

ERROR ESTIMATE OF GENERALIZED SHANNON SAMPLING OPERATORS  
IN WEIGHTED  $L_{p,\alpha}$  SPACE ( $\alpha > 0$ ,  $0 < p < 1$ )

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(Received On: 22-06-15; Revised & Accepted On: 26-07-15)

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ABSTRACT

The aim of this work is to provide some approximation by generalized Shannon sampling operators, which are defined by band-limited kernels, and they are linear combinations of translated sinc- functions.

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INTRODUCTION

The generalized sampling operators for the uniformly continuous and bounded functions  $f \in C(R)$  ([8], [11]) are given by

$$(S_G f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{G}\right) s(Gt - k), \quad (t \in R; G > 0) \quad (1)$$

The operator  $S_G: C(R) \rightarrow C(R)$  to be well-defined where the condition

$$\sum_{k=-\infty}^{\infty} |s(v - k)| < \infty \quad (v \in R) \quad (2)$$

is satisfied. A systematic study of sampling operators (1) for arbitrary kernel functions  $s$  with (2) was initiated at WTH Aachen by P. L. Butzer and his students since 1977([12],[14]).

**Definition 1.1** [14]: If  $s: R \rightarrow R$  is a bounded function such that (2) the absolute convergence being uniform on compact subsets of

$$\sum_{k=-\infty}^{\infty} |s(v - k)| = 1 \quad (v \in R) \quad (3)$$

Then  $s$  is said to be a kernel for sampling operators (1).

If the kernel function is  $s(t) = \text{sinc}(t) := \frac{\sin \pi t}{\pi t}$ , which do not satisfy (2), we get the classical Shannon operator

$$(S_G^{\text{sinc}} f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{G}\right) \text{sinc}(Gt - k). \quad (t \in R; G > 0) \quad (4)$$

In this paper we estimate the order of approximation in terms of modulus of smoothness in weighted  $L_{p,\alpha}$  space ( $\alpha > 0$ ,  $0 < p < 1$ ).

2. PRELIMINARY RESULTS

2.1 The modulus of smoothness in  $L_{p,\alpha}$  space ( $\alpha > 0$ ,  $0 < p < 1$ )

**Definition 2.1.1 (Weight function)** [1]: An integrable function  $w$  is called a weight function on the interval  $[a, b]$  if  $w(x) > 0$  for all  $x \in [a, b]$ . For example  $w(x) = e^{\alpha x}$ ,  $\alpha > 0$ .

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Consider the space  $L_{p,\alpha}(X), 0 < p < 1$  of all unbounded functions  $f$  on  $X$  such that  $|f(x)| \leq Me^{\alpha x}$ , where  $M$  is a positive real number, which are equipped with the following quasi norm

$$\|f\|_{p,\alpha}^p = \left( \int_X \left| \frac{f(x)}{e^{\alpha x}} \right|^p dx \right)^{\frac{1}{p}} < \infty \tag{5}$$

Let  $f$  be unbounded function on  $\mathbb{R}$ , and  $\delta \geq 0$ , the modulus of smoothness  $\omega_k(f; \delta)_{p,\alpha}$  in  $L_{p,\alpha}(X), \alpha > 0, 0 < p < 1$  is defined exactly as in case  $1 \leq p \leq \infty$ : [22]

$$\omega_k(f; \delta)_{p,\alpha} = \sup_{0 \leq h < \delta} \left( \int_a^{b-kh} \left| \Delta_h^k \left( \frac{f(x)}{e^{\alpha x}} \right) \right|^p dx \right)^{\frac{1}{p}} \tag{6}$$

### 2.2 The space $\Lambda^{p,\alpha}$

**Definition 2.2.1 ([10]):**

(a) A sequence  $\Sigma := (x_j)_{j \in \mathbb{Z}} \subset \mathbb{R}$  is called an admissible partition of  $\mathbb{R}$  or an admissible sequence, if it satisfies  $0 < \inf_{j \in \mathbb{Z}} \Delta_j \leq \sup_{j \in \mathbb{Z}} \Delta_j < \infty$ .

