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PERIODIC SOLUTION OF INTEGRO-DIFFERENTIAL EQUATIONS WITH OPERATORS

Raad N. Butris*[1], Dawoud S.Abdullah[2]

[1]Faculty of Educational Science School of Basic Education, Mathematics Department - University Of Duhok.

[2]Faculty of Science-University of Zahko.

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ABSTRACT

In this paper, we investigate the existence, uniqueness and stability of a periodic solution of integro-differential equations with the operators by using the method of Samoilenko. These investigations lead us to improving and extending the above method. . Thus the integro-differential equations with the operators are more general and detailed than those introduced by Butris.

Keywords: *Numerical-analytic method, nonlinear system, existence, uniqueness and stability of periodic solution, integro-differential equations with the operators.*

I. INTRODUCTION

The theory of integro- differential equations has been of great interest for many years. It plays an important role in different subjects, such as physics, biology, chemistry, etc, and the study of periodic solutions for non-linear system of integro- differential equations is very important branch in the integro- differential equations theory [1, 2, 3, 8, 13, 14]. Many results about the existence, uniqueness and stability of periodic solutions for system of non–linear integro- differential equations have been obtained by the numerical analytic methods that were proposed by Samoilenko [12] which had been later applied in many studies [5,6,7,9,10].

Butris [3] used numerical–analytic method for investigating a periodic solution for studying the periodic existence and uniqueness solutions of integro–differential equations which has the form

$$\frac{dx(t)}{dt} = f\left(t, x(t), \int_t^{t+T} g\left(s, x(s)\right) ds\right)$$

where $x \in D \subseteq R^n$, D is a closed and bounded domain.

In this paper, we investigate the existence, uniqueness and stability of periodic solution of integro-differential equations with the operators by using the method of Samoilenko. [12].

Consider the following problem:

$$\frac{dx}{dt} = f\left(t, x, Ax, \int_0^{h(t)} g\left(s, x(s), Bx(s)\right) ds\right) \tag{1}$$

Suppose that the vector functions:

$f(t,x,y,z) = (\, f_1(t,x,y,z),\ f_2(t,x,y,z) \dots\ f_n(t,x,y,z))$
$g(t,x,w) = (\, g_1(t,x,y,z),\ g_2(t,x,y,z) \dots\ g_n(t,x,y,z))$

and defined on the domains:

$$\left.\begin{aligned} &(t,x,y,z) \in R^1 \times D \times D_1 \times D_2 = (-\infty,\infty) \times D \times D_1 \times D_2 \\ &(t,x,w) \in R^1 \times D \times D^* = (-\infty,\infty) \times D \times D^* \end{aligned}\right\} \tag{2}$$

Corresponding Author: Raad N. Butris*[1]
[1]Faculty of Educational Science School of Basic Education
Mathematics department - University of Duhok.

which are continuous vector functions in t, x, y, z, w and periodic in t of a period T.

where $x \in D \subset R^n$, D is compact domain subset of Euclidean space R^n and D_1, D_2, D^* are bounded domains subset of Euclidean space R^m.

Assume that the vector functions $f(t, x, y, z)$ and $g(t, x, w)$ satisfy the following inequalities:

$$\|f(t,x,y,z)\| \le M_1, \ \|g(t,x,w)\| \le M_2, \tag{3}$$

$$\|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\| \le K_1\|x_1 - x_2\| + K_2\|y_1 - y_2\| + K_3\|z_1 - z_2\| \tag{4}$$

$$\|g(t,x_1,w_1) - g(t,x_2,w_2)\| \le P_1\|x_1 - x_2\| + P_2\|w_1 - w_2\| \tag{5}$$

$$\| h(t) \| \le h < \infty \tag{6}$$

$$\|Ax_1 - Ax_2\| \le Q_1\|x_1 - x_2\| \tag{7}$$

$$\|Bx_1 - Bx_2\| \le Q_2\|x_1 - x_2\| \tag{8}$$

for all $t \in R^1$, $(x, x_1, x_2) \in D$, $(y, y_1, y_2) \in D_1$, $(z, z_1, z_2) \in D_2$.

for all $t \in R^1$, $x, x_1, x_2 \in D$, $y, y_1, y_2 \in D_1$, $z, z_1, z_2 \in D_2$, $w, w_1, w_2 \in D^*$.

where $M_1, M_2, K_1, K_2, P_1, P_2, h$, are positive constants, A and B are operators

where $A : R^1 \longrightarrow R^1$ and also $B : R^1 \longrightarrow R^1$.

