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ON THE RIESZ BASIS AND BASIS PROPERTY OF THE EIGENFUNCTIONS OF THE MODIFIED FRANKL PROBLEM WITH A NONLOCAL ODDNESS CONDITION OF THE TIRED KIND

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ABSTRACT

In the present paper, we consider a new boundaries conditions of the tired kind. we prove the basis property, completeness, and the minimality of the eigen functions with a nonlocal Oddness condition of the tired kind.

Keywords and phrases: *Frankl problem, Lebesgue integral, Holder inequality, Bessel equation, Sobolev space.*

1. INTRODUCTION

The classical Frankl problem was considered in [3]. The problem was further developed in [2, pp.339-345], [8, pp.235-252]. The modified Frankl problem with a nonlocal boundary condition of the first kind was studied in [1, 6]. The basis property of an eigen functions of the Frankl problem with a nonlocal parity conditions in the space sobolev was studied in[7]. In the present paper, we consider a new boundaries conditions of the tired kind and prove the completeness, the basis property, and the minimality of the eigen functions in the space L^2. This analysis may be of interest in itself.

2. PRELIMINARIES

Definition 2.1: In the domain $D=(D_+\cup D_{-1}\cup D_{-2})$, we seek a solution of the modified generalizedFrankl problem

$$u_{xx}+\operatorname{sgn}(y)u_{yy}+\mu^2\operatorname{sgn}(x+y)u=0 \quad \text{in} \quad (D_+\cup D_{-1}\cup D_{-2}), \tag{1}$$

with the boundary conditions

$$u(1,\theta)=0, \theta\in\left[0,\frac{\pi}{2}\right], \tag{2}$$

$$\frac{\partial u}{\partial x}(0,y)=0, y\in(-1,0)\cup(0,1) \tag{3}$$

$$ku(0,y)=u(0,-y), y\in[0,1], ku(0,+0)=u(x,-0). \tag{4}$$

where $u(x,y)$ is a regular solution in the class

$$u\in C^0(\overline{D_+\cup D_{-1}\cup D_{-2}})\cap C^2(D_{-1})\cap C^2(D_{-2}),$$

and where

$$D_+=\left\{(r,\theta):0<r<1, 0<\theta<\frac{\pi}{2}\right\},$$

$$D_{-1}=\left\{(x,y):-y<x<y+1, \frac{-1}{2}<y<0\right\},$$

$$D_{-2}=\left\{(x,y):x-1<y<-x, 0<x<\frac{1}{2}\right\},$$

$$\kappa\frac{\partial u}{\partial y}(x,+0)=\frac{\partial u}{\partial y}(x,-0), -\infty<\kappa<\infty, 0<x<1. \tag{5}$$

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Definition 2.2: .System $\{x_n\}_{n\in N} \subset X$ is called complete in X if $\overline{L[\{x_n\}_{n\in N}]} = X$.

Definition 2.3: .System $\{x_n\}_{n\in N} \subset X$ is called minimal in X if $x_k \notin \overline{L[\{x_n\}_{n\in N}]}, \forall k \in N.$

Remark 2.1: If the system $\{x_n\}_{n\in N} \subset X$ minimal in $L_p(I),$ then it is also minimal in $L_p(J),$ for $J \supset I$, and if it is complete in $L_p(J)$ for $J \subset I.$

Theorem 2.5 ([5]): The eigenvalues and eigenfunctions of problem (1-5) can be written out in two series.

In the first series, the eigenvalues $\lambda = \mu_{nk}^2$ are found from the equation

$$J_{4n}(\mu_{nk}) = 0, \tag{6}$$

where $\mu_{nk}, n, k = 1, 2, \ldots,$ are roots of the Bessel equation (6), $J_\alpha(z)$, is the Bessel function [4], and the eigenfunctions are given by the formula

$$u_{nk} = \begin{cases} A_{nk} J_{4n}(\mu_{nk} r)\cos(4n)\left(\dfrac{\pi}{2} - \theta\right), & \text{in} \quad D^+; \\ kA_{nk} J_{4n}(\mu_{nk}\rho)\cosh(4n)\psi, & \text{in} \quad D_{-1}; \\ kA_{nk} J_{4n}(\mu_{nk} R)\cosh(4n)\varphi, & \text{in} \quad D_{-2}, \end{cases} \tag{7}$$

where $x = r\cos\theta, y = r\sin\theta$ for $0 \le \theta \le \dfrac{\pi}{2}, r^2 = x^2 + y^2$ in D_+, $x = \rho\cosh\psi, y = \rho\sinh\psi,$ for, $0 < \rho < 1,$ $-\infty < \psi < 0, \rho^2 = x^2 - y^2,$ in D_{-1}, and, $x = R\sinh\varphi, y = -R\cosh\varphi,$ for, $0 < \varphi < +\infty, R^2 = y^2 - x^2$ in D_{-2}.