(b) Let  $\Sigma := (x_j)_{j \in \mathbb{Z}}$  be an admissible partition of  $\mathbb{R}$ , and let  $\Delta_j = x_j - x_{j-1}$ . The discrete  $\ell_p(\Sigma)$  – norm of a sequence of function values  $f_\Sigma$  on the partition  $(\Sigma)$  of the function  $f: \mathbb{R} \rightarrow \mathbb{C}$  is defined for  $1 \leq p < \infty$  by

$$\|f\|_{\ell_p(\Sigma)} = \left\{ \sum_{j \in \mathbb{Z}} |f(x_j)|^p \Delta_j \right\}^{\frac{1}{p}} \tag{7}$$

(c) The space  $\Lambda^p$  for  $1 \leq p < \infty$  is defined by

$$\Lambda^p := \left\{ f; \|f\|_{\ell_p(\Sigma)} < \infty \right\} \text{ for each admissible sequence } (\Sigma).$$

We can defined  $\Lambda^{p,\alpha}, \alpha > 0, 0 < p < 1$  the space of all unbounded functions  $f$  on admissible sequence  $(\Sigma)$  of  $\mathbb{R}$ , such that

$$\|f\|_{\ell_p(\Sigma),\alpha}^p = \left\{ \sum_{j \in \mathbb{Z}} \left| \frac{f(x)}{e^{\alpha x}} \right|^p \Delta_j \right\}^{\frac{1}{p}} \tag{8}$$

**Theorem 2.1.2:** For  $f \in \Lambda^{p,\alpha}, (\alpha > 0, 0 < p < 1)$ , and  $k \in \mathbb{R}$  then

$$\|S_G^{sinc} f - f\|_{p,\alpha}^p \leq C \omega_k \left( f; \frac{1}{G} \right)_{p,\alpha} \tag{9}$$

**Proof:**

$$\begin{aligned} \|S_G^{sinc} f - f\|_{p,\alpha}^p &= \left\{ \sum_{j \in \mathbb{Z}} \left| \frac{S_G^{sinc} f(x) - f(x)}{e^{\alpha x}} \right|^p \Delta_j \right\}^{\frac{1}{p}} \\ &= \left\{ \sum_{j \in \mathbb{Z}} \left| \frac{\sum_{k=-\infty}^{\infty} f\left(\frac{k}{G}\right) \text{sinc}(Gt-k) - f(x)}{e^{\alpha x}} \right|^p \Delta_j \right\}^{\frac{1}{p}} \\ &\leq \sup \left\{ \sum_{j \in \mathbb{Z}} \left| \frac{\sum_{k=-\infty}^{\infty} f\left(\frac{k}{G}\right) \text{sinc}(Gt-k) - f(x)}{e^{\alpha x}} \right|^p \Delta_j \right\}^{\frac{1}{p}} \\ &\leq \sup \left\{ \sum_{j \in \mathbb{Z}} \left| \frac{f\left(\frac{k}{G}\right) \sum_{k=-\infty}^{\infty} \text{sinc}(Gt-k) - f(x)}{e^{\alpha x}} \right|^p \Delta_j \right\}^{\frac{1}{p}} \end{aligned}$$

By using (3) we have

$$\begin{aligned} \|S_G^{sinc} f - f\|_{p,\alpha}^p &\leq \sup \left\{ \sum_{j \in \mathbb{Z}} \left| \frac{f\left(\frac{k}{G}\right) - f(x)}{e^{\alpha x}} \right|^p \Delta_j \right\}^{\frac{1}{p}} \\ &\leq C(P) \left\{ \sum_{j \in \mathbb{Z}} \left| \Delta_h^k f\left(\frac{1}{G}\right) e^{-\alpha x} \right|^p \Delta_j \right\}^{\frac{1}{p}} \\ &\leq C(P) \left\| \Delta_h^k f\left(\frac{1}{G}\right) \right\|_{p,\alpha}^p \\ &\leq C(P) \omega_k \left( f; \frac{1}{G} \right)_{p,\alpha} \end{aligned}$$

### 2.3 Band-limited kernels

By (see [2], [4], [5], [6],[7]) an even window function  $\lambda \in C_{[-1,1]}$ ,  $\lambda(0) = 1, \lambda(u) = 0$  ( $|u| \geq 1$ ), an even band-limited kernel  $s$ , defined the equality

$$s(t) := s_\lambda(t) := \int_0^1 \lambda(u) \cos(\pi t u) du. \tag{10}$$

We studied the generalized sampling operators  $S_W: C(\mathbb{R}) \rightarrow C(\mathbb{R})$  with the kernels in form (9) in ([2], [3], [5], [6], and [7]).

