

pg**- CLOSED SETS IN TOPOLOGICAL SPACES

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ABSTRACT

*In this paper we introduce a new class of sets called pg**- closed sets in topological spaces which is properly placed in between the class of closed sets and gsp-closed sets. As an application, we introduce new spaces namely, ${}_pT_{1/2}^{**}$ - space, ${}_{ap}T_c^{**}$ -space, ${}_pT_{1/2}^{**}$ -space, ${}^{**}T_{1/2}$ -space and ${}_pT_c^{**}$ -space. Further, pg**-continuous, pg**-irresolute mappings are also introduced and investigated.*

Key words: pg**-closed set, pg**-continuous map, pg**-irresolute map, ${}_pT_{1/2}^{**}$ - space, ${}_{ap}T_c^{**}$ -space, ${}_pT_{1/2}^{**}$ -space, ${}^{**}T_{1/2}$ -space and ${}_pT_c^{**}$ -spaces.

1. INTRODUCTION

Levine [11] introduced the class of g-closed sets in 1970. Arya and Tour [3] defined gs-closed sets in 1990. Dontchev [9], Gnanambal [10] Palaniappan and Rao [17] introduced gsp-closed sets, gpr-closed sets and rg-closed sets respectively. Veerakumar [18] introduced g*-closed sets in 1991. Dontchev [8] introduced gsp-closed sets in 1995. P M Helen [20] introduced g**-closed sets. Levine [11] Devi [6,8] introduced $T_{1/2}$ -spaces, T_b spaces and ${}_aT_b$ spaces respectively. The purpose of this paper is to introduce the concepts of pg**-closed set, pg**-continuous map, pg**-irresolute maps. ${}_pT_{1/2}^{**}$ - space, ${}_{ap}T_c^{**}$ -space, ${}_pT_{1/2}^{**}$ -space, ${}^{**}T_{1/2}$ -space and ${}_pT_c^{**}$ -space are introduced and investigated.

2. PRILIMINARIES

Throughout this paper $(X, \tau), (Y, \sigma)$ and (Z, η) represent non-empty topological spaces of which no separation axioms are assumed unless otherwise stated. For a subset A of a space (X, τ) , $cl(A)$ and $int(A)$ denote the closure and the interior of A respectively. The class of all closed subsets of a space (X, τ) is denoted by $C(X, \tau)$. The smallest semi-closed (resp. pre-closed and α -closed) set containing a subset A of (X, τ) is called the semi-closure (resp. pre-closure and α -closure) of A and is denoted by $scl(A)$ (resp. $pcl(A)$ and $\alpha cl(A)$).

Definition 2.1: A subset A of a topological space (X, τ) is called

- (1) a pre-open set [14] if $A \subseteq int(cl(A))$ and a pre-closed set if $cl(int(A)) \subseteq A$.
- (2) a semi-open set [12] if $A \subseteq cl(int(A))$ and a semi-closed set if $int(cl(A)) \subseteq A$.
- (3) a semi-preopen set [1] if $A \subseteq cl(int(cl(A)))$ and a semi-preclosed set [1] if $int(cl(int(A))) \subseteq A$.
- (4) an α -open set [16] if $A \subseteq int(cl(int(A)))$ and an α -closed set [16] if $cl(int(cl(A))) \subseteq A$.
- (5) a regular-open set [14] if $int(cl(A)) = A$ and regular-closed set [14] if $A = int(cl(A))$.

Definition 2.2: A subset A of topological space (X, τ) is called

- (1) a generalized closed set (briefly g-closed) [1] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (2) generalized semi-closed set (briefly gs-closed) [3] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (3) an α -generalized closed set (briefly α g-closed) [19] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (4) a generalized semi pre-closed set (briefly gsp-closed) [9] if $sp cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (5) a regular generalized closed set (briefly rg-closed) [17] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in (X, τ) .

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- (6) a generalized pre-closed set (briefly gp-