizes the notion of even, odd and k-symmetric functions. We consider the quotient of analytic represe- ntation of starlikeness and convexity. We present the criteria that embed a normalized analytic function in the class of functions that are starlike with respect to (j,k)-symmetric points.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$\mathcal{U} = \{z : z \in \mathbb{C} \; and \; |z| < 1\}$, and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$.

We define a function $f \in \mathcal{A}$ to be strongly starlike order α, $(0 < \alpha \leq 1)$ if

$$\left| arg \frac{zf'(z)}{f(z)} \right| < \frac{\alpha\pi}{2}, \quad z \in \mathcal{U}.$$

The family of all strongly starlike functions is denoted $\widetilde{S^*}(\alpha)$.

We now introduce the concept of (j,k)-symmetrical functions which generalizes the concept of even, odd and k-symmetric functions. Consider,

$$e^{-i\alpha} f(e^{i\alpha} z) = z + \sum_{n=2}^{\infty} a_n e^{i(n-1)\alpha} z^n, \; \alpha \in R. \tag{2}$$

and

$$[f(z^k)]^{\frac{1}{k}} = z + \frac{a_2}{k} z^{k+1} + \frac{1}{2k^2}[2ka_3 - (k-1)a_2^2] z^{2k-1} + \ldots \tag{3}$$

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where k is a positive integer. The transformation (2) is a rotation of f because it rotates the unit disc in the z -plane through an angle α and rotates the image domain in the w -plane in the reverse direction through an angle of the same magnitude. In (3) the transformation $u = z^k$ maps $\mathbb{U}$ onto k copies of $\mathbb{U}$ and $f(z)$ carries this surface onto k copies of $f(\mathbb{U})$ joined by a suitable branch point at $w = 0$. It is intuitively clear that the k^{th} root merely unwinds the symmetry. Precisely, we have

Definition 1.1: *Let k be a positive integer. A domain D is said to be k -fold symmetric if a rotation of D about the origin through an angle $\frac{2\pi}{k}$ carries D onto itself. A function f is said to be k -fold symmetric in $\mathbb{U}$ if for every z in $\mathbb{U}$*

$$f(e^{\frac{2\pi i}{k}} z) = e^{\frac{2\pi i}{k}} f(z).$$

The family of all k -fold symmetric function is denoted by $\mathbb{S}^k$ and for $k = 2$ we get the odd univalent function.

The notion of (j,k) -symmetrical functions $(k = 2,3,...;\ j = 0,1,2,...,k-1)$ is a generalization of the notion of even, odd, k -symmetrical functions and generalize the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function.

The theory of (j,k) -symmetrical functions has many interesting applications, for instance in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan uniqueness theorem for holomorphic mappings [9].

Definition 1.2: *Let $\varepsilon = (e^{\frac{2\pi i}{k}})$ and $j = 0,1,2,..,k-1$ where $k \geq 2$ is a natural number. A function $f : \mathbb{U} \mapsto \mathbb{C}$ is called (j,k) -symmetrical if*

$$f(\varepsilon z) = \varepsilon^j f(z), \quad z \in \mathbb{U}.$$

The family of all (j,k) -symmetrical functions is denoted be $\mathbb{S}^{(j,k)}$. $\mathbb{S}^{(0,2)}$, $\mathbb{S}^{(1,2)}$ and $\mathbb{S}^{(1,k)}$ are respectively the classes of even, odd and k -symmetric functions. We have the following decomposition theorem.

Theorem 1.3: *[9] For every mapping $f : \mathbb{U} \mapsto \mathbb{C}$, there exists exactly the sequence of (j,k) -symmetrical functions $f_{j,k}$,*

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z)$$

where

$$f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} f(\varepsilon^v z) \tag{4}$$

.

$$(f \in \mathbb{A};\ k = 1,2,...;\ j = 0,1,2,...,k-1)$$

Definition 1.4: *A function f in $\mathbb{A}$ is said to belong to the class $\mathbb{S}^*_{(j,k)}$* if

$$\Re\left\{\frac{zf'(z)}{f_{j,k}(z)}\right\} > 0, \quad z \in \mathbb{U}.$$

where $f_{j,k}(z)$ is defined by (4)

Definition 1.5: *A function f in $\mathbb{A}$ is said to belong to the class $\mathbb{K}_{(j,k)}$*, if

$$\Re\left\{\frac{(zf'(z))'}{f'_{j,k}(z)}\right\} > 0, \quad z \in \mathbb{U}.$$ where $f_{j,k}(z)$ is defined by (4).

In our work we study the classes $\mathcal{S}^*_{(j,k)}$ and $\mathcal{K}_{(j,k)}$ and obtain conditions for starlikeness with respect to (j,k)-symmetric points, in terms of the operator.

$$I(f,j,k,a,b;z)=\frac{a+bzf''(z)/f'(z)}{zf'_{j,k}(z)/f_{j,k}(z)}.$$

For special choices $a=b=1$ we get

$$I(f,j,k,1,1;z)=\frac{1+zf''(z)/f'(z)}{zf'_{j,k}(z)/f_{j,k}(z)}=\frac{[zf'(z)]'/f'_{j,k}(z)}{zf'(z)/f_{j,k}(z)}.$$

which is the quotient of analytic representations of starlikeness and convexity with respect to (j,k) -symmetric points.

