International Journal of Mathematical Archive-6(7), 2015, 168-177
IJMA Available online through www.ijma.info ISSN 2229 – 5046

ON ANALYTIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVE AND A NEW GENERALIZED MULTIPLIER DIFFERENTIAL OPERATOR

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(Received On: 01-07-15; Revised & Accepted On: 31-07-15)

ABSTRACT

New subclasses of analytic functions, containing the linear operator obtained as a linear combination of Ruscheweyh derivative and a new generalized multiplier transformation have been considered. Sharp results concerning coefficients, distortion theorems of functions belonging to these classes are determined. Furthermore, Functions with negative coefficients belonging to these classes are also discussed.

2010 Mathematics Subject Classification: *30C45, 30A20, 34A40.*

Key words and phrases: *Analytic function, differential operator, multiplier transformation, distortion theorem.*

1. INTRODUCTION

Denote by U the open unit disc of the complex plane, $U = \{z \in C : |z| < 1\}$. Let $H(U)$ be the space of holomorphic functions in U. Let A denote the family of functions in $H(U)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

The author in [14], has introduced a new generalized multiplier differential operator as follows.

Definition 1.1: Let $m \in N_0 = N \cup \{0\}, \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then for $f \in A$, a new generalized multiplier operator $I_{\alpha,\beta}^m$ was defined by

$$I_{\alpha,\beta}^0 f(z) = f(z), I_{\alpha,\beta}^1 f(z) = \frac{\alpha f(z) + \beta z f'(z)}{\alpha + \beta}, \ldots, I_{\alpha,\beta}^m f(z) = I_{\alpha,\beta}(I_{\alpha,\beta}^{m-1} f(z)).$$

Remark 1.2: Observe that for $f(z)$ given by (1.1), we have

$$I_{\alpha,\beta}^m f(z) = z + \sum_{k=2}^{\infty} A_k(\alpha, \beta, m) a_k z^k, \tag{1.2}$$

where

$$A_k(\alpha, \beta, m) = \left(\frac{\alpha + k\beta}{\alpha + \beta}\right)^m. \tag{1.3}$$

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We note that: i) $I^{m}_{1-\beta,\beta,0} f(z) = D^{m}_{\beta} f(z),\ \beta \geq 0$ (See F. M. Al-Oboudi [1]), ii) $I^{m}_{l+1-\beta,\beta,0} f(z) = I^{m}_{l,\beta} f(z)$, $l > -1,\ \beta \geq 0$ (See A. Catas [6] and he has considered for $l \geq 0$) and iii) $I^{m}_{\alpha,1} f(z) = I^{m}_{\alpha} f(z)$, $\alpha > -1$, (See Cho and Srivastava [7]) and Cho and Kim [8]).

Remark 1.3: $D^{m}_{1} f(z)$ was introduced by Salagean [11] and was considered for $m \geq 0$ in [3].

Definition 1.4: ([10]) For $m \in N_0$, $f \in A$, the operator R^m is defined by $R^m : A \to A$,

$$R^0 f(z) = f(z)\ , R^1 f(z) = zf'(z), \ldots,$$
$$(m+1)R^{m+1} f(z) = z(R^m f(z))' + mR^m f(z), z \in U.$$

Remark 1.5: If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$, then $R^m f(z) = z + \sum_{k=2}^{\infty} B_k(m) a_k z^k,\ \ z \in U$,
where

$$B_k(m) = \frac{(m+k-1)!}{m!(k-1)!}. \tag{1.4}$$

The author in [15] has introduced the following operator:

Definition 1.6: Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0,1), \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0.$
Denote by $RI^{m}_{\alpha,\beta,,\delta}$, the operator given by $RI^{m}_{\alpha,\beta,\delta} : A \to A,$

$$RI^{m}_{\alpha,\beta,\delta} f(z) = (1-\delta)R^m f(z) + \delta I^{m}_{\alpha,\beta} f(z), z \in U.$$

The operator was studied also in [13]. Clearly $RI^{m}_{\alpha,\beta,,0} = R^m$ and $RI^{m}_{\alpha,\beta,,1} = I^{m}_{\alpha,\beta}$.