We define the non-empty sets as follows:

$$\left.\begin{aligned} D_f &= D - \frac{T}{2} M_1 \\ D_{1f} &= D_1 - \frac{T}{2} Q_1 M_1 \\ D_{2f} &= D_2 - \frac{T}{2} h M_1 (P_1 + P_2 Q_2) \end{aligned}\right\} \tag{9}$$

Furthermore, we assume that the following condition holds:

$$q = \frac{T}{2}[K_1 + K_2 Q_1 + K_3 h(P_1 + P_2 Q_2)] < 1. \tag{10}$$

Lemma 1: Let f(t) be a continuous vector function in the interval $0 \le t \le T$. Then

$$\left\| \int_0^t \left(f(s) - \frac{1}{T}\int_0^T f(s)ds\right)ds \right\| \le \alpha(t) \max_{t \in [0,T]} \|f(t)\|,$$

Where $\alpha(t) = 2t(1 - \frac{t}{T})$. (For the proof see [12]).

II. APPROXIMATE PERIODIC SOLUTION

The study of the approximate periodic solution of problem (1) be introduced by the following theorem:

Theorem 1: Let t the function $f(t, x, y, z)$ and $g(t, x, w)$ be defined and continuous on the domain (2), satisfy the inequalities (3) to (8) and the condition (10). Then there exist a sequence of functions.

$$\begin{aligned} x_{m+1}(t,x_0) = {} & x_0 + \int_0^t f\left(s, x_m(s,x_0), Ax_m(s,x_0), \int_0^{h(s)} g\big(\tau, x_m(\tau,x_0), Bx_m(\tau,x_0)\big)d\tau\right)ds \\ & - \frac{1}{T}\int_0^T f\left(s, x(s,x_0), Ax(s,x_0), \int_0^{h(s)} g\big(\tau, x(\tau,x_0), Bx(\tau,x_0)\big)d\tau\right) ds \end{aligned} \tag{11}$$

with $x_0(t, x_0) = x_0$, m=0,1,2,…periodic in t of period T ,convergent uniformly as $m \to \infty$ in the domain

$$(t, x_0) \in [0,T] \times D_f \tag{12}$$

to the limit function $x^0(t, x_0)$ which is defined on the domain (2) and satisfy the following integral equation:

$$\begin{aligned} x(t,x_0) = {} & x_0 + \int_0^t f\left(s, x(s,x_0), Ax(s,x_0), \int_0^{h(s)} g(\tau, x(\tau,x_0), Bx(\tau,x_0))d\tau\right)ds \\ & - \frac{1}{T}\int_0^T f\left(s, x(s,x_0), Ax(s,x_0), \int_0^{h(s)} g\big(\tau, x(\tau,x_0), Bx(\tau,x_0)\big)d\tau\right)ds \end{aligned} \tag{13}$$

which is a periodic solution of the problem (1). Provided that:

$$\|x^0(t,x_0) - x_0\| \le \frac{M_1 T}{2} \tag{14}$$

and
$\|x^0(t,x_0) - x_m(t,x_0)\| \le q^m\ (1-q)^{-1} M_1,$ (15)
for all $m \ge 1$ and $t \in R^1$.

Proof: By the lemma (1) and using the sequence of functions (11) when m = 0, we get:

$$\|x_1(t,x_0) - x_0\| = \Big\| x_0 + \int_0^t f(s,x_0,Ax_0,\int_0^{h(s)} g(\tau,x_0,Bx_0)d\tau)ds - \frac{1}{T}\int_0^T f(s,x(s,x_0),Ax(s,x_0),\int_0^{h(s)} g\big(\tau,x(\tau,x_0),Bx(\tau,x_0)\big)d\tau)\,ds - x_0 \Big\|$$

$$\le (1-\tfrac{t}{T})\int_0^t \Big\| f(s,x_0,Ax_0,\int_0^{h(s)} g(\tau,x_0,Bx_0)d\tau \Big\| ds + \frac{t}{T}\int_t^T \Big\| f(s,x(s,x_0),Ax(s,x_0),\int_0^{h(s)} g\big(\tau,x(\tau,x_0),Bx(\tau,x_0)\big)d\tau) \Big\| ds$$

So that
$\|x_1(t,x_0) - x_0\| \le \alpha(t)\, M_1$
and hence
$$\|x_1(t,x_0) - x_0\| \le \frac{T}{2}\, M_1.$$

Therefore, $x_1(t,x_0) \in D$, for all $t \in [0,T]$.

Then by mathematical induction we can prove that:
$\|x_m(t,x_0) - x_0\| \le \frac{T}{2}\, M_1$ (16)

From (16) we obtain the estimate
$$\|Ax_m(t,x_0) - A\,x_0\| \le \frac{T}{2}\, Q_1 M_1$$

which given $x_m(t,x_0) \in D$, $Ax_m(t,x_0) \in D_1$ for all $t \in [0,T]$ and $x_0 \in D_f$, $Ax_0(t,x_0) \in D_{1f}$.