Many the band-limited kernel have been used in applications ([15], [16], [17], [18]). Many kernels can be defined by (9), e.g.

- 1)  $\lambda_{(r)}(u) = 1 - u^2, r \geq 1$  defines the Zygmund (or Riesz) kernel, denoted by  $Z_r = Z_r(t)$ , which special case  $r = 1$ , the Fej'er kernel (see[19])

$$s_F(t) = \frac{1}{2} \text{sinc}^2\left(\frac{t}{2}\right) \tag{11}$$

- 2)  $\lambda_{(j)}(u) := \cos\pi(j + 1/2)u, j = 0,1,2, \dots$  defines the Rogosinski-type kernel (see [4]) in the form  $r_j(t) := (\text{sinc}(t + j + 1/2) + \text{sinc}(t - j - 1/2))$

$$\lambda_{(H)}(u) := \cos^2\left(\frac{\pi u}{2}\right) = \frac{1}{2}(1 + \cos \pi u) \text{ defines the Hann kernel (see[5])}$$

$$s_H(t) = \frac{1}{2} \frac{\text{sinc } t}{1-t^2}. \tag{13}$$

Powers of the Hann window (see [15])

$$\lambda_{H,m}(u) := \cos^m\left(\frac{\pi u}{2}\right) = \frac{1}{2^m} \sum_{k=0}^m \binom{m}{k} \cos\left(\left(k - \frac{m}{2}\right) \pi u\right) \tag{14}$$

- 3) Give the general Hann kernel in the form

$$s_{H,m}(t) = 2^{-m} \frac{\Gamma(1+m)}{\Gamma(1+\frac{m}{2}-t)\Gamma(1+\frac{m}{2}+t)} \tag{15}$$

From ([5], Proposition 2) we have that for  $m = 0,1,2, \dots$ , and  $\ell \leq m$

$$s_{H,m}(t) = \frac{1}{2^{m-\ell}} \sum_{k=0}^{m-\ell} \binom{m-\ell}{k} s_{H,\ell}\left(t + k - \frac{m-\ell}{2}\right) \tag{16}$$

- 4) The Blackman-Harris window function

$$\lambda_{Ca}(u) = \sum_{k=0}^m a_k \cos k\pi u = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} a_{2k} \cos(2k\pi u) + \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} a_{2k+1} \cos((2k+1)\pi u) \tag{17}$$

Where ( $\lfloor x \rfloor$ ) is the largest integer less than or equal to  $x \in \mathbb{R}$ )

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} a_{2k} = \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} a_{2k+1} = \frac{1}{2} \tag{18}$$

Defines through (9) the Blackman-Harris kernel (see [7])

$$s_{Ca}(t) = \frac{1}{2} \sum_{k=0}^m a_k (\text{sinc}(t - k) + \text{sinc}(t + k)) \tag{19}$$

Proposition 2.3.1([20])

For  $m \in \mathbb{N}, 1 \leq \ell \leq m$  the kernel

$$s(t) = \text{sinc}(t) - \frac{1}{2^{2\ell+1}} \sum_{k=0}^{m-\ell} (-1)^{k+\ell} q_k [\Delta_1^{2\ell} \text{sinc}(t - k) + \Delta_1^{2\ell} \text{sinc}(t + k)] \tag{20}$$

With  $q \in \mathbb{R}^{m-\ell+1}, \sum_{k=0}^{m-\ell} q_k = 1$  is a Blackman Harris kernel  $s_{C,a(q)}$  with parameter vector  $a(q) \in \mathbb{R}^{m+1}$ .

### 3. MAIN RESULTS

In this suction we shall estimates error approximation of some sampling operators  $S_G f: C(\mathbb{R}) \rightarrow C(\mathbb{R})$  which is linear combinations of translated sinc-functions.