Remark 1.7: i) If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$, then from (1.2) and Remark 1.5, we have

$$RI^{m}_{\alpha,\beta,\delta} f(z) = z + \sum_{k=2}^{\infty} \{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\} a_k z^k,\ \ z \in U,$$

where $A_k(\alpha,\beta,m)$ and $B_k(m)$ are as defined in (1.3) and (1.4), respectively.

By making use of the generalized operator $RI^{m}_{\alpha,\beta,\delta}$ we introduce new classes as follows.

Definition 1.8: Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0,1), \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$ .Then $f(z)$ is in the class $S^{m}_{\alpha,\beta,\delta}(\rho)$ if and only if

$$\mathrm{Re}\left(\frac{z(RI^{m}_{\alpha,\beta,\delta} f(z))'}{RI^{m}_{\alpha,\beta,\delta} f(z)} \right) > \rho, z \in U\ . \tag{1.5}$$

Definition 1.9: Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0,1), \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $K^{m}_{\alpha,\beta,\delta}(\rho)$ if and only if

$$\mathrm{Re}\left(\frac{[z^2(RI^{m}_{\alpha,\beta,\delta} f(z))']'}{(zRI^{m}_{\alpha,\beta,\delta} f(z))'} \right) > \rho, z \in U\ . \tag{1.6}$$

Definition 1.10: Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0,1), \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$.
Then $f(z)$ is in the class $C^{m}_{\alpha,\beta,\delta}(\rho)$ if and only if

$$\mathrm{Re}\left(\frac{[z(RI^{m}_{\alpha,\beta,\delta} f(z))']'}{(RI^{m}_{\alpha,\beta,\delta} f(z))'} \right) > \rho, z \in U\ . \tag{1.7}$$

Definition 1.11: Let $f \in A, m \in N_0 = N \cup \{0\}, \lambda \ge 0, \delta \ge 0, \rho \in [0,1), \beta \ge 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $\mathrm{P}^m_{\alpha,\beta,\lambda,\delta}(\rho)$ if and only if

$$\mathrm{Re}\left((1-\lambda)\frac{RI^m_{\alpha,\beta,\delta} f(z)}{z} + \lambda(RI^m_{\alpha,\beta,\delta} f(z))'\right) > \rho, z \in U. \tag{1.8}$$

Definition 1.12: Let $f \in A, m \in N_0 = N \cup \{0\}, \lambda \ge 0, \delta \ge 0, \rho \in [0,1), \beta \ge 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $\mathrm{H}^m_{\alpha,\beta,\lambda,\delta}(\rho)$ if and only if

$$\mathrm{Re}\left((RI^m_{\alpha,\beta,\delta} f(z))' + \lambda z(RI^m_{\alpha,\beta,\delta} f(z))''\right) > \rho, z \in U. \tag{1.9}$$

In view of the above definitions of the classes $S^m_{\alpha,\beta,\delta}(\rho)$, $K^m_{\alpha,\beta,\delta}(\rho)$, $C^m_{\alpha,\beta,\delta}(\rho)$, $\mathrm{P}^m_{\alpha,\beta,\lambda,\delta}(\rho)$ and $\mathrm{H}^m_{\alpha,\beta,\lambda,\delta}(\rho)$, we deem it worthwhile to point out the relevance of these classes of functions with some known classes. Indeed we have i) $S^m_{1-\beta,\beta,\delta}(\rho) = S^m_{\beta,\delta}(\rho)$ considered in [2], ii) $C^m_{1-\beta,\beta,\delta}(\rho) = C^m_{\beta,\delta}(\rho)$ introduced in [2], iii) $\mathrm{P}^m_{\alpha,0,\lambda,1}(\rho) = P_\lambda(\rho)$ examined in [5] for functions with negative coefficients and iv) $\mathrm{H}^m_{\alpha,0,\lambda,1}(\rho) = H_\lambda(\rho)$ investigated for functions with negative coefficients in [4].

In section 2 we study the characterization properties for the function $f \in A$ to belong to the classes $S^m_{\alpha,\beta,\delta}(\rho)$, $K^m_{\alpha,\beta,\delta}(\rho)$, $C^m_{\alpha,\beta,\delta}(\rho)$, $\mathrm{P}^m_{\alpha,\beta,\lambda,\delta}(\rho)$ and $\mathrm{H}^m_{\alpha,\beta,\lambda,\delta}(\rho)$, by obtaining the coefficient bounds. Distortion theorems of functions belonging to these classes are obtained in section 3. Analytic functions with negative coefficients belonging to the above classes are considered in section 4.

