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ERROR ESTIMATE OF GENERALIZED SHANNON SAMPLING OPERATORS IN WEIGHTED $L_{P,\alpha}$ SPACE $(\alpha > 0,\ 0 < P < 1)$

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ABSTRACT

The aim of this work is to provide some approximation by generalized Shannon sampling operators, which are defined by band-limited kernels, and they are linear combinations of translated sinc- functions.

INTRODUCTION

The generalized sampling operators for the uniformly continuous and bounded functions $f \in C(R)$ ([8], [11] are given by

$$(S_G f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{G}\right) s(Gt-k), \quad (t \in \mathrm{R}; G > 0) \tag{1}$$

The operator $S_G: C(R) \to C(R)$ to be well-defined where the condition

$$\sum_{k=-\infty}^{\infty} |s(v-k)| < \infty \quad (v \in R) \tag{2}$$

is satisfied. A systematic study of sampling operators (1) for arbitrary kernel functions s with (2) was initiated at WTH Aachen by P. L. Butzer and his students since 1977([12],[14]).

Definition 1.1 [14]: If $s: R \to R$ is a bounded function such that (2) the absolute convergence being uniform on compact subsets of

$$\sum_{k=-\infty}^{\infty} |s(v-k)| = 1 \quad (v \in R) \tag{3}$$

Then s is said to be a kernel for sampling operators (1).

If the kernel function is $s(t) = sinc(t) := \frac{\sin \pi t}{\pi t}$, which do not satisfy (2), we get the classical Shannon operator

$$\left(S_G^{sinc} f\right)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{G}\right) sinc(Gt-k). \quad (t \in R\ ; G > 0) \tag{4}$$

In this paper we estimate the order of approximation in terms of modulus of smoothness in weighted $L_{P,\alpha}$ space $(\alpha > 0,\ 0 < p < 1)$.

2. PRELIMINARY RESULTS

2.1The modulus of smoothness in $L_{P,\alpha}$ space $(\alpha > 0,\ 0 < P < 1)$

Definition 2.1.1(Weight function) [1]: An integrable function **w** is called a weight function on the interval $[a, b]$ if $w(x) > 0$ for all $x \in [a, b]$. For example $w(x) = e^{\alpha x}$, $\alpha > 0$.

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Consider the space $L_{p,\alpha}(X), 0 < p < 1$ of all unbounded functions f on X such that $|f(x)| \le Me^{\alpha x}$, where M is a positive real number, which are equipped with the following quasi norm

$$\|f\|_{p,\alpha}^{p} = \left(\int_X \left|\frac{f(x)}{e^{\alpha x}}\right|^p dx\right)^{\frac{1}{p}} < \infty \tag{5}$$

.

Let f be unbounded function on R, and $\delta \ge 0$, the modulus of smoothness $\omega_k(f;\delta)_{p,\alpha}$ in $L_{p,\alpha}(X)$, $\alpha > 0$, $0 < p < 1$ is defined exactly as in case $1 \le p \le \infty$:[22]

$$\omega_k(f;\delta)_{p,\alpha} = \underbrace{sup}_{0 \le h < \delta} \left(\int_a^{b-kh} \left|\Delta_h^k\left(\frac{f(x)}{e^{\alpha x}}\right)\right|^p\right)^{\frac{1}{p}} \tag{6}$$

2.2 The space $\Lambda^{p,\alpha}$

Definition 2.2.1 ([10]):

(a) A sequence $\sum := (x_j)_{j \in Z} \subset R$ is called an admissible partition of R or an admissible sequence, if it satisfies $0 < \inf_{j\in Z}\Delta_j \le \sup_{j \in Z}\Delta_j < \infty$.

(b) Let $\sum := (x_j)_{j\in Z}$ be an admissible partition of R, and let $\Delta_j = x_j - x_{j-1}$. The discrete $\ell_p(\sum)$ − norm of a sequence of function values $f_{\sum}$ on the partition $(\sum)$ of the function $f: R \to C$ is defined for $1 \le p < \infty$ by

$$\|f\|_{\ell_p(\Sigma)} = \left\{\sum_{j\in Z}\left|f(x_j)\right|^p \Delta_j\right\}^{\frac{1}{p}} \tag{7}$$

(c) The space Λ^p for $1 \le p < \infty$ is defined by
$\Lambda^p := \left\{f; \|f\|_{\ell_p(\Sigma)} < \infty\right\}$ for each admissible sequence $(\sum)$.