In the second series, the eigenvalues $\tilde{\lambda} = \tilde{\mu}_{nk}^2$ are found from the equation

$$J_{4(n+\vartriangle)}(\tilde{\mu}_{nk}) = 0. \tag{8}$$

where $n, k = 1, 2, \ldots$ and the $(\tilde{\mu}_{nk})$ are the roots of the Bessel equation (8).

$$u_{nk} = \begin{cases} \tilde{A}_{nk} J_{4(n+\vartriangle)}(\tilde{\mu}_{nk} r)\cos 4(n+\vartriangle)\left(\dfrac{\pi}{2} - \theta\right), & \text{in} \quad D^+; \\ \tilde{A}_{nk} J_{4(n+\vartriangle)}(\tilde{\mu}_{nk}\rho)[\cosh 4(n+\vartriangle)\varphi\cos 4(n+\vartriangle)\dfrac{\pi}{2} + \kappa\sinh 4(n+\vartriangle)\psi\cos 4(n+\vartriangle)], & \text{in} \quad D_{-1}; \\ k\tilde{A}_{nk} J_{4(n+\vartriangle)}(\tilde{\mu}_{nk} R)\cosh 4(n+\vartriangle)\varphi[\cos 4(n+\vartriangle)\dfrac{\pi}{2} - \sin 4(n+\vartriangle)\dfrac{\pi}{2}], & \text{in} \quad D_{-2}, \end{cases} \tag{9}$$

where $\Delta = \dfrac{1}{\pi}\arcsin\dfrac{\kappa}{\sqrt{1+\kappa^2}}, \Delta \in \left(0, \dfrac{1}{2}\right)$, and

$$A_{nk}^2 \int_0^1 J_{4n}^2(\mu_{nk} r) r dr = 1,$$

$$\tilde{A}_{nk}^2 \int_0^1 J_{4n+\vartriangle}^2(\tilde{\mu}_{nk} r) r dr = 1,\ A_{nk} > 0 \text{ and } \tilde{A}_{nk} > 0.$$

3. THE COMPLETENESS, THE BASIS PROPERTY, and MINIMALITY of THE EIGENFUNCTIONS

Theorem 3.1: The function system

$$\left\{\cos(4n)\left(\frac{\pi}{2} - \theta\right)\right\}_{n=0}^{\infty}, \left\{\cos 4(n+\vartriangle)\left(\frac{\pi}{2} - \theta\right)\right\}_{n=1}^{\infty}, \tag{10}$$

is complete and a Riesz basis in $L_2\left(0, \dfrac{\pi}{2}\right)$, provided that $\vartriangle \in \left(\dfrac{-1}{4}, \dfrac{1}{2}\right)$.

Proof: In order to prove this theorem we use the method in [1, 6] by considering convergence function

$$f(\theta)=\sum_{n=0}^{\infty}A_n\cos 4n\left(\frac{\pi}{2}-\theta\right)+\sum_{n=1}^{\infty}B_n\cos 4(n+\vartriangle)\left(\frac{\pi}{2}-\theta\right), \tag{11}$$

In $L_2\left(0,\frac{\pi}{2}\right)$ and Riesz basis the system $\left(\sin 4(n+\vartriangle)\left(\frac{\pi}{2}-\theta\right)\right)$ for $\vartriangle\in\left(\frac{-1}{4},\frac{3}{4}\right)$.

Remark 3.2: For $\vartriangle<\frac{-1}{4}$ the system (10) is not complete but is minimal, for $\vartriangle>\frac{3}{4}$ is complete but isnot minimal, and if $\vartriangle=\frac{-1}{4}$, is complete and minimal.

Theorem 3.3: The system of eigenfunctions

$$u_{nk}(r,\theta)=A_{nk}J_{4n}(\mu_{nk}r)\cos(4n)\left(\frac{\pi}{2}-\theta\right),$$

$$\tilde{u}_{nk}(r,\theta)=\tilde{A}_{nk}J_{4(n+\vartriangle)}(\tilde{\mu}_{nk}r)[\cosh 4(n+\vartriangle)\varphi\cos 4(n+\vartriangle)\frac{\pi}{2},$$

is complete and basis in the space $L_2\left(0,\frac{\pi}{2}\right)$, therefore

$$\int_0^{\frac{\pi}{2}} f(r,\theta)u_{nk}(r,\theta)rdrd\theta=0,$$

$$\int_0^{\frac{\pi}{2}} f(r,\theta)\tilde{u}_{nk}(r,\theta)rdrd\theta=0,$$

and $f\in L\left(0,\frac{\pi}{2}\right)$ then $f=0$ in $\left(0,\frac{\pi}{2}\right)$.