#### 3.1 Rogosinski-type sampling operators

Let consider the Rogosinski-type sampling operators  $R_{G,j}$  defined by the kernel functions  $r_j$  in (9). These kernel functions are deduced by the window functions  $\lambda_{(j)}(u) := \cos\pi(j + 1/2)u, j \in \mathbb{N}$ . (see[20])

**Theorem 3.1.1:** Assume that a Rogosinski-type sampling operator  $R_{G,j}$  ( $j = 0,1,2, \dots$ ),  $G > 0$  defined by (1) with the kernel (10). Then for  $f \in \Lambda^{p,\alpha}, (\alpha > 0, 0 < p < 1)$  we have

$$\|R_{G,j} f - f\|_{p,\alpha}^p \leq C_j \omega_2\left(f; \frac{1}{G}\right)_{p,\alpha}. \tag{20}$$

where the constant  $C_j$  is independent of  $f$  and  $G$ .

**Proof:** Since the Rogosinski-type kernel in (10) is a linear composition of translated sinc-function. Then we give this operator  $R_{G,j}$  which is representation

$$\begin{aligned} (R_{G,j}f)(t) &= \sum_{j \in \mathbb{Z}} f\left(\frac{j}{G}\right) \left[ \frac{1}{2} \left( \text{sinc}\left(t + j + \frac{1}{2}\right) + \text{sinc}\left(t - j - \frac{1}{2}\right) \right) \right] \\ &= \frac{1}{2} \left[ \sum_{j \in \mathbb{Z}} f\left(\frac{j}{G}\right) \text{sinc}\left(t + j + \frac{1}{2}\right) + \sum_{j \in \mathbb{Z}} f\left(\frac{j}{G}\right) \text{sinc}\left(t - j - \frac{1}{2}\right) \right] \\ &= \frac{1}{2} \left[ (S_G^{\text{sinc}} f)\left(t + \frac{2j+1}{2G}\right) + (S_G^{\text{sinc}} f)\left(t - \frac{2j+1}{2G}\right) \right] \end{aligned}$$

We obtain

$$\begin{aligned} (R_{G,j}f)(t) - f(t) &= \frac{1}{2} \left[ (S_G^{\text{sinc}} f)\left(t + \frac{2j+1}{2G}\right) + (S_G^{\text{sinc}} f)\left(t - \frac{2j+1}{2G}\right) \right] - f(t) \\ &= \frac{1}{2} \left[ (S_G^{\text{sinc}} f)\left(t + \frac{2j+1}{2G}\right) + (S_G^{\text{sinc}} f)\left(t - \frac{2j+1}{2G}\right) - f\left(t + \frac{2j+1}{2G}\right) \right. \\ &\quad \left. + f\left(t + \frac{2j+1}{2G}\right) - f\left(t - \frac{2j+1}{2G}\right) + f\left(t - \frac{2j+1}{2G}\right) - 2f(t) \right] \\ &= \frac{1}{2} \left[ (R_{G,j}f)\left(t + \frac{2j+1}{2G}\right) - f\left(t + \frac{2j+1}{2G}\right) + (S_G^{\text{sinc}} f)\left(t - \frac{2j+1}{2G}\right) \right. \\ &\quad \left. - f\left(t - \frac{2j+1}{2G}\right) + \left(f\left(t + \frac{2j+1}{2G}\right) - 2f(t) + f\left(t - \frac{2j+1}{2G}\right)\right) \right] \end{aligned}$$

Since  $0 < p < 1$  then by properties of trigonometric inequality for quasi norm we have

$$\begin{aligned} \|(R_{G,j}f) - f\|_{p,\alpha}^p &\leq 2^p \left[ \frac{1}{2} \left( \|S_G^{\text{sinc}} f - f\|_{p,\alpha}^p + \|S_G^{\text{sinc}} f - f\|_{p,\alpha}^p + \left\| \Delta_{\frac{2j+1}{2G}}^2 f \right\|_{p,\alpha}^p \right) \right] \\ &\leq 2^p \left[ \|S_G^{\text{sinc}} f - f\|_{p,\alpha}^p + \frac{1}{2} \omega_2\left(f; \frac{1}{G}\right)_{p,\alpha} \right] \end{aligned}$$