We can defined $\Lambda^{p,\alpha}$ $\alpha > 0, 0 < p < 1$ the space of all unbounded functions f on admissible sequence $(\sum)$ of R, such that

$$\|f\|_{\ell_p(\Sigma),\alpha}^{p} = \left\{\sum_{j\in Z}\left|\frac{f(x)}{e^{\alpha x}}\right|^p \Delta_j\right\}^{\frac{1}{p}} \tag{8}$$

Theorem 2.1.2: For $f \in \Lambda^{p,\alpha}$, $(\alpha > 0, 0 < p < 1)$, and $k \in R$ then

$$\left\|S_G^{sinc}f - f\right\|_{p,\alpha}^{p} \le C\omega_k\left(f;\frac{1}{G}\right)_{p,\alpha} \tag{9}$$

Proof:

$$\begin{aligned}
\left\|S_G^{sinc}f - f\right\|_{p,\alpha}^{p} &= \left\{\sum_{j\in Z}\left|\frac{S_G^{sinc} f(x) - f(x)}{e^{\alpha x}}\right|^p \Delta_j\right\}^{\frac{1}{p}} \\
&= \left\{\sum_{j\in Z}\left|\frac{\sum_{k=-\infty}^{\infty} f\left(\frac{k}{G}\right) sinc\,(Gt-k) - f(x)}{e^{\alpha x}}\right|^p \Delta_j\right\}^{\frac{1}{p}} \\
&\le sup\left\{\sum_{j\in Z}\left|\frac{\sum_{k=-\infty}^{\infty} f\left(\frac{k}{G}\right) sinc\,(Gt-k) - f(x)}{e^{\alpha x}}\right|^p \Delta_j\right\}^{\frac{1}{p}} \\
&\le \sup\left\{\sum_{j\in Z}\left|\frac{f\left(\frac{k}{G}\right)\sum_{k=-\infty}^{\infty} \mathrm{sinc}\,(Gt-k) - f(x)}{e^{\alpha x}}\right|^p \Delta_j\right\}^{\frac{1}{p}}
\end{aligned}$$

By using (3) we have

$$\begin{aligned}
\left\|S_G^{sinc}f - f\right\|_{p,\alpha}^{p} &\le sup\left\{\sum_{j\in Z}\left|\frac{f\left(\frac{k}{G}\right) - f(x)}{e^{\alpha x}}\right|^p \Delta_j\right\}^{\frac{1}{p}} \\
&\le C(P)\left\{\sum_{j\in Z}\left|\Delta_h^k f\left(\frac{1}{G}\right)e^{-\alpha x}\right|^p \Delta_j\right\}^{\frac{1}{p}} \\
&\le C(P)\left\|\Delta_h^k f\left(\frac{1}{G}\right)\right\|_{P,\alpha}^{P} \\
&\le C(P)\omega_k\left(f;\frac{1}{G}\right)_{p,\alpha}
\end{aligned}$$

2.3 Band-limited kernels

By (see [2], [4], [5], [6],[7]) an even window function $\lambda \in C_{[-1,1]}$, $\lambda(0) = 1, \lambda(u) = 0$ $(|u| \ge 1)$, an even band-limited kernel s, defined the equality

$$s(t) := s_\lambda(t) := \int_0^1 \lambda(u)\cos(\pi t u)\,du. \tag{10}$$

We studied the generalized sampling operators $S_W: C(R) \to C(R)$ with the kernels in form (9) in ([2], [3], [5], [6], and [7]).

Many the band-limited kernel have been used in applications ([15], [16], [17], [18]). Many kernels can be defined by (9), e.g.