For special choices of the parameters we get classes studied by Tuneski, Irmak, Singh, Obradovic, Sakaguchi and Silverman.

2. MAIN RESULTS

The following lemma is due to Miller and Mocanu [2]

Lemma 2.1: *[2] Let* Ω *be a subset of the complex plan* $\mathbb{C}$ *and let the function* $\psi:\mathbb{C}^2\times\mathbb{U}\to\mathbb{C}$ *satisfy* $\psi(ix,y;z)\notin\Omega$ *for all real* $x,y\le-\frac{1+x^2}{2}$ *and for all* $z\in\mathbb{U}$*, if the function* $p(z)$ *is analytic in* $\mathbb{U}$*,* $p(0)=1$ *and* $\psi(p(z),zp'(z);z)\in\Omega$ *for all* $z\in\mathbb{U}$ *then* $\Re\{p(z)\}>0$ *,* $z\in\mathbb{U}$.

Lemma 2.2: *Let* $f\in\mathcal{A}, k\ge 2$ *is a natural number,* $j=0,1,2,..,k-1,$ *and* $a,b\in\mathbb{R}$*. Also, let* $\Omega=\mathbb{C}\setminus\Omega_1$*, where*

$$\Omega_1=\left\{b+\frac{f_{j,k}(z)}{zf'_{j,k}(z)}(a-b+bui):\ z\in\mathbb{U},u\in\mathbb{R},|u|\ge 1\right\}$$

If

$$I(f,j,k,a,b;z)\in\Omega,\ z\in\mathbb{U}\ \text{then}\ f\in\mathcal{S}^*_{(j,k)}.$$

Proof: Let $p(z)=\frac{zf'(z)}{f_{j,k}(z)}$ then $p(z)$ is analytic in $\mathbb{U}$ and $p(0)=1$. Further, for the the function

$$\psi(r,s;z)=b+\frac{f_{j,k}(z)}{zf'_{j,k}(z)}(a-b+b\frac{s}{r})\ \text{then}$$

$$\psi(p(z),zp'(z);z)=b+\frac{f_{j,k}(z)}{zf'_{j,k}(z)}\left(a-b+b\frac{zp'(z)}{p(z)}\right)=I(f,j,k,a,b;z).$$

So, by Lemma 2.1 for proving $f\in\mathcal{S}^*_{(j,k)}$ or equivalently $\Re p(z)>0$, $z\in\mathbb{U}$, it is enough to show that $\psi(ix,y;z)\in\Omega_1$ for all real $x,y\le-\frac{1+x^2}{2}$ and $z\in\mathbb{U}$. For

$$\psi(ix,y;z)=b+\frac{f_{j,k}(z)}{zf'_{j,k}(z)}\left(a-b+b\frac{y}{x}i\right)\in\Omega_1$$

since $|y/x|\ge 1$.

This Lemma leads to following criteria for starlikeness with respect to (j,k) -symmetric points.

Theorem 2.3: *Let* $f \in \mathcal{A}, k \geq 2$ *is a natural number ,* $j = 0,1,2,..,k-1,$ *and* $a,b \in \mathbb{R}$.

(i) If $f_{j,k} \in \widetilde{\mathcal{S}^*}(\alpha)\,(0<\alpha<1)$, $\dfrac{|b|}{a-b} > tan\left(\dfrac{\alpha\pi}{2}\right)$ when $a-b>0$ and if

$$|arg[I(f,j,k,a,b;z)-b]| < \lambda_1 \equiv \begin{cases} \arctan\left(\dfrac{|b|}{a-b}\right) - \alpha\dfrac{\pi}{2}, & a-b>0 \\ (1-\alpha)\dfrac{\pi}{2}, & a-b \leq 0. \end{cases}$$

for all $z \in \mathcal{U}$ then $f \in \mathcal{S}^*_{(j,k)}$.

(ii) If $\left|\dfrac{zf'_{j,k}(z)}{f_{j,k}(z)}\right| > \dfrac{1}{\mu}\,(\mu>1)$ and $|I(f,j,k,a,b;z)-b| < \lambda_2 \equiv \mu\sqrt{(a-b)^2+b^2}$ for all $z \in \mathcal{U}$ then $f \in \mathcal{S}^*_{(j,k)}$.

Proof: Let $\Sigma_1 = \{w : |arg(w-b)| < \lambda_1\}$ and $\Sigma_2 = \{w : |w-b| < \lambda_2\}$ be a subsets defined in the complex plan $\mathbb{C}$. By Lemma 2.2, for proving (i) and (ii) it is enough to show that $\Sigma_1 \subseteq \Omega$ and $\Sigma_2 \subseteq \Omega,$ respectively.