Now taking

$$\|z_1(t,x_0) - z_0(t,x_0)\| = \Big\| \int_0^{h(t)} g\big(s,x_1(s,x_0),,Bx_1(s,x_0)\big)ds - \int_0^{h(t)} g(s,x_0,Bx_0)ds \Big\|$$
$$\le \int_0^{h(t)} \big\| g\big(s,x_1(s,x_0),\ Bx_1(s,x_0)\big) - g(s,x_0,Bx_0) \big\|\ ds$$
$$\le \int_0^{h(t)} \|\ [\ P_1\|x_1 - x_0\| + P_2\|Bx_1 - Bx_0\|\ ]\|\ ds$$
$$\le \frac{T}{2} h M_1\ (\ P_1 + P_2 Q_2\ ).$$

That is $z_1(t,x_0) \in D_2$ for all $t \in [0,T]$ and $z_0 \in D_{2f}$.

Then, by mathematical induction we can prove that:
$$\|z_m(t,x_0) - z_0(t,x_0)\| \le \frac{T}{2} h M_1\ (\ P_1 + P_2 Q_2\ )$$

That is $z_m(t,x_0) \in D_2$ for all $t \in [0,T]$ and $z_0 \in D_{2f}$.

Now, we shall prove that the sequence of functions (11) converges uniformly on the domain (2). By the lemma (1) and using the sequence of functions (11) when m = 1, we get:

$$\|x_2(t,x_0) - x_1(t,x_0)\| \le \Big(1-\frac{t}{T}\Big)\int_0^t \Big\| f(s,x_1(s,x_0),Ax_1(s,x_0),\int_0^{h(s)} g\big(\tau,x_1(\tau,x_0),Bx_1(\tau,x_0)\big)d\tau - f(s,x_0,Ax_0,\int_0^{h(s)} g(\tau,x_0,Bx_0)d\tau)ds \Big\| ds + \frac{t}{T}\int_t^T \| f(s,x_1(s,x_0),Ax_1(s,x_0), \int_0^{h(s)} g\big(\tau,x_1(\tau,x_0),Bx_1(\tau,x_0)\big)d\tau - f(s,x_0,Ax_0,\int_0^{h(s)} g(\tau,x_0,Bx_0)d\tau \Big\| ds$$
$$\le \Big(1-\frac{t}{T}\Big)\int_0^t [\ K_1\|x_1(s,x_0) - x_0\| + K_2\, Q_1\|x_1(s,x_0) - x_0\| + K_3\Big(\int_0^{h(s)} g\big(\tau,x_1(\tau,x_0),Bx_1(\tau,x_0)\big)d\tau - \int_0^{h(s)} g(\tau,x_0,Bx_0)d\tau\Big)]ds$$
$$+ \frac{t}{T}\int_t^T [\ K_1\|x_1(s,x_0) - x_0\| + K_2\, Q_1\|x_1(s,x_0) - x_0\| + K_3\Big(\int_0^{h(s)} g\big(\tau,x_1(\tau,x_0),Bx_1(\tau,x_0)\big)d\tau - \int_0^{h(s)} g(\tau,x_0,Bx_0)d\tau\Big)]ds$$
$$\le \alpha(t)\ [\ K_1 + K_2Q_1 + K_3 h(\ P_1 + P_2Q_2)\ ]\ \|x(t,x_0) - x_0\|$$
$$\le \frac{T}{2}\ [\ K_1 + K_2Q_1 + K_3 h(\ P_1 + P_2Q_2)\ ]\ \|x(t,x_0) - x_0\|$$
$$\le q\ \|x_1(t,x_0) - x_0\| \qquad (17)$$

By mathematical induction and by (17) the following inequality holds:
$\|x_{m+1}(t,x_0) - x_m(t,x_0)\| \le q^m \|x_1(t,x_0) - x_0\|$. (18)

We can clue that from m ≥ 0, we have the following inequality:

$$\|x_{m+p}(t,x_0) - x_m(t,x_0)\| \le \|x_{m+p}(t,x_0) - x_{m+p-1}(t,x_0)\| + \|x_{m+p-1}(t,x_0) - x_{m+p-2}(t,x_0)\| + \ldots + \|x_{m+1}(t,x_0) - x_m(t,x_0)\|$$

$$\le q^{m+p-1}\|x_1(t,x_0) - x_0\| + \|x_1(t,x_0) - x_0\| + \ldots + q^m\|x_1(t,x_0) - x_0\|$$

Therefore
$\|x_{m+p}(t,x_0) - x_m(t,x_0)\| \le q^m (1-q)^{-1} \|x_1(t,x_0) - x_0\|$ (19)
for all $t \in [0,T]$ and $x_0 \in D_f$.

Since $q < 1$ and $\lim_{m\to\infty} q^m = 0$, So that the right side of (19) tends to zero and thus sequence of functions (11) is convergent uniformly on the domain (2) to the limit function $x^0(t,x_0)$ which is defined on the same domain.