1) $\lambda_{(r)}(u) = 1 - u^2, r \geq 1$ defines the Zygmund (or Riesz) kernel, denoted by $Z_r = Z_r(t)$, which special case $r = 1$, the Fej´er kernel (see[19])
$$s_F(t) = \frac{1}{2}\operatorname{sinc}^2\left(\frac{t}{2}\right) \tag{11}$$
2) $\lambda_{(j)}(u) := \cos\pi(j + 1/2)u, j = 0,1,2, \ldots$ defines the Rogosinski-type kernel (see [4]) in the form
$$r_j(t) := \left(\operatorname{sinc}(t + j + 1/2) + \operatorname{sinc}(t - j - 1/2)\right) \tag{12}$$
$\lambda_{(H)}(u) := \cos^2\left(\frac{\pi u}{2}\right) = \frac{1}{2}(1 + \cos\pi u)$ defines the Hann kernel (see[5])
$$s_H(t) = \frac{1}{2}\frac{\operatorname{sinc}\ t}{1-t^2}. \tag{13}$$
Powers of the Hann window (see [15])
$$\lambda_{H,m}(u) := \cos^m\left(\frac{\pi u}{2}\right) = \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}\cos\left(\left(k - \frac{m}{2}\right)\pi u\right) \tag{14}$$
3) Give the general Hann kernel in the form
$$s_{H,m}(t) = 2^{-m}\frac{\Gamma(1+m)}{\Gamma\left(1+\frac{m}{2}-t\right)\Gamma\left(1+\frac{m}{2}+t\right)} \tag{15}$$

From ([5], Prorposition 2) we have that for $m = 0,1,2, \ldots,$ and $\ell \leq m$
$$s_{H,m}(t) = \frac{1}{2^{m-\ell}}\sum_{k=0}^{m-\ell}\binom{m-\ell}{k}s_{H,\ell}\left(t + k - \frac{m-\ell}{2}\right) \tag{16}$$
4) The Blackman-Harris window function
$$\lambda_{Ca}(u) = \sum_{k=0}^{m} a_k \cos k\pi u = \sum_{k=0}^{\left\lfloor\frac{m}{2}\right\rfloor} a_{2k}\cos(2k\pi u) + \sum_{k=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor} a_{2k+1}\cos\left((2k+1)\pi u\right) \tag{17}$$

Where ($\lfloor x \rfloor$ is the largest integer less than or equal to $x \in R$)
$$\sum_{k=0}^{\left\lfloor\frac{m}{2}\right\rfloor} a_{2k} = \sum_{k=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor} a_{2k+1} = \frac{1}{2} \tag{18}$$

Defines through (9) the Blackman-Harris kernel (see [7])
$$s_{Ca}(t) = \frac{1}{2}\sum_{k=0}^{m} a_k\left(sinc(t-k) + sinc(t+k)\right) \tag{19}$$

Proposition 2.3.1([20])

For $m \in N, 1 \leq \ell \leq m$ the kernel
$$s(t) = \operatorname{sinc}(t) - \frac{1}{2^{2\ell+1}}\sum_{k=o}^{m-\ell}(-1)^{k+\ell}q_k\left[\Delta_1^{2\ell}\operatorname{sinc}(t-k) + \Delta_1^{2\ell}\operatorname{sinc}(t+k)\right] \tag{20}$$
With $q \in R^{m-\ell+1}, \sum_{k=0}^{m-\ell} q_k = 1$ is a Blackman Harris kernel $s_{C,a(q)}$ with parameter vector $a(q) \in R^{m+1}$.

3. MAIN RESULTS

In this suction we shall estimates error approximation of some sampling operators $S_G f: C(R) \to C(R)$ which is linear combinations of translated sinc-functions.

3.1 Rogosinski-type sampling operators

Let consider the Rogosinski-type sampling operators $R_{G,j}$ defined by the kernel functions r_j in (9). These kernel functions are deduced by the window functions $\lambda_{(j)}(u) := \cos\pi(j + 1/2)u, j \in N$. (see[20])

Theorem 3.1.1: Assume that a Rogosinski-type sampling operator $R_{G,j}$ $(j = 0,1,2, \ldots), G > 0$ defined by (1) with the kernel (10). Then for $f \in \Lambda^{p,\alpha}$, $(\alpha > 0, 0 < p < 1)$ we have
$$\left\|R_{G,j}f - f\right\|_{p,\alpha}^{p} \leq C_j\omega_2\left(f; \frac{1}{G}\right)_{P,\alpha}. \tag{20}$$
where the constant C_j is independent of f and G.