By using theorem (2.1.2) we have

$$\begin{aligned} \|(R_{G,j}f) - f\|_{p,\alpha}^p &\leq 2^p \left[ \omega_2\left(f; \frac{1}{G}\right)_{p,\alpha} + \frac{1}{2} \left(1 + \frac{2j+1}{2G}\right) \omega_2\left(f; \frac{1}{G}\right)_{p,\alpha} \right] \\ &\leq \omega_2\left(f; \frac{1}{G}\right)_{p,\alpha} \left[ 2^p + \frac{1}{2} \left(1 + \frac{2j+1}{2G}\right) \right] \\ &\leq C_j \omega_2\left(f; \frac{1}{G}\right)_{p,\alpha}. \end{aligned}$$

### 3.2 Hann sampling operators

Consider Hann sampling operators  $H_{W,m}$  ( $m = 0, 1, 2, \dots$ ). The Hann kernel (see [21]  $s_{H,m}(t) = O(|t|^{-m-1})$  as  $|t| \rightarrow \infty$ . from (15) if  $\ell = 0$  we have aliner combination of sinc -function because  $H_{W,0} = \text{sinc}$ .

**Theorem 3.2.1:** For the Hann sampling operator  $H_{G,m}$  ( $m = 1, 2, \dots$ ) defined by (1) with the kernel (14). Then for  $f \in \Lambda^{p,\alpha}$ , ( $\alpha > 0, 0 < p < 1$ ) we have

$$\|H_{G,m}f - f\|_{p,\alpha}^p \leq C_m \omega_2\left(f; \frac{1}{G}\right)_{p,\alpha}. \tag{21}$$

where the constant  $C_m$  is independent of  $f$  and  $G$ .

**Proof:** According to ([20] equation (9)) we give this operator ( $H_{G,m}$ ) which has the form

$$(H_{G,m}f)(t) = \frac{1}{2} \left[ (H_{G,m-1}f)\left(t - \frac{1}{2G}\right) + (H_{G,m-1}f)\left(t + \frac{1}{2G}\right) \right]$$

Hence

$$\begin{aligned} (H_{G,m}f)(t) - f(t) &= \frac{1}{2} \left[ (H_{G,m-1}f)\left(t - \frac{1}{2G}\right) + (H_{G,m-1}f)\left(t + \frac{1}{2G}\right) \right] - f(t) \\ &= \frac{1}{2} \left[ (H_{G,m-1}f)\left(t - \frac{1}{2G}\right) - f\left(t - \frac{2j+1}{2G}\right) + f\left(t - \frac{2j+1}{2G}\right) + (H_{G,m-1}f)\left(t + \frac{1}{2G}\right) \right. \\ &\quad \left. - f\left(t + \frac{2j+1}{2G}\right) + f\left(t + \frac{2j+1}{2G}\right) - 2f(t) \right] \\ &= \frac{1}{2} \left[ (H_{G,m-1}f)\left(t - \frac{1}{2G}\right) - f\left(t - \frac{2j+1}{2G}\right) + (H_{G,m-1}f)\left(t + \frac{1}{2G}\right) \right. \\ &\quad \left. - f\left(t + \frac{2j+1}{2G}\right) + \left(f\left(t + \frac{2j+1}{2G}\right) - 2f(t) + f\left(t - \frac{2j+1}{2G}\right)\right) \right] \end{aligned}$$

Since  $0 < p < 1$  we have from properties of quasi norm,

$$\begin{aligned} \|(H_{G,m}f) - f\|_{p,\alpha}^p &\leq 2^p \left[ \frac{1}{2} \left( \|H_{G,m-1}f - f\|_{p,\alpha}^p + \|H_{G,m-1}f - f\|_{p,\alpha}^p + \left\| \Delta_{\frac{1}{2G}}^2 f \right\|_{p,\alpha}^p \right) \right] \\ &\leq 2^p \left[ \|H_{G,m-1}f - f\|_{p,\alpha}^p + \frac{1}{2} \omega_2\left(f; \frac{1}{2G}\right)_{p,\alpha} \right] \end{aligned}$$