(i) we will show that $\Sigma_1 \subseteq \Omega$ by verifying $\Sigma_1 \bigcap \Omega_1 = \emptyset.$ If $w \in \Omega_1$ then for some $z \in \mathcal{U}$, $|u| \geq 1$ and $u \in \mathbb{R}$ we have

$$|arg(w-b)| = \left|arg\frac{f_{j,k}(z)}{zf'_{j,k}(z)} + arg(a-b+bui)\right|$$

$$\geq \left|\left|arg\frac{f_{j,k}(z)}{zf'_{j,k}(z)}\right| - |arg(a-b+bui)|\right|. \tag{5}$$

For $a-b>0$ we have

$$\arctan\frac{|b|}{a-b} < |arg(a-b+bui)| < \frac{\pi}{2}$$

and

$$|arg(w-b)| \geq \arctan\frac{|b|}{a-b} - \alpha\frac{\pi}{2} = \lambda_1$$

i.e. $w \notin \Sigma_1$. In the case $a-b \leq 0$ we have $w \notin \Sigma_1$ because

$$|arg(w-b)| \geq \left|\alpha\frac{\pi}{2} - \left(\frac{\pi}{2} + \arctan\frac{|a-b|}{b}\right)\right| \geq (1-\alpha)\frac{\pi}{2} = \lambda_1.$$

If $w \in \Omega_1$ then $w \notin \Sigma_2$ because of

$$|w-b| = \left|\frac{f_{j,k}(z)}{zf'_{j,k}(z)}.(a-b+bui)\right|$$

$$= \left|\frac{f_{j,k}(z)}{zf'_{j,k}(z)}\right|.\sqrt{(a-b)^2+b^2u^2} \geq \mu\sqrt{(a-b)^2+b^2} = \lambda_2.$$

Analogous to (i) we can prove (ii) .

For special choice of a and b we get the following Corollaries.

Corollary 2.4: *Let* $f \in \mathcal{A}, k \geq 2$ *is a natural number,* $j = 0,1,2,..,k-1,$.

(i) If $f_{j,k} \in \widetilde{\mathcal{S}}^*(\alpha)$ for some $0 < \alpha < 1$ and ,

$$\left| arg\left[\frac{[zf'(z)]'/f'_{j,k}(z)}{zf'(z)/f_{j,k}(z)} - 1\right]\right| < (1-\alpha)\frac{\pi}{2} \text{ for all } z \in \mathcal{U} \text{ then } f \in \mathcal{S}^*_{(j,k)}.$$

(ii) If $\left|\frac{zf'_{j,k}(z)}{f_{j,k}(z)}\right| > \frac{1}{\mu}$ for some $\mu > 1$ and $\left|\frac{[zf'(z)]'/f'_{j,k}(z)}{zf'(z)/f_{j,k}(z)} - 1\right| < \mu$ for all $z \in \mathcal{U}$ then $f \in \mathcal{S}^*_{(j,k)}$.

Corollary 2.5: *Let* $f \in \mathcal{A}, k \geq 2$ *is a natural number*, $j = 0,1,2,..,k-1,$.

(i) If $f_{j,k} \in \widetilde{\mathcal{S}}^*(\alpha)$ for some $0 < \alpha < 1$ and

$$\left| arg\left[\frac{f_{j,k}(z)f''(z)}{f'_{j,k}(z)f'(z)} - 1\right]\right| < (1-\alpha)\frac{\pi}{2} \text{ for all } z \in \mathcal{U} \text{ then } f \in \mathcal{S}^*_{(j,k)}.$$

(ii) If $\left|\frac{zf'_{j,k}(z)}{f_{j,k}(z)}\right| > \frac{1}{\mu}$ for some $\mu > 1$ and $\left|\frac{f_{j,k}(z)f''(z)}{f'_{j,k}(z)f'(z)} - 1\right| < \mu\sqrt{2}$ for all $z \in \mathcal{U}$ then $f \in \mathcal{S}^*_{(j,k)}$.

Corollary 2.6: *Let* $f \in \mathcal{A}, k \geq 2$ *is a natural number*, $j = 0,1,2,..,k-1,$.

(i) If $f_{j,k} \in \widetilde{\mathcal{S}}^*(\alpha)$ for some $0 < \alpha < 1$ such that $tan\left(\frac{\alpha\pi}{2}\right) < \frac{1}{2}$ and if

$$\left| arg\left[\frac{1 - zf''(z)/f'(z)}{zf'_{j,k}(z)/f_{j,k}(z)} + 1\right]\right| < \arctan\frac{1}{2} - \frac{\alpha\pi}{2} \text{ for all } z \in \mathcal{U} \text{ then } f \in \mathcal{S}^*_{(j,k)}.$$

(ii) If $\left|\frac{zf'_{j,k}(z)}{f_{j,k}(z)}\right| > \frac{1}{\mu}$ for some $\mu > 1$ and

$$\left|\frac{1 - zf''(z)/f'(z)}{zf'_{j,k}(z)/f_{j,k}(z)} + 1\right| < \mu\sqrt{5}$$

for all $z \in \mathcal{U}$ then $f \in \mathcal{S}^*_{(j,k)}$.

Remark 2.7: *For* $j = 1$ *and* $k = N$ *we get the results of Nikola Tuneski in [10]*

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