Proof: Since the Rogosinski-type kernel in (10) is a linear componation of translated sinc-function. Then we give this operator $R_{G,j}$ which is representation

$$\begin{aligned}(R_{G,j}f)(t) &= \sum_{j\in Z} f\left(\tfrac{j}{G}\right)\left[\tfrac{1}{2}\left(\operatorname{sinc}\left(t+j+\tfrac{1}{2}\right)+\operatorname{sinc}\left(t-j-\tfrac{1}{2}\right)\right)\right]\\ &= \tfrac{1}{2}\left[\sum_{j\in Z} f\left(\tfrac{j}{G}\right)\operatorname{sinc}\left(t+j+\tfrac{1}{2}\right)+\sum_{j\in Z} f\left(\tfrac{j}{G}\right)\operatorname{sinc}\left(t-j-\tfrac{1}{2}\right)\right]\\ &= \tfrac{1}{2}\left[\left(S_G^{sinc} f\right)\left(t+\tfrac{2j+1}{2G}\right)+\left(S_G^{sinc} f\right)\left(t-\tfrac{2j+1}{2G}\right)\right]\end{aligned}$$

We obtain

$$\begin{aligned}(R_{G,j}f)(t)-f(t) &= \tfrac{1}{2}\left[\left(S_G^{sinc} f\right)\left(t+\tfrac{2j+1}{2G}\right)+\left(S_G^{sinc} f\right)\left(t-\tfrac{2j+1}{2G}\right)\right]-f(t)\\ &= \tfrac{1}{2}\begin{bmatrix}\left(S_G^{sinc} f\right)\left(t+\tfrac{2j+1}{2G}\right)+\left(S_G^{sinc} f\right)\left(t-\tfrac{2j+1}{2G}\right)-f\left(t+\tfrac{2j+1}{2G}\right)\\ +f\left(t+\tfrac{2j+1}{2G}\right)-f\left(t-\tfrac{2j+1}{2G}\right)+f\left(t-\tfrac{2j+1}{2G}\right)-2f(t)\end{bmatrix}\\ &= \tfrac{1}{2}\begin{bmatrix}\left(R_{G,j} f\right)\left(t+\tfrac{2j+1}{2G}\right)-f\left(t+\tfrac{2j+1}{2G}\right)+\left(S_G^{sinc} f\right)\left(t-\tfrac{2j+1}{2G}\right)\\ -f\left(t-\tfrac{2j+1}{2G}\right)+\left(f\left(t+\tfrac{2j+1}{2G}\right)-2f(t)+f\left(t-\tfrac{2j+1}{2G}\right)\right)\end{bmatrix}\end{aligned}$$

Since $0 < p < 1$ then by properties of trigonometric inequality for quasi norm we have

$$\begin{aligned}\left\|(R_{G,j}f)-f\right\|_{p,\alpha}^{p} &\le 2^{p}\left[\left|\tfrac{1}{2}\right|\left(\left\|S_G^{sinc} f-f\right\|_{p,\alpha}^{p}+\left\|S_G^{sinc} f-f\right\|_{p,\alpha}^{p}+\left\|\Delta^2_{\frac{2j+1}{2G}} f\right\|_{p,\alpha}^{p}\right)\right]\\ &\le 2^{p}\left[\left\|S_G^{sinc} f-f\right\|_{p,\alpha}^{p}+\tfrac{1}{2}\omega_2\left(f;\tfrac{1}{G}\right)_{P,\alpha}\right]\end{aligned}$$

By using theorem (2.1.2) we have

$$\begin{aligned}\left\|(R_{G,j}f)-f\right\|_{p,\alpha}^{p} &\le 2^{p}\left[\omega_2\left(f,\tfrac{1}{G}\right)_{P,\alpha}+\tfrac{1}{2}\left(1+\tfrac{2j+1}{2G}\right)\omega_2\left(f;\tfrac{1}{G}\right)_{P,\alpha}\right]\\ &\le \omega_2\left(f;\tfrac{1}{G}\right)_{P,\alpha}\left[2^{p}+\tfrac{1}{2}\left(1+\tfrac{2j+1}{2G}\right)\right]\\ &\le C_j\,\omega_2\left(f;\tfrac{1}{G}\right)_{P,\alpha}.\end{aligned}$$

3.2 Hann sampling operators

Consider Hann sampling operators $H_{W,m}$ $(m = 0,1,2,\ldots)$. The Hann kernel (see [21] $s_{H,m}(t) = O(|t|^{-m-1})$ as $|t| \to \infty$. from (15) if $\ell = 0$ we have aliner combination of sinc –function because $H_{W,0} = \operatorname{sinc}$.