By using induction the proof give

$$\|H_{G,m}f - f\|_{p,\alpha}^p \leq 2^p \left[ \|H_{G,0}f - f\|_{p,\alpha}^p + \frac{m}{2} \omega_2 \left( f; \frac{1}{2G} \right)_{p,\alpha} \right]$$

From (15) if we take  $\ell = 0$  then we have a linear combination of sinc-function because  $S_{H,0} = \text{sinc}$ . hence  $H_{G,0} = S_G^{\text{sinc}}$ ,

Therefore by using theorem (2.1.2) we have

$$\begin{aligned} \|H_{G,m}f - f\|_{p,\alpha}^p &\leq 2^p \left[ C_2 \omega_2 \left( f; \frac{1}{G} \right)_{p,\alpha} + \frac{m}{2} \omega_2 \left( f; \frac{1}{2G} \right)_{p,\alpha} \right] \\ &\leq C_m \left( \omega_2 \left( f; \frac{1}{2G} \right)_{p,\alpha} \right). \end{aligned}$$

### 3.3 Blackman-Harris sampling operators

Before over 52 years ([15], [16], [17], [18]) The Blackman window has been used in signal analysis. Recently Lasser and Obermaier [13] studied the role of the Blackman window for defining approximative identities in Fourier approximation.

**Theorem 3.3.1:** consider  $C_{G,a}$  be the blackman-harris sampling operator defined by (1) with the kernel (18).

Then for  $f \in \Lambda^{p,\alpha}$ , ( $\alpha > 0, 0 < p < 1$ ) we have

$$\|C_{G,a}f - f\|_{p,\alpha}^p \leq T_a \omega_2 \left( f; \frac{1}{G} \right)_{p,\alpha} \tag{22}$$

where the constant  $T_a$  is independent of  $f$  and  $G$ .

**Proof:** from (18) we show that the blackman-harris kernel is a linear combination of translated sinc-functions.

Therefore the operator  $C_{G,a}$  can be representation by

$$\begin{aligned} (C_{G,a}f)(t) &= \frac{1}{2} \sum_{j \in \mathbb{Z}} f \left( \frac{j}{G} \right) \sum_{k=0}^m a_k \left( \text{sinc}(Gt - j + k) + \text{sinc}(Gt - j - k) \right) \\ &= \frac{1}{2} \sum_{k=0}^m a_k \left[ \sum_{j \in \mathbb{Z}} f \left( \frac{j}{G} \right) \left( \text{sinc}(Gt - j + k) + \sum_{j \in \mathbb{Z}} f \left( \frac{j}{G} \right) \text{sinc}(Gt - j - k) \right) \right] \\ &= \frac{1}{2} \sum_{k=0}^m a_k \left[ (S_G^{\text{sinc}} f) \left( t + \frac{k}{G} \right) + (S_G^{\text{sinc}} f) \left( t - \frac{k}{G} \right) \right] \end{aligned}$$

Therefore

$$\begin{aligned} (C_{G,a}f)(t) - f(t) &= \frac{1}{2} \sum_{k=0}^m a_k \left[ (S_G^{\text{sinc}} f) \left( t + \frac{k}{G} \right) + (S_G^{\text{sinc}} f) \left( t - \frac{k}{G} \right) \right] - f(t) \\ &= \frac{1}{2} \sum_{k=0}^m a_k \left[ (S_G^{\text{sinc}} f) \left( t + \frac{k}{G} \right) - f \left( t + \frac{k}{G} \right) + (S_G^{\text{sinc}} f) \left( t - \frac{k}{G} \right) - f \left( t - \frac{k}{G} \right) \right] \\ &\quad + \left( f \left( t + \frac{k}{G} \right) - 2f(t) + f \left( t - \frac{k}{G} \right) \right) \end{aligned}$$