Let
$\lim_{m\to\infty} x_m(t,x_0) = x^0(t,x_0)$ (20)

Now, we show that $x^0(t,x_0) \in D$, for all $t \in [0,T]$

By using (19) and (20), that is:

$$\Big\| \int_0^t f\Big(s, x_m(s,x_0), Ax_m(s,x_0), \int_0^{h(s)} g\big(\tau, x_m(\tau,x_0), Bx_m(\tau,x_0)\big)d\tau\Big)ds - \frac{1}{T}\int_0^T f\Big(s, x_m(s,x_0), Ax_m(s,x_0), \int_0^{h(s)} g\big(\tau, x_m(\tau,x_0), Bx_m(\tau,x_0)\big)d\tau\Big)ds - \int_0^t f\Big(s, x^0(s,x_0), Ax^0(s,x_0), \int_0^{h(s)} g\big(\tau, x^0(\tau,x_0), Bx^0(\tau,x_0)\big)d\tau\Big)ds - \frac{1}{T}\int_0^T f\Big(s, x^0(s,x_0), Ax^0(s,x_0), \int_0^{h(s)} g\big(\tau, x^0(\tau,x_0), Bx^0(\tau,x_0)\big)d\tau\Big)ds \Big\|$$

$$\le \Big(1-\frac{t}{T}\Big)\int_0^t [K_1\|x_m(s,x_0) - x^0(s,x_0)\| + K_2 Q_1\|x_m(s,x_0) - x^0(s,x_0)\| + K_3 h\,(P_1\|x_m(s,x_0) - x^0(s,x_0)\| - P_2Q_2\|x_m(s,x_0) - x^0(s,x_0)\|)]\,ds + \frac{t}{T}\int_t^T [K_1\|x_m(s,x_0) - x^0(s,x_0)\| + K_2 Q_1\|x_m(s,x_0) - x^0(s,x_0)\| + K_3 h\,(P_1\|x_m(s,x_0) - x^0(s,x_0)\| - P_2Q_2\|x_m(s,x_0) - x^0(s,x_0)\|)]\,ds$$

$$\le \alpha(t)\,[K_1 + K_2Q_1 + K_3h(P_1 + P_2Q_2)]\,\|x_m(s,x_0) - x^0(s,x_0)\|$$

$$\le \frac{T}{2}[K_1 + K_2Q_1 + K_3h(P_1 + P_2Q_2)]\,\|x_m(s,x_0) - x^0(s,x_0)\|$$

From (20) we have $\|x_m(s,x_0) - x^0(s,x_0)\| \le \epsilon_1$

Thus

$$\Big\| \int_0^t f\Big(s, x_m(s,x_0), Ax_m(s,x_0), \int_0^{h(s)} g\big(\tau, x_m(\tau,x_0), Bx_m(\tau,x_0)\big)d\tau\Big)ds - \frac{1}{T}\int_0^T f\Big(s, x_m(s,x_0), Ax_m(s,x_0), \int_0^{h(s)} g\big(\tau, x_m(\tau,x_0), Bx_m(\tau,x_0)\big)d\tau\Big)ds - \int_0^t f\Big(s, x^0(s,x_0), Ax^0(s,x_0), \int_0^{h(s)} g\big(\tau, x^0(\tau,x_0), Bx^0(\tau,x_0)\big)d\tau\Big)ds - \frac{1}{T}\int_0^T f\Big(s, x^0(s,x_0), Ax^0(s,x_0), \int_0^{h(s)} g\big(\tau, x^0(\tau,x_0), Bx^0(\tau,x_0)\big)d\tau\Big)ds \Big\|$$

$$\le \frac{T}{2}[K_1 + K_2Q_1 + K_3h(P_1 + P_2Q_2)]\,\frac{\epsilon}{\frac{T}{2}[K_1 + K_2Q_1 + K_3h(P_1 + P_2Q_2)]}$$

$$\le \epsilon\,,\ \textit{for all}\ \ m \ge 0\ ,\ \textit{Putting}\ \epsilon_1 = \frac{\epsilon}{\frac{T}{2}[K_1 + K_2Q_1 + K_3h(P_1 + P_2Q_2)]}$$

So that

$$\lim_{m\to\infty}\int_0^t f\Big(s, x_m(s,x_0), Ax_m(s,x_0), \int_0^{h(s)} g\big(\tau, x(\tau,x_0), Bx(\tau,x_0)\big)d\tau\Big)ds = \int_0^t f\Big(s, x^0(s,x_0), Ax^0(s,x_0), \int_0^{h(s)} g\big(\tau, x^0(\tau,x_0), Bx^0(\tau,x_0)\big)d\tau\Big)ds$$

So $x^0(t,x_0) \in D$ and is a periodic solution of the integral equation (13) and hence $x^0(t,x_0) = x(t,x_0)$, that is $x(t,x_0)$ is a periodic solution of the problem (1).

Moreover, by the hypotheses and conditions to the theorem the inequality (14) and (15) are satisfied for all $m \ge 1$.

III. UNIQUENESS PERIODIC SOLUTION

The study of the uniqueness periodic solution of problem (1) is introduced by:

Theorem 2: If the right side of problem (1) satisfying all conditions and inequalities of theorem1.Then there exists a unique continuous periodic solution of the problem (1).