Theorem 3.2.1: For the Hann sampling operator $H_{G,m}(m = 1,2,\ldots)$ defined by (1) with the kernel (14). Then for $f \in \Lambda^{p,\alpha}$, $(\alpha > 0, 0 < p < 1)$ we have

$$\left\|H_{G,m} f-f\right\|_{p,\alpha}^{p} \le C_m\,\omega_2\left(f;\tfrac{1}{G}\right)_{P,\alpha}. \tag{21}$$

where the constant C_m is independent of f and G.

Proof: According to ([20] equation (9)) we give this operator $(H_{G,m})$ which has the form

$$(H_{G,m}f)(t) = \tfrac{1}{2}\left[\left(H_{G,m-1}f\right)\left(t-\tfrac{1}{2G}\right)+\left(H_{G,m-1}f\right)\left(t+\tfrac{1}{2G}\right)\right]$$

Hence

$$\begin{aligned}(H_{G,m}f)(t)-f(t) &= \tfrac{1}{2}\left[\left(H_{G,m-1}f\right)\left(t-\tfrac{1}{2G}\right)+\left(H_{G,m-1}f\right)\left(t+\tfrac{1}{2G}\right)\right]-f(t)\\ &= \tfrac{1}{2}\begin{bmatrix}\left(H_{G,m-1}f\right)\left(t-\tfrac{1}{2G}\right)-f\left(t-\tfrac{2j+1}{2G}\right)+f\left(t-\tfrac{2j+1}{2G}\right)+\left(H_{G,m-1}f\right)\left(t+\tfrac{1}{2G}\right)\\ -f\left(t+\tfrac{2j+1}{2G}\right)+f\left(t+\tfrac{2j+1}{2G}\right)-2f(t)\end{bmatrix}\\ &= \tfrac{1}{2}\begin{bmatrix}\left(H_{G,m-1}f\right)\left(t-\tfrac{1}{2G}\right)-f\left(t-\tfrac{2j+1}{2G}\right)+\left(H_{G,m-1}f\right)\left(t+\tfrac{1}{2G}\right)\\ -f\left(t+\tfrac{2j+1}{2G}\right)+\left(f\left(t+\tfrac{2j+1}{2G}\right)-2f(t)+f\left(t-\tfrac{2j+1}{2G}\right)\right)\end{bmatrix}\end{aligned}$$

Since $0 < p < 1$ we have from properties of quasi norm,

$$\begin{aligned}\left\|(H_{G,m}f)-f\right\|_{p,\alpha}^{p} &\le 2^{p}\left[\left|\tfrac{1}{2}\right|\left(\left\|H_{G,m-1}f-f\right\|_{p,\alpha}^{p}+\left\|H_{G,m-1}f-f\right\|_{p,\alpha}^{p}+\left\|\Delta^2_{\frac{1}{2G}} f\right\|_{p,\alpha}^{p}\right)\right]\\ &\le 2^{p}\left[\left\|H_{G,m-1}f-f\right\|_{p,\alpha}^{p}+\tfrac{1}{2}\omega_2\left(f;\tfrac{1}{2G}\right)_{P,\alpha}\right]\end{aligned}$$

By using induction the proof give

$$\left\|H_{G,m}f - f\right\|_{p,\alpha}^{p} \le 2^{p}\left[\left\|H_{G,0}f - f\right\|_{p,\alpha}^{p} + \frac{m}{2}\omega_2\left(f;\frac{1}{2G}\right)_{P,\alpha}\right]$$