2. GENE1RAL PROPERTIES

In this section we study the characterization properties following the paper of M. Darus and R. Ibrahim [9]. Unless otherwise mentioned we shall assume that $A_k(\alpha, \beta, m)$ and $B_k(m)$ are as defined in (1.3) and (1.4) respectively, throughout this paper.

Theorem 2.1: Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \ge 0, \rho \in [0,1), \beta \ge 0, \alpha$ a real number such that $\alpha + \beta > 0$.
i) If

$$\sum_{k=2}^{\infty} (k-\rho)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \le 1-\rho, \tag{2.1}$$

then $f(z) \in S^m_{\alpha,\beta,\delta}(\rho)$.
ii) If

$$\sum_{k=2}^{\infty} (k+1)(k-\rho)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \le 2(1-\rho), \tag{2.2}$$

then $f(z) \in K^m_{\alpha,\beta,\delta}(\rho)$.
iii) If

$$\sum_{k=2}^{\infty} k(k-\rho)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \le 1-\rho, \tag{2.3}$$

then $f(z) \in C^m_{\alpha,\beta,\delta}(\rho)$.The results (2.1), (2.2) and (2.3)are sharp.

Proof: i) It suffices to show that the values of $z(RI^m_{\alpha,\beta,\delta} f(z))' / RI^m_{\alpha,\beta,\delta} f(z)$ lie in a circle centred at 1 with radius $1-\rho$. We have

$$\left|\frac{z(RI^m_{\alpha,\beta,\delta} f(z))'}{RI^m_{\alpha,\beta,\delta} f(z)} - 1\right| = \left|\frac{z(RI^m_{\alpha,\beta,\delta} f(z))' - RI^m_{\alpha,\beta,\delta} f(z)}{RI^m_{\alpha,\beta,\delta} f(z)}\right|$$

$$= \left| \frac{\sum_{k=2}^{\infty}(k-1)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k z^k}{z+\sum_{k=2}^{\infty}(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)a_k z^k} \right|$$

$$\leq \frac{\sum_{k=2}^{\infty}(k-1)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k||z|^{k-1}}$$

$$\leq \frac{\sum_{k=2}^{\infty}(k-1)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k|}{1-\sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k|}.$$

The last expression is bounded above by $1-\rho$ if

$$\sum_{k=2}^{\infty}(k-1)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \leq (1-\rho)\left(1-\sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k|\right),$$

which is equivalent to (2.1). Hence $\left|\frac{z(RI_{\alpha,\beta,\delta}^{m}f(z))'}{RI_{\alpha,\beta,\delta}^{m}f(z)}-1\right| < 1-\rho,$ and the theorem is proved.

The proofs of the remaining two parts of the theorem are similar and so omitted.

The assertions (2.1), (2.2) and (2.3) are sharp and extremal functions are given by

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\rho}{(k-\rho)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}z^k,\ z\in U,$$

$$f(z)=z+\sum_{k=2}^{\infty}\frac{2(1-\rho)}{(k+1)(k-\rho)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}z^k,\ z\in U,$$

and

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\rho}{k(k-\rho)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}z^k,\ z\in U,$$ respectively.

The following two theorems immediately follow by appealing to Theorem 2.1.

Theorem 2.2: Let $f\in A, m\in N_0=N\cup\{0\}, \lambda\geq 0, \delta\geq 0, \rho\in[0,1), \beta\geq 0, \alpha$ a real number such that $\alpha+\beta>0$.
i) If

$$\sum_{k=2}^{\infty}(1+(k-1)\lambda)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \leq 1-\rho, \tag{2.4}$$

then $f(z)\in \mathrm{P}_{\alpha,\beta,\lambda,\delta}^{m}(\rho)$ and
ii) If

$$\sum_{k=2}^{\infty}k(1+(k-1)\lambda)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \leq 1-\rho, \tag{2.5}$$

then $f(z)\in \mathrm{H}_{\alpha,\beta,\lambda,\delta}^{m}(\rho)$.