Proof: Let $y(t,x_0)$ be another periodic solution of (*1*), that is

$$y(t,x_0) = x_0 + \int_0^t f(s, y(s,x_0), Ay(s,x_0), \int_0^{h(s)} g(\tau, y(\tau,x_0), By(\tau,x_0))d\tau)ds - \frac{1}{T}\int_0^T f(s, y(s,x_0), Ay(s,x_0), \int_0^{h(s)} g(\tau, y(\tau,x_0), By(\tau,x_0))d\tau)\, ds$$

and hence

$$\|x(t,x_0) - y(t,x_0)\| = \Big\| x_0 + \int_0^t f(s, x(s,x_0), Ax(s,x_0), \int_0^{h(s)} g(\tau, x(\tau,x_0), Bx(\tau,x_0))d\tau)ds - \frac{1}{T}\int_0^T f(s, x(s,x_0), Ax(s,x_0), \int_0^{h(s)} g(\tau, x(\tau,x_0), Bx(\tau,x_0))d\tau)\, ds - x_0 - \int_0^t f(s, y(s,x_0), Ay(s,x_0), \int_0^{h(s)} g(\tau, y(\tau,x_0), By(\tau,x_0))d\tau)ds - \frac{1}{T}\int_0^T f(s, y(s,x_0), Ay(s,x_0), \int_o^{h(s)} g(\tau, y(\tau,x_0), By(\tau,x_0))d\tau)ds \Big\|$$

$$\leq \left(1 - \frac{t}{T}\right)\int_0^t \Big\| f(s, x(s,x_0), Ax(s,x_0), \int_0^{h(s)} g(\tau, x(\tau,x_0), Bx(\tau,x_0))d\tau - f(s, y(s,x_0), Ay(s,x_0), \int_o^{h(s)} g(\tau, y(\tau,x_0), By(\tau,x_0))d\tau \Big\|\, ds + \frac{t}{T}\int_t^T \Big\| f(s, x(s,x_0), Ax(s,x_0), \int_0^{h(s)} g(\tau, x(\tau,x_0), Bx(\tau,x_0))d\tau - f(s, y(s,x_0), Ay(s,x_0), \int_o^{h(s)} g(\tau, y(\tau,x_0), By(\tau,x_0))d\tau \Big\|\, ds$$

$$\leq \alpha(t)\,[\,K_1 + K_2Q_1 + K_3h(\,P_1 + P_2Q_2)\,]\,\|x(t,x_0) - y(t,x_0)\|$$

$$\leq \frac{T}{2}[\,K_1 + K_2Q_1 + K_3h(\,P_1 + P_2Q_2)\,]\,\|x(t,x_0) - y(t,x_0)\|$$

$$\leq q\,\|x(t,x_0) - y(t,x_0)\|$$

Since $q < 1$, then

$$\|x(t,x_0) - y(t,x_0)\| < \|x(t,x_0) - y(t,x_0)\|$$

That is contradiction.

So that:

$$\|x(t,x_0) - y(t,x_0)\| \longrightarrow 0$$

Thus

$$x(t,x_0) = y(t,x_0)$$

Therefore, $x(t,x_0)$ is a unique continuous periodic solution on the domain (2) of the problem (1).

V. EXISTENCE OF PERIODIC SOLUTION

The problem of existence of periodic solution of (1) is uniquely connected with the existence of zeros of the function $\Delta\,(t,x_0)$ which has the form:

$$\Delta: D_f \longrightarrow R^1$$

$$\Delta\,(0,x_0) = \frac{1}{T}\int_0^T f(s, x^0(s,x_0), Ax^0(s,x_0), \int_0^{h(s)} g(\tau, x^0(\tau,x_0), Bx^0(\tau,x_0))d\tau)\, ds \quad (21)$$

where $x^0(t,x_0)$ is the limiting function of (11) and the equation (21) is approximation determined from the sequence of functions:

$$\Delta_m: D_f \longrightarrow R^1$$

$$\Delta_m(0,x_0) = \frac{1}{T}\int_0^T f(s, x_m(s,x_0), Ax_m(s,x_0), \int_0^{h(s)} g(\tau, x_m(\tau,x_0), Bx_m(\tau,x_0))d\tau)\, ds \quad (22)$$

where m=0,1,2,…

Theorem 3: Under the hypothesis of theorem1 and 2, the following inequality:

$$\|\Delta(0,x_0)-\Delta_m(0,x_0)\| \le q^{\,m+1}(1-q)^{-1} M_1 \tag{23}$$

Is hold for all $m \ge 0$, $x_0 \in D_f$.