From (15) if we take $\ell = 0$ then we have a linear combination of sinc-function because $S_{H,0} = \text{sinc}$. hence $H_{G,0} = S_G^{\text{sinc}}$,

Therefore by using theorem (2.1.2) we have

$$\left\|H_{G,m}f - f\right\|_{p,\alpha}^{p} \le 2^{p}\left[C_2\omega_2\left(f;\frac{1}{G}\right)_{P,\alpha} + \frac{m}{2}\omega_2\left(f;\frac{1}{2G}\right)_{P,\alpha}\right]$$
$$\le C_m\left(\omega_2\left(f;\frac{1}{2G}\right)_{P,\alpha}\right).$$

3.3 Blackman-Harris sampling operators

Before over 52 years ([15], [16], [17], [18]) The Blackman window has been used in signal analysis. Recently Lasser and Obermaier [13] studied the role of the Blackman window for defining approximative identities in Fourier approximation.

Theorem 3.3.1: consider $C_{G,a}$ be the blackman-harris sampling operator defined by (1) with the kernel (18).
Then for $f \in \Lambda^{p,\alpha}$, $(\alpha > 0, 0 < p < 1)$ we have

$$\left\|C_{G,a}f - f\right\|_{p,\alpha}^{p} \le T_a\omega_2\left(f;\frac{1}{G}\right)_{P,\alpha} \tag{22}$$

where the constant T_a is independent of f and G.

Proof: from (18) we show that the blackman-harris kernel is a linear combination of translated sinc-functions.

Therefore the operator $C_{G,a}$ can be representation by

$$\left(C_{G,a}f\right)(t) = \frac{1}{2}\sum_{j\in Z} f\left(\frac{j}{G}\right)\sum_{k=o}^{m} a_k\left(\text{sinc}(Gt - j + k) + \text{sinc}(Gt - j - k)\right)$$
$$= \frac{1}{2}\sum_{k=o}^{m} a_k\left[\sum_{j\in Z} f\left(\frac{j}{G}\right)\left(\text{sinc}(Gt - j + k) + \sum_{j\in Z} f\left(\frac{j}{G}\right)\text{sinc}(Gt - j - k)\right)\right]$$
$$= \frac{1}{2}\sum_{k=o}^{m} a_k\left[\left(S_G^{\text{sinc}} f\right)\left(t + \frac{k}{G}\right) + \left(S_G^{\text{sinc}} f\right)\left(t - \frac{k}{G}\right)\right]$$

Therefore

$$\left(C_{G,a}f\right)(t) - f(t) = \frac{1}{2}\sum_{k=o}^{m} a_k\left[\left(S_G^{\text{sinc}} f\right)\left(t + \frac{k}{G}\right) + \left(S_G^{\text{sinc}} f\right)\left(t - \frac{k}{G}\right)\right] - f(t)$$
$$= \frac{1}{2}\sum_{k=o}^{m} a_k\left[\begin{array}{c}\left(S_G^{\text{sinc}} f\right)\left(t + \frac{k}{G}\right) - f\left(t + \frac{k}{G}\right) + \left(S_G^{\text{sinc}} f\right)\left(t - \frac{k}{G}\right) - f\left(t - \frac{k}{G}\right) \\ + \left(f\left(t + \frac{k}{G}\right) - 2f(t) + f\left(t - \frac{k}{G}\right)\right)\end{array}\right]$$

Since $0 < p < 1$, by properties of quasi norm we obtain

$$\left\|C_{G,a}f - f\right\|_{p,\alpha}^{p} \le 2^{p}\frac{1}{2}\sum_{k=o}^{m}|a_k|\left(\left\|C_{G,a}f - f\right\|_{p,\alpha}^{p} + \left\|C_{G,a}f - f\right\|_{p,\alpha}^{p} + \left\|\Delta_{\frac{k}{G}}^{2}f\right\|_{p,\alpha}^{p}\right)$$
$$\le 2^{p}\frac{1}{2}\sum_{k=o}^{m}|a_k|\left(\left\|C_{G,a}f - f\right\|_{p,\alpha}^{p} + \frac{1}{2}\omega_2\left(f;\frac{k}{G}\right)_{P,\alpha}\right)$$