The results (2.4) and (2.5) are sharp and extremal functions are given by

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\rho}{(1+(k-1)\lambda)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}z^k,\ z\in U$$

and

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\rho}{k(1+(k-1)\lambda)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}z^k,\ z\in U,$$ respectively.

Theorem 2.3: Let $0 \le \rho_1 \le \rho_2 < 1$. Then

a) i) $S^{m}_{\alpha,\beta,\delta}(\rho_2) \subseteq S^{m}_{\alpha,\beta,\delta}(\rho_1)$, ii) $K^{m}_{\alpha,\beta,\delta}(\rho_2) \subseteq K^{m}_{\alpha,\beta,\delta}(\rho_1)$ and ii) $C^{m}_{\alpha,\beta,\delta}(\rho_2) \subseteq C^{m}_{\alpha,\beta,\delta}(\rho_1)$.

b) i) $\mathrm{P}^{m}_{\alpha,\beta,\lambda,\delta}(\rho_2) \subseteq \mathrm{P}^{m}_{\alpha,\beta,\lambda,\delta}(\rho_1)$ and ii) $\mathrm{H}^{m}_{\alpha,\beta,\lambda,\delta}(\rho_2) \subseteq \mathrm{H}^{m}_{\alpha,\beta,\lambda,\delta}(\rho_1)$.

3. DISTORTION THEOREMS

Theorem 3.1: Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \ge 0, \rho \in [0,1), \beta \ge 0, \alpha$ a real number such that $\alpha + \beta > 0$.

i) If $\sum_{k=2}^{\infty}(k-\rho)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \le 1-\rho$, then

$$|z| - \frac{1-\rho}{2-\rho}|z|^2 \le \left|RI^{m}_{\alpha,\beta,\delta} f(z)\right| \le |z| + \frac{1-\rho}{2-\rho}|z|^2, z \in U.$$

ii) If $\sum_{k=2}^{\infty}(k+1)(k-\rho)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \le 2(1-\rho)$, then

$$|z| - \frac{2(1-\rho)}{3(2-\rho)}|z|^2 \le \left|RI^{m}_{\alpha,\beta,\delta} f(z)\right| \le |z| + \frac{2(1-\rho)}{3(2-\rho)}|z|^2, z \in U.$$

iii) If $\sum_{k=2}^{\infty}k(k-\rho)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \le 1-\rho$, then

$$|z| - \frac{1-\rho}{2(2-\rho)}|z|^2 \le \left|RI^{m}_{\alpha,\beta,\delta} f(z)\right| \le |z| + \frac{1-\rho}{2(2-\rho)}|z|^2$$

Proof: i) Note that $(2-\rho)\sum_{k=2}^{\infty}\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \le$

$$\sum_{k=2}^{\infty}(k-\rho)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \le 1-\rho,$$

by Theorem 2.1. Thus $\sum_{k=2}^{\infty}\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \le \frac{1-\rho}{2-\rho}$.Hence we obtain

$$\begin{aligned}\left|RI^{m}_{\alpha,\beta,\delta} f(z)\right| &\le |z| + \sum_{k=2}^{\infty}\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k||z|^k \\ &\le |z| + |z|^2\sum_{k=2}^{\infty}\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \\ &\le |z| + \frac{1-\rho}{2-\rho}|z|^2.\end{aligned}$$

Similarly

$$\begin{aligned}\left|RI^{m}_{\alpha,\beta,\delta} f(z)\right| &\ge |z| - \sum_{k=2}^{\infty}\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k||z|^k \\ &\ge |z| - |z|^2\sum_{k=2}^{\infty}\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \\ &\ge |z| - \frac{1-\rho}{2-\rho}|z|^2.\end{aligned}$$

This completes the proof of (i). (ii) and (iii) can be proved on similar lines.

Theorem 3.2: Let $f \in A, m \in N_0 = N \cup \{0\}, \lambda \geq 0, \delta \geq 0, \rho \in [0,1), \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$.

i) If $\sum_{k=2}^{\infty} (1+(k-1)\lambda)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \leq 1-\rho$, then

$$|z| - \frac{1-\rho}{1+\lambda}|z|^2 \leq \left|RI^m_{\alpha,\beta,\delta} f(z)\right| \leq |z| + \frac{1-\rho}{1+\lambda}|z|^2 .$$

ii) If $\sum_{k=2}^{\infty} k(1+(k-1)\lambda)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \leq 1-\rho$, then

$$|z| - \frac{1-\rho}{2(1+\lambda)}|z|^2 \leq \left|RI^m_{\alpha,\beta,\delta} f(z)\right| \leq |z| + \frac{1-\rho}{2(1+\lambda)}|z|^2 .$$

Proof: The proof follow on the lines of the proof of Theorem 3.1. The details are omitted.