Since  $0 < p < 1$ , by properties of quasi norm we obtain

$$\begin{aligned} \|C_{G,a}f - f\|_{p,\alpha}^p &\leq 2^p \frac{1}{2} \sum_{k=0}^m |a_k| \left( \|C_{G,a}f - f\|_{p,\alpha}^p + \|C_{G,a}f - f\|_{p,\alpha}^p + \left\| \Delta_{\frac{k}{G}}^2 f \right\|_{p,\alpha}^p \right) \\ &\leq 2^p \frac{1}{2} \sum_{k=0}^m |a_k| \left( \|C_{G,a}f - f\|_{p,\alpha}^p + \frac{1}{2} \omega_2 \left( f; \frac{k}{G} \right)_{p,\alpha} \right) \end{aligned}$$

By using theorem (2.1.2) we have

$$\begin{aligned} \|C_{G,a}f - f\|_{p,\alpha}^p &\leq 2^p \sum_{k=0}^m |a_k| \left[ C \omega_2 \left( f; \frac{1}{G} \right)_{p,\alpha} + \frac{k^2}{2} \omega_2 \left( f; \frac{1}{G} \right)_{p,\alpha} \right] \\ &\leq \omega_2 \left( f; \frac{1}{G} \right)_{p,\alpha} \left[ 2^p \sum_{k=0}^m |a_k| \left( C + \frac{k^2}{2} \right) \right] \\ &\leq M_a \omega_2 \left( f; \frac{1}{G} \right)_{p,\alpha}. \end{aligned}$$

**Theorem 3.3.2:** For  $C_{G,a}$  ( $a \in \mathbb{R}^{m+1}$ ) let  $\ell, 1 \leq \ell \leq m$  be fixed. For a parameter vector  $q \in \mathbb{R}^{m-\ell+1}$ , such that we have for the kernel (13) a representation via central differences(4) inform (14),

Then for  $f \in \Lambda^{p,\alpha}$ , ( $\alpha > 0, 0 < p < 1$ ) we have

$$\|C_{G,a}f - f\|_{p,\alpha}^p \leq D_{a,\ell} \omega_{2\ell} \left( f; \frac{1}{G} \right)_{p,\alpha} \tag{22}$$

where the constant  $D_{a,\ell}$  is independent of  $f$  and  $G$ .

**Proof:** For kernel (20) we get operator  $C_{G,a}$  which is representation

$$\begin{aligned} (C_{G,a}f)(t) &= \sum_{j \in \mathbb{Z}} f\left(\frac{j}{G}\right) \left[ \text{sinc}(t) - \frac{1}{2^{2\ell+1}} \sum_{j=0}^{m-\ell} (-1)^{j+\ell} q_j [\Delta_1^{2\ell} \text{sinc}(t-j) + \Delta_1^{2\ell} \text{sinc}(t-j)] \right] \\ &= \sum_{j \in \mathbb{Z}} f\left(\frac{j}{G}\right) \text{sinc}(Gt-k) - \sum_{j \in \mathbb{Z}} f\left(\frac{j}{G}\right) \left[ \frac{1}{2^{2\ell+1}} \sum_{j=0}^{m-\ell} (-1)^{j+\ell} q_j [\Delta_1^{2\ell} \text{sinc}(t-j) + \Delta_1^{2\ell} \text{sinc}(t-j)] \right] \\ &= (S_G^{\text{sinc}} f)(t) - \frac{1}{2^{2\ell+1}} \sum_{j=0}^{m-\ell} (-1)^{j+\ell} q_j \sum_{j \in \mathbb{Z}} f\left(\frac{j}{G}\right) [\Delta_1^{2\ell} \text{sinc}(t-j) + \Delta_1^{2\ell} \text{sinc}(t-j)] \\ &= (S_G^{\text{sinc}} f)(t) - \frac{1}{2^{2\ell+1}} \sum_{j=0}^{m-\ell} (-1)^{j+\ell} q_j \left[ \Delta_1^{2\ell} \sum_{j \in \mathbb{Z}} f\left(\frac{j}{G}\right) \text{sinc}(Gt-j) + \Delta_1^{2\ell} \sum_{j \in \mathbb{Z}} f\left(\frac{j}{G}\right) \text{sinc}(Gt-j) \right] \end{aligned}$$