By using theorem (2.1.2) we have

$$\left\|C_{G,a}f - f\right\|_{p,\alpha}^{p} \le 2^{p}\sum_{k=o}^{m}|a_k|\left[C\omega_2\left(f;\frac{1}{G}\right)_{P,\alpha} + \frac{k^2}{2}\omega_2\left(f;\frac{1}{G}\right)_{P,\alpha}\right]$$
$$\le \omega_2\left(f;\frac{1}{G}\right)_{P,\alpha}\left[2^{p}\sum_{k=o}^{m}|a_k|\left(C + \frac{k^2}{2}\right)\right]$$
$$\le M_a\omega_2\left(f;\frac{1}{G}\right)_{P,\alpha}.$$

Theorem 3.3.2: For $C_{G,a}$ $(a \in R^{m+1})$ let $\ell, 1 \le \ell \le m$ be fixed. For a parameter vector $q \in R^{m-\ell+1}$, such that we have for the kernel (13) a representation via central differences(4) inform (14),

Then for $f \in \Lambda^{p,\alpha}$, $(\alpha > 0, 0 < p < 1)$ we have

$$\left\|C_{G,a}f - f\right\|_{p,\alpha}^{p} \le D_{a,\ell}\omega_{2\ell}\left(f;\frac{1}{G}\right)_{P,\alpha} \tag{22}$$

where the constant $D_{a,\ell}$ is independent of f and G.

Proof: For kernel (20) we get operator $C_{G,a}$ which is representation

$$\begin{aligned}(C_{G,a}f)(t) &= \sum_{j\in Z} f\left(\tfrac{j}{G}\right)\left[\operatorname{sinc}(t) - \tfrac{1}{2^{2\ell+1}}\sum_{j=o}^{m-\ell}(-1)^{j+\ell}\, q_j\left[\Delta_1^{2\ell}\operatorname{sinc}(t-j) + \Delta_1^{2\ell}\operatorname{sinc}(t-j)\right]\right]\\
&= \sum_{j\in Z} f\left(\tfrac{j}{G}\right)\operatorname{sinc}(Gt-k) - \sum_{j\in Z} f\left(\tfrac{j}{G}\right)\left[\tfrac{1}{2^{2\ell+1}}\sum_{j=o}^{m-\ell}(-1)^{j+\ell}\, q_j\left[\Delta_1^{2\ell}\operatorname{sinc}(t-j) + \Delta_1^{2\ell}\operatorname{sinc}(t-j)\right]\right]\\
&= \left(S_G^{\operatorname{sinc}} f\right)(t) - \tfrac{1}{2^{2\ell+1}}\sum_{j=o}^{m-\ell}(-1)^{j+\ell}\, q_j \sum_{j\in Z} f\left(\tfrac{j}{G}\right)\left[\Delta_1^{2\ell}\operatorname{sinc}(t-j) + \Delta_1^{2\ell}\operatorname{sinc}(t-j)\right]\\
&= \left(S_G^{\operatorname{sinc}} f\right)(t) - \tfrac{1}{2^{2\ell+1}}\sum_{j=0}^{m-\ell}(-1)^{j+\ell}\, q_j\left[\Delta_1^{2\ell}\sum_{j\in Z} f\left(\tfrac{j}{G}\right)\operatorname{sinc}(Gt-j) \quad + \Delta_1^{2\ell}\sum_{j\in Z} f\left(\tfrac{j}{G}\right)\operatorname{sinc}(Gt-j)\right]\end{aligned}$$

Therefore

$$(C_{G,a}f)(t) = \left(S_G^{\operatorname{sinc}} f\right)(t) - \tfrac{1}{2^{2\ell+1}}\sum_{j=0}^{m-\ell}(-1)^{j+\ell}\, q_j\left[\Delta_1^{2\ell}\left(S_G^{\operatorname{sinc}} f\right)\left(t-\tfrac{j}{G}\right) + \Delta_1^{2\ell}\left(S_G^{\operatorname{sinc}} f\right)\left(t+\tfrac{j}{G}\right)\right]$$