Theorem 3.3: Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0,1), \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$.

i) If $\sum_{k=2}^{\infty} (k-\rho)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \leq 1-\rho$, then

$$|f(z)| \geq |z| - \frac{1-\rho}{(2-\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{(2-\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U .$$

ii) If $\sum_{k=2}^{\infty} (k+1)(k-\rho)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \leq 2(1-\rho)$, then

$$|f(z)| \geq |z| - \frac{2(1-\rho)}{3(2-\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{2(1-\rho)}{3(2-\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U .$$

iii) If $\sum_{k=2}^{\infty} k(k-\rho)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \leq 1-\rho$, then

$$|f(z)| \geq |z| - \frac{1-\rho}{2(2-\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{2(2-\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U .$$

Proof: i) In virtue of Theorem 2.1, we have

$$(2-\rho)[(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\sum_{k=2}^{\infty}|a_k| \leq$$

$$\sum_{k=2}^{\infty}(k-\rho)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \leq 1-\rho .$$

Thus $\sum_{k=2}^{\infty}|a_k| \leq \frac{1-\rho}{(2-\rho)[(m+1)(1-\delta)+\delta A_2(\alpha,\beta,\delta)]}$. So we get

$$|f(z)| \leq |z| + |z|^2 \sum_{k=2}^{\infty}|a_k| \leq |z| + \frac{1-\rho}{(2-\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2 .$$

On the other hand

$$|f(z)| \geq |z| - |z|^2 \sum_{k=2}^{\infty} |a_k| \geq |z| - \frac{1-\rho}{(2-\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}} |z|^2 .$$

This completes the proof. The proofs of (ii) and (iii) are similar.

In the same way we can prove the following result, using Theorem 3.3.

Theorem 3.4: Let $f \in A, m \in N_0 = N \cup \{0\}, \lambda \geq 0, \delta \geq 0, \rho \in [0,1), \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$.

i) If $\sum_{k=2}^{\infty} (1+(k-1)\lambda)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \leq 1-\rho,$ then

$$|f(z)| \geq |z| - \frac{1-\rho}{(1+\lambda)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}} |z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{(1+\lambda)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}} |z|^2, z \in U .$$

ii) If $\sum_{k=2}^{\infty} k(1+(k-1)\lambda)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \leq 1-\rho,$ then

$$|f(z)| \geq |z| - \frac{1-\rho}{2(1+\lambda)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}} |z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{2(1+\lambda)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}} |z|^2, z \in U .$$

4. FUNCTIONS WITH NEGATIVE COEFFICIENTS

Let T denote the subclass of A consisting of functions of the form $f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0$.

We denote by $TS^{m}_{\alpha,\beta,\delta}(\rho)$, $TK^{m}_{\alpha,\beta,\delta}(\rho)$, $TC^{m}_{\alpha,\beta,\delta}(\rho)$, $TP^{m}_{\alpha,\beta,\lambda,\delta}(\rho)$ and $TH^{m}_{\alpha,\beta,\lambda,\delta}(\rho)$, the classes of functions $f(z) \in T$ satisfying (1.5), (1.6), (1.7),(1.8) and (1.9) respectively. We study the coefficient estimates, distortion theorems and other properties of these classes, following the paper of H. Silverman [12].

For functions in T, the converses of Theorem 2.1 and Theorem 2.2 are also true.

Theorem 4.1:

i) A function $f(z) \in T$ is in $TS^{m}_{\alpha,\beta,\delta}(\rho)$ if and only if

$$\sum_{k=2}^{\infty} (k-\rho)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k \leq 1-\rho . \tag{4.1}$$

ii) A function $f(z) \in T$ is in $TK^{m}_{\alpha,\beta,\delta}(\rho)$ if and only if

$$\sum_{k=2}^{\infty} (k+1)(k-\rho)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k \leq 2(1-\rho) . \tag{4.2}$$

iii) A function $f(z) \in T$ is in $TC^{m}_{\alpha,\beta,\delta}(\rho)$ if and only if

$$\sum_{k=2}^{\infty} k(k-\rho)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k \leq 1-\rho . \tag{4.3}$$

The results are sharp.