Proof: From (21) and (22), we have the estimate:

$$\begin{aligned}\|\Delta(0,x_0)-\Delta_m(0,x_0)\| &\le \frac{1}{T}\int_0^T \|f(s,x^0(t,x_0),Ax^0(t,x_0),\\ &\quad \int_0^{h(s)} g(\tau,x^0(\tau,x_0),Bx^0(\tau,x_0))\,d\tau - f(s,x_m(s,x_0),Ax_m(s,x_0),\\ &\quad \int_0^{h(s)} g\big(\tau,x_m(\tau,x_0),Bx_m(\tau,x_0)\big)\,ds\Big\|\;ds\\ &\le \frac{1}{T}\int_0^T [K_1\,\|x^0(s,x_0)-x_m(s,x_0)\| + K_2\,Q_1\|x^0(s,x_0)-x_m(s,x_0)\|\\ &\quad +K_3h(P_1-P_1Q_2)\|x^0(s,x_0)-x_m(s,x_0)\|]\,ds\\ &\le \frac{T}{2}\,[\,K_1+K_2Q_1+K_3h(\,P_1+\,P_2Q_2)\,]\|x^0(t,x_0)-x_m(t,x_0)\|.\end{aligned}$$

From (15) we get:

$$\|\Delta(0,x_0)-\Delta_m(0,x_0)\| \le \frac{T}{2}\,[\,K_1+K_2Q_1+K_3h(\,P_1+\,P_2Q_2)\,]\quad q^{\,m}(1-q)^{-1} M_1$$

Since $q = \frac{T}{2}\,[\,K_1+K_2Q_1+K_3h(\,P_1+\,P_2Q_2)\,]$, then the above inequality can be written as:

$$\|\Delta(0,x_0)-\Delta_m(0,x_0)\| \le q^{\,m+1}(1-q)^{-1} M_1$$

Thus the inequality (23) holds for all $m \ge 0$.

By using theorem 3, we can state and proof the following theorem.

Theorem 4: Let the functions $f(t,x,y,z)$ and $g(t,x,w)$ be defined on the domain $G=\{0\le s\le t\le T, a\le x\le b, e\le y,z\le f\}\subseteq R^1$, suppose that the sequence of functions $\Delta_m(0,x_0)$ is defined as in (4.22) and satisfies the inequalities:

$$\left.\begin{aligned} &min_{a+P_1\le x_0\le b-P_1}\,\Delta_m(0,x_0) \le -\eta_m,\\ &max_{a+P_1\le x_0\le b-P_1}\,\Delta_m(0,x_0) \le \eta_m.\end{aligned}\right\} \tag{24}$$

for all $m \ge 0$, where $P_1 = M(h_2-h_1)$ and $\eta_m = \|\Omega^{m+1}(1-\Omega)^{-1}M\|$

Then the system (1) has periodic solution $x = x(t,x_0)$ for which $x_0 \in [a+P_1, b-P_1]$.

Proof: Let x_1, x_2 be any points in the interval $x_0 \in [a+P_1, b-P_1]$ such that:

$$\left.\begin{aligned} &\Delta_m(0,x_1) = min_{a+P_1\le x_0\le b-P_1}\,\Delta_m(0,x_0),\\ &\Delta_m(0,x_2) = max_{a+P_1\le x_0\le b-P_1}\,\Delta_m(0,x_0),\end{aligned}\right\} \tag{25}$$

From the inequalities (23) and (24), we have:

$$\left.\begin{aligned} &\Delta(0,x_1) = \Delta_m(0,x_1) + [\Delta(0,x_1)-\Delta_m(0,x_1)] \le 0,\\ &\Delta(0,x_2) = \Delta_m(0,x_2) + [\Delta(0,x_2)-\Delta_m(0,x_2)] \ge 0.\end{aligned}\right\} \tag{26}$$

It follows from (26) and the continuity of the function $\Delta(0,x_0)$, that there exists an isolated singular point $x^0, x^0 \in [x_1,x_2]$, such that $\Delta(0,x^0)=0$. This means that the system (1) has a periodic solution $x = x(t,x_0)$ for which $x_0 \in [a+P_1, b-P_1]$.

VI. STABILITY OF PERIODIC SOLUTION

In this section, we prove a theorem on stability of periodic solution for the problem (1).

Theorem 5: If the function $\Delta(0,x_0)$ be defined by (21), where $x^0(t,x_0)$ is a limit function of $\{x_m(0,x_0)\}_{m=0}^{\infty}$, then the following inequalities:

$\|\Delta(0,x_0)\| \le M_1$ and $\|\Delta(0,x_0^1)-\Delta(0,x_0^1)\| \le \frac{2}{T}\,q(1-q)^{-1}\|x_0^1(t)-x_0^2(t)\|$ are holds for all $x^0, x_0^1, x_0^2 \in D_f$.