We obtain

$$\begin{aligned}(C_{G,a}f)(t) - f(t) &= \left(S_G^{\operatorname{sinc}} f\right)(t) - \tfrac{1}{2^{2\ell+1}}\sum_{j=0}^{m-\ell}(-1)^{j+\ell}\, q_j\left[\Delta_1^{2\ell}\left(S_G^{\operatorname{sinc}} f\right)\left(t-\tfrac{j}{G}\right) + \Delta_1^{2\ell}\left(S_G^{\operatorname{sinc}} f\right)\left(t+\tfrac{j}{G}\right)\right] - t(t)\\
&= \left(S_G^{\operatorname{sinc}} f\right)(t) - f(t) - \tfrac{1}{2^{2\ell+1}}\sum_{j=0}^{m-\ell}(-1)^{j+\ell}\, q_j\begin{bmatrix}\Delta_1^{2\ell}\left(\left(S_G^{\operatorname{sinc}} f\right)\left(t-\tfrac{j}{G}\right) - f\left(t-\tfrac{j}{G}\right) + f\left(t-\tfrac{j}{G}\right)\right)\\ +\Delta_1^{2\ell}\left(\left(S_G^{\operatorname{sinc}} f\right)\left(t+\tfrac{j}{G}\right) - f\left(t+\tfrac{j}{G}\right) + f\left(t+\tfrac{j}{G}\right)\right)\end{bmatrix}\\
&= \left(S_G^{\operatorname{sinc}} f\right) - \tfrac{1}{2^{2\ell+1}}\sum_{j=0}^{m-\ell}(-1)^{j+\ell}\, q_j\begin{bmatrix}\Delta_{\frac{1}{G}}^{2\ell}\left(\left(S_G^{\operatorname{sinc}} f\right)\left(t-\tfrac{j}{G}\right) - f\left(t-\tfrac{j}{G}\right)\right)\\ +\Delta_{\frac{1}{G}}^{2\ell}\left(\left(S_G^{\operatorname{sinc}} f\right)\left(t+\tfrac{j}{G}\right) - f\left(t+\tfrac{j}{G}\right)\right)\end{bmatrix}\\
&\qquad - \tfrac{1}{2^{2\ell+1}}\sum_{j=0}^{m-\ell}(-1)^{j+\ell}\, q_j\left[\left(\Delta_{\frac{1}{G}}^{2\ell} f\left(t-\tfrac{j}{G}\right)\right) + \left(\Delta_{\frac{1}{G}}^{2\ell} f\left(t+\tfrac{j}{G}\right)\right)\right]\end{aligned}$$

Since $0 < p < 1$ from properties of quasi norm we obtain

$$\begin{aligned}\left\|C_{G,a}f - f\right\|_{p,\alpha}^{p} &\le 2^p\left[\left\|S_G^{\operatorname{sinc}} f - f\right\|_{p,\alpha}^{p} + \sum_{j=0}^{m-\ell}|q_j|\left\|S_G^{\operatorname{sinc}} f - f\right\|_{p,\alpha}^{p} + \tfrac{1}{2^{2\ell}}\sum_{j=0}^{m-\ell}|q_j|\left\|\Delta_{\frac{1}{G}}^{2\ell} f\right\|_{p,\alpha}^{p}\right]\\
&\le 2^p\left[\left(1 + \sum_{j=0}^{m-\ell}|q_j|\right)\left(\left\|S_G^{\operatorname{sinc}} f - f\right\|_{p,\alpha}^{p}\right) + \tfrac{1}{2^{2\ell}}\sum_{j=0}^{m-\ell}|q_j|\left(\omega_{2\ell}\left(f;\tfrac{1}{G}\right)_{P,\alpha}\right)\right]\end{aligned}$$

By using theorem (2.1.2) we have

$$\begin{aligned}\left\|C_{G,a}f - f\right\|_{p,\alpha}^{p} &\le 2^p\left(1 + \sum_{j=0}^{m-\ell}|q_j| + \tfrac{1}{2^{2\ell}}\sum_{j=0}^{m-\ell}|q_j|\right)\omega_{2\ell}\left(f;\tfrac{1}{G}\right)_{P,\alpha}\\
&\le D_{a,\ell}\,\omega_{2\ell}\left(f;\tfrac{1}{G}\right)_{P,\alpha}.\end{aligned}$$

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