Proof: Using fobini theorem and Lebesgue‘s integral for any $n,k=1,2,\ldots$ we have

$$0=\int_0^{\frac{\pi}{2}} f(r,\theta)u_{nk}(r,\theta)rd\theta dr$$

$$\int_0^1(rJ_{4n}(\mu_{nk}r)\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)\left(\frac{\pi}{2}-\theta\right)d\theta)dr,$$

Again since $f\in L^2\left(0,\frac{\pi}{2}\right)$ so;

$$\int_0^1\int_0^{\frac{\pi}{2}}|f(r,\theta)|^2\,d\theta dr<\infty.$$

In so much system $\{\sqrt{r}J_{4n}(\mu_{nk}r)\}_{k=1}^{\infty}$ in $L^2(0,1)$ is orthogonal and complete, it is enough to prove:

$$\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)\left(\frac{\pi}{2}-\theta\right)d\theta\in L^2(0,1).$$

Using the Holder inequality

$$|\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)\left(\frac{\pi}{2}-\theta\right)d\theta|^2<\frac{1}{2}r\int_0^{\frac{\pi}{2}}|f^2(r,\theta)|\,d\theta\int_0^{\frac{\pi}{2}}d\theta$$

$$=\frac{\pi}{4}r\int_0^{\frac{\pi}{2}}|f(r,\theta)|^2\,d\theta=\frac{\pi}{4}r\int_0^{\frac{\pi}{2}}|f(r,\theta)|^2\,d\theta,$$

with the integration interval (0,1).

$$\int_0^1 |\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)\left(\frac{\pi}{2}-\theta\right)d\theta|^2\, dr < \frac{\pi}{4}\int_0^1\int_0^{\frac{\pi}{2}} r\,|\,f(r,\theta)|^2\, drd\theta < \infty.$$

This inequality is equivalent to

$$\left\{\int_0^1 \sqrt{r}\,|\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)\left(\frac{\pi}{2}-\theta\right)d\theta|^2\, dr\right\}^{\frac{1}{2}} < \infty.$$

Also system $\{\sqrt{r}J_{4n}(\mu_{nk}r)\}_{k=1}^{\infty}$ is orthogonal and complete in $L^2\left(0,\frac{\pi}{2}\right)$ of relation

$$\int_0^1 (\sqrt{r}J_{4n}(\mu_{nk}r)\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)\left(\frac{\pi}{2}-\theta\right)d\theta)dr = 0,$$

imply that

$$\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)\left(\frac{\pi}{2}-\theta\right)d\theta = 0.$$

According to theorem 2,we conclude that $f(r,\theta)=0$ in $L^2(0,1).$ Similarly, if we consider the above calculations for sequence $\left\{\cos 4(n+\vartriangle)\left(\frac{\pi}{2}-\theta\right)\right\}_{n=1}^{\infty}$,

We have

$$\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos 4(n+\vartriangle)\left(\frac{\pi}{2}-\theta\right)d\theta = 0.$$

Because completeness $\left\{\cos 4(n+\vartriangle)\left(\frac{\pi}{2}-\theta\right)\right\}_{n=0}^{\infty}$, $f(r,\theta)=0$ in $L^2(0,1).$

The proof of the theorem is complete.

Theorem 3.4: The system of eigenfunctions $u_{nk}(r,\theta)$ and $\tilde{u}_{nk}(r,\theta)$ of the problem (1)-(5) is a Riesz basis in the space $L\left(0,\frac{\pi}{2}\right)$, where, $A_{nk}^2 = \left(\int_0^1 J_{4n}^2(\mu_{nk}r)rdr\right)^{-1}, \widetilde{A_{nk}^2} = \left(\int_0^1 J_{4(n+\vartriangle)}^2(\widetilde{\mu_{nk}}r)rdr\right)^{-1}.$

Proof: Theorem 3.3 results from Theorem 3.2 and the completeness and orthogonality of the system $\{A_{nk}J_{4n}(\mu_{nk}r)\}_{k=1}^{\infty}$, for $n>0$ and $\{\widetilde{A_{nk}}J_{4(n+\vartriangle)}(\widetilde{\mu_{nk}})r\}_{k=1}^{\infty}$ for $n>1$ in $L^2(0,1).$

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