Therefore

$$(C_{G,a}f)(t) = (S_G^{\text{sinc}} f)(t) - \frac{1}{2^{2\ell+1}} \sum_{j=0}^{m-\ell} (-1)^{j+\ell} q_j \left[ \Delta_1^{2\ell} (S_G^{\text{sinc}} f)\left(t - \frac{j}{G}\right) + \Delta_1^{2\ell} (S_G^{\text{sinc}} f)\left(t + \frac{j}{G}\right) \right]$$

We obtain

$$\begin{aligned} (C_{G,a}f)(t) - f(t) &= (S_G^{\text{sinc}} f)(t) - \frac{1}{2^{2\ell+1}} \sum_{j=0}^{m-\ell} (-1)^{j+\ell} q_j \left[ \Delta_1^{2\ell} (S_G^{\text{sinc}} f)\left(t - \frac{j}{G}\right) + \Delta_1^{2\ell} (S_G^{\text{sinc}} f)\left(t + \frac{j}{G}\right) \right] - f(t) \\ &= (S_G^{\text{sinc}} f)(t) - f(t) - \frac{1}{2^{2\ell+1}} \sum_{j=0}^{m-\ell} (-1)^{j+\ell} q_j \left[ \Delta_1^{2\ell} \left( (S_G^{\text{sinc}} f)\left(t - \frac{j}{G}\right) - f\left(t - \frac{j}{G}\right) + f\left(t - \frac{j}{G}\right) \right) \right. \\ &\quad \left. + \Delta_1^{2\ell} \left( (S_G^{\text{sinc}} f)\left(t + \frac{j}{G}\right) - f\left(t + \frac{j}{G}\right) + f\left(t + \frac{j}{G}\right) \right) \right] \\ &= (S_G^{\text{sinc}} f) - \frac{1}{2^{2\ell+1}} \sum_{j=0}^{m-\ell} (-1)^{j+\ell} q_j \left[ \Delta_1^{2\ell} \left( (S_G^{\text{sinc}} f)\left(t - \frac{j}{G}\right) - f\left(t - \frac{j}{G}\right) \right) \right. \\ &\quad \left. + \Delta_1^{2\ell} \left( (S_G^{\text{sinc}} f)\left(t + \frac{j}{G}\right) - f\left(t + \frac{j}{G}\right) \right) \right] \\ &\quad - \frac{1}{2^{2\ell+1}} \sum_{j=0}^{m-\ell} (-1)^{j+\ell} q_j \left[ \left( \Delta_1^{2\ell} f\left(t - \frac{j}{G}\right) \right) + \left( \Delta_1^{2\ell} f\left(t + \frac{j}{G}\right) \right) \right] \end{aligned}$$

Since  $0 < p < 1$  from properties of quasi norm we obtain

$$\begin{aligned} \|C_{G,a}f - f\|_{p,\alpha}^p &\leq 2^p \left[ \|S_G^{\text{sinc}} f - f\|_{p,\alpha}^p + \sum_{j=0}^{m-\ell} |q_j| \|S_G^{\text{sinc}} f - f\|_{p,\alpha}^p + \frac{1}{2^{2\ell}} \sum_{j=0}^{m-\ell} |q_j| \left\| \Delta_1^{2\ell} f \right\|_{p,\alpha}^p \right] \\ &\leq 2^p \left[ \left(1 + \sum_{j=0}^{m-\ell} |q_j|\right) \left( \|S_G^{\text{sinc}} f - f\|_{p,\alpha}^p \right) + \frac{1}{2^{2\ell}} \sum_{j=0}^{m-\ell} |q_j| \left( \omega_{2\ell} \left( f; \frac{1}{G} \right)_{p,\alpha} \right) \right] \end{aligned}$$

By using theorem (2.1.2) we have

$$\begin{aligned} \|C_{G,a}f - f\|_{p,\alpha}^p &\leq 2^p \left( 1 + \sum_{j=0}^{m-\ell} |q_j| + \frac{1}{2^{2\ell}} \sum_{j=0}^{m-\ell} |q_j| \right) \omega_{2\ell} \left( f; \frac{1}{G} \right)_{p,\alpha} \\ &\leq D_{a,\ell} \omega_{2\ell} \left( f; \frac{1}{G} \right)_{p,\alpha}. \end{aligned}$$

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**Source of support: Nil, Conflict of interest: None Declared**

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