Proof:
i) In view of Theorem 2.1, it suffices to prove the only if part. Assume that

$$\mathrm{Re}\left(\frac{z(RI^{m}_{\alpha,\beta,\delta}f(z))'}{RI^{m}_{\alpha,\beta,\delta}f(z)}\right)=\mathrm{Re}\left(\frac{z-\sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}ka_k z^k}{z-\sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k z^k}\right)>\rho. \tag{4.4}$$

Clearing the denominator in (4.4) and letting $z\to 1^-$ through real values, we obtain

$$1-\sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}ka_k\geq\rho\left(1-\sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k\right).$$

Hence we obtain (4.1), and the proof is complete. The proofs of (ii) and (iii) are similar.

Finally, we note that assertions (4.1), (4.2) and (4.3) of Theorem 4.1 are sharp, extremal functions being

$$f(z)=z-\sum_{k=2}^{\infty}\frac{1-\rho}{(k-\rho)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}z^k,\ z\in U,$$

$$f(z)=z-\sum_{k=2}^{\infty}\frac{2(1-\rho)}{(k+1)(k-\rho)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}z^k,\ z\in U,$$

and

$$f(z)=z-\sum_{k=2}^{\infty}\frac{1-\rho}{k(k-\rho)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}z^k,\ z\in U, \text{ respectively.}$$

Proceeding similarly, we now obtain

Theorem 4.2: i) A function $f(z)\in T$ is in $T\mathrm{P}^{m}_{\alpha,\beta,\lambda,\delta}(\rho)$ if and only if

$$\sum_{k=2}^{\infty}(1+(k-1)\lambda)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k\leq 1-\rho. \tag{4.5}$$

ii) A function $f(z)\in T$ is in $TH^{m}_{\alpha,\beta,\lambda,\delta}(\rho)$ if and only if

$$\sum_{k=2}^{\infty}k(1+(k-1)\lambda)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k\leq 1-\rho. \tag{4.6}$$

The results are sharp.

The assertions (4.5) and (4.6) are sharp and extremal functions are given by

$$f(z)=z-\sum_{k=2}^{\infty}\frac{1-\rho}{(1+(k-1)\lambda)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}z^k,\ z\in U$$

and

$$f(z)=z-\sum_{k=2}^{\infty}\frac{1-\rho}{k(1+(k-1)\lambda)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}z^k,\ z\in U, \text{ respectively.}$$

Our coefficient bounds enable us to prove the following.

Theorem 4.3: i) If $f\in TS^{m}_{\alpha,\beta,\delta}(\rho)$, then

$$|f(z)|\geq|z|-\frac{1-\rho}{(2-\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2,\ z\in U$$

and

$$|f(z)|\leq|z|+\frac{1-\rho}{(2-\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2,\ z\in U.$$

ii) If $f \in TK^{m}_{\alpha,\beta,\delta}(\rho)$, then

$$|f(z)| \geq |z| - \frac{2(1-\rho)}{3(2-\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{2(1-\rho)}{3(2-\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U .$$

iii) If $f \in TC^{m}_{\alpha,\beta,\delta}(\rho)$, then

$$|f(z)| \geq |z| - \frac{1-\rho}{2(2-\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{2(2-\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U .$$

The bounds in i), ii) and iii) are sharp.

Theorem 4.4:

i) If $f \in T\mathrm{P}^{m}_{\alpha,\beta,\lambda,\delta}(\rho)$, then

$$|f(z)| \geq |z| - \frac{1-\rho}{(1+\lambda)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{(1+\lambda)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U .$$

ii) If $f \in T\mathrm{H}^{m}_{\alpha,\beta,\lambda,\delta}(\rho)$, then

$$|f(z)| \geq |z| - \frac{1-\rho}{2(1+\lambda)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{2(1+\lambda)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U .$$

The bounds in i) and ii) are sharp.

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Source of support: Nil, Conflict of interest: None Declared