Proof: From the equation (21), we get:

$$\begin{aligned}\|\Delta(0,x_0)\| &\le \frac{1}{T}\int_0^T \|f(s,x^0(s,x_0),Ax^0(s,x_0),\; \int_0^{h(s)} g(\tau,x^0(\tau,x_0),Bx^0(\tau,x_0))\,d\tau\Big\|\,ds\\ &\le \frac{1}{T}\int_0^T M_1\,ds\\ &\le M_1\end{aligned}$$

And by using (22), we find that:

$$\begin{aligned}\|\Delta(0,x_0^1)-\Delta(0,x_0^2)\| &\le \tfrac{1}{T}\int_0^T \|f(s,x^0(s,x_0^1),Ax^0(s,x_0^1),\\ &\quad \int_0^{h(s)} g(\tau,x^0(\tau,x_0^1),Bx^0(\tau,x_0^1))d\tau)\,ds - f(s,x^0(t,x_0^2),Ax^0(t,x_0^2),\\ &\quad \int_0^{h(s)} g(\tau,x^0(\tau,x_0^2),Bx^0(\tau,x_0^2))\,d\tau\|\,ds\\ &\le \tfrac{1}{T}\int_0^T (\,K_1\|x^0(s,x_0^1)-x^0(s,x_0^2)\| + K_2Q_1\|x^0(s,x_0^1)-x^0(s,x_0^2)\|\\ &\quad + K_3h(\,P_1+\,P_2Q_2)\|x^0(s,x_0^1)-x^0(s,x_0^2)\|)ds\\ &\le [\,K_1+K_2Q_1+K_3h(\,P_1+\,P_2Q_2)\,]\|x^0(t,x_0^1)-x^0(t,x_0^2)\|\end{aligned}$$

Hence

$$\|\Delta(0,x_0^1)-\Delta(0,x_0^2)\| \le \frac{2}{T}\,q\,\|x^0(t,x_0^1)-x^0(t,x_0^2)\| \tag{27}$$

where $x^0(t,x_0^1)$, $x^0(t,x_0^2)$ are the solutions of the integral equation:

$$\begin{aligned}x(t,x_0^k) = x_0^k(t) + \int_0^t f(s,x^0(s,x_0^k),Ax^0(s,x_0^k),\\ \int_0^{h(s)} g(\tau,x^0(\tau,x_0^k),Bx^0(\tau,x_0^k))\,d\tau - \tfrac{1}{T}\int_0^T f(s,x^0(s,x_0^k),Ax^0(s,x_0^k),\\ \int_0^{h(s)} g(\tau,x^0(\tau,x_0^k),Bx^0(\tau,x_0^k))d\tau)\,ds\end{aligned} \tag{28}$$

where $k = 1, 2$.

Now, by using (28), we have:

$$\begin{aligned}\|x^0(t,x_0^1)-x^0(t,x_0^2)\| &\le \|x_0^1(t)-x_0^2(t)\| + (1-\frac{t}{T})\int_0^t (K_1\|x^0(s,x_0^1)-x^0(s,x_0^2)\|\\ &\quad +K_2Q_1\|x^0(s,x_0^1)-x^0(s,x_0^2)\| + K_3h(\,P_1+\,P_2Q_2)\|x^0(s,x_0^1)-x^0(s,x_0^2)\|ds\\ &\quad +\tfrac{t}{T}\int_t^T K_1\|x^0(s,x_0^1)-x^0(s,x_0^2)\|+K_2Q_1\|x^0(s,x_0^1)-x^0(s,x_0^2)\|\\ &\quad +K_3h(\,P_1+\,P_2Q_2)\|x^0(t,x_0^1)-x^0(t,x_0^2)\|)ds\\ &\le \tfrac{T}{2}[\,K_1+K_2Q_1+K_3h(\,P_1+\,P_2Q_2)\,]\|x^0(t,x_0^1)-x^0(t,x_0^2)\|\\ &\le \|x_0^1(t)-x_0^2(t)\| + q\,\|x^0(t,x_0^1)-x^0(t,x_0^2)\|\end{aligned}$$

So that:

$$\|x^0(t,x_0^1)-x^0(t,x_0^2)\| \le (1-q)^{-1}\|x_0^1(t)-x_0^2(t)\| \tag{29}$$

By substituting inequality (4.27) in (4.29), we get

$\|\Delta(0,x_0^1)-\Delta(0,x_0^1)\| \le \frac{2}{T}\,q(1-q)^{-1}\|x_0^1(t)-x_0^2(t)\|.$ *for all* $x_0^1, x_0^2 \in D_f$.

VII. BANACH FIXED POINT THEOREM

In this section, we prove the existence and uniqueness theorem for the problem (1) by using Banach fixed point theorem [11].

Theorem 6: Let the functions $f(t,x,y,z)$ and $g(t,x,w)$ in the problem (1) are defined and continuous on the domain (2) periodic in t of period $T > 0$ and satisfies assumptions and conditions of theorem 1, Then the problem (1) has a unique continuous periodic solution on the domain (2).

Proof: Let (C[0,T], ‖.‖) be a Banach space and T^* be a mapping on C[0,T] as follows:

$$\begin{aligned}T^*x(t,x_0) = x_0 + \int_0^t f(s,x(s,x_0),Ax(s,x_0),\int_0^{h(s)} g(\tau,x(\tau,x_0),Bx(\tau,x_0))d\tau)ds\\ -\tfrac{1}{T}\int_0^T f(s,x(s,x_0),Ax(s,x_0),\int_0^{h(s)} g(\tau,x(\tau,x_0),Bx(\tau,x_0))d\tau)\,ds\end{aligned}$$

Since $x(t,x_0)$ is continuous on the domain (2), then $\int_0^{h(t)} g(s,x(s,x_0),Bx(s,x_0))ds$ is also continuous on the same domain.

Thus we get:

$\int_0^t f(s,x(s,x_0),Ax(s,x_0),\int_0^{h(s)} g(\tau,x(\tau,x_0),Bx(\tau,x_0))d\tau)ds$ is continuous on the domain (2)

i.e.: $T^* : C[0,T] \longrightarrow C[0,T]$

Now, we shall prove that T* is a contraction mapping on C[0,T] .

Let $x(t, x_0)$ and $z(t, x_0)$ be any vector functions on C[0,T], then

$$\| T^*x(t,x_0) - T^*z(t,x_0) \| = \max_{t\in[0,T]} \{|T^*x(t,x_0) - T^*z(t,x_0)|\}$$

$$\begin{aligned}
&\le \max_{t\in[0,T]} \{ |x_0 + \int_0^t f(s,x(s,x_0),Ax(s,x_0),\int_0^{h(s)} g(\tau,x(\tau,x_0),Bx(\tau,x_0))d\tau)ds \\
&\quad -\frac{1}{T}\int_0^T f(s,x(s,x_0),Ax(s,x_0),\int_0^{h(s)} g(\tau,x(\tau,x_0),Bx(\tau,x_0))d\tau)\,ds \\
&\quad -x_0 - \int_0^t f(s,z(s,x_0),Az(s,x_0),\int_0^{h(s)} g(\tau,z(\tau,x_0),Bz(\tau,x_0))d\tau)ds \\
&\quad -\frac{1}{T}\int_0^T f(s,z(s,x_0),Az(s,x_0),\int_0^{h(s)} g(\tau,z(\tau,x_0),Bz(\tau,x_0))d\tau)\,ds\ | \\
&\le \left(1-\frac{t}{T}\right)\int_0^t \Big\| f(s,x(s,x_0),Ax(s,x_0),\int_0^{h(s)} g(\tau,x(\tau,x_0),Bx(\tau,x_0))d\tau \\
&\quad -f(s,z(s,x_0),Az(s,x_0),\int_0^{h(s)} g(\tau,z(\tau,x_0),Bz(\tau,x_0))d\tau\Big\|\,ds \\
&\quad +\frac{t}{T}\int_t^T \Big\| f(s,x(s,x_0),Ax(s,x_0),\int_0^{h(s)} g(\tau,x(\tau,x_0),Bx(\tau,x_0))d\tau\,, \\
&\quad -f(s,z(s,x_0),Az(s,x_0),\int_0^{h(s)} g(\tau,z(\tau,x_0),Bz(\tau,x_0))d\tau\Big\|\,ds
\end{aligned}$$

$$\begin{aligned}
&\le (1-\tfrac{t}{T})\int_0^t [\,K_1\,\|x(s,x_0)-z(s,x_0)\| + K_2\,Q_1\|x(s,x_0)-z(s,x_0)\| \\
&\quad + K_3 h\,(P_1\,\|x(s,x_0)-z(s,x_0)\| - P_1Q_2\,\|x(s,x_0)-z(s,x_0)\|)\,]\,ds \\
&\quad +\frac{t}{T}\int_t^T [\,K_1\|x(s,x_0)-z(s,x_0)\| + K_2\,Q_1\|x(s,x_0)-z(s,x_0)\| \\
&\quad +K_3 h\,(P_1\,\|x(s,x_0)-z(s,x_0)\| - P_1Q_2\,\|x(s,x_0)-z(s,x_0)\|)]\,ds
\end{aligned}$$

$$\le \alpha(t)\,[\,K_1 + K_2Q_1 + K_3h(\,P_1 + P_2Q_2)\,]\,\|x(t,x_0)-z(t,x_0)\|$$

$$\le \frac{T}{2}[\,K_1 + K_2Q_1 + K_3h(\,P_1 + P_2Q_2)\,]\,\|x(t,x_0)-z(t,x_0)\|$$

$$\| Tx(t,x_0) - Tz(t,x_0) \| \le q\,\|\,x(t,x_0) - z(t,x_0)\,\|$$

So T* is a contraction mapping if $0 < q < 1$; thus, by Banach fixed point theorem, there exists a fixed point x(t) in C[0,T] such that

$$T^*x(t,x_0) = x(t,x_0)$$

There fore

$$\begin{aligned}
x(t,x_0) &= x_0 + \int_0^t f(s,x(s,x_0),Ax(s,x_0),\int_0^{h(s)} g(\tau,x(\tau,x_0),Bx(\tau,x_0))d\tau)ds \\
&\quad -\frac{1}{T}\int_0^T f(s,x(s,x_0),Ax(s,x_0),\int_0^{h(s)} g(\tau,x(\tau,x_0),Bx(\tau,x_0))d\tau)\,ds
\end{aligned}$$

is a unique periodic continuous solution of the problem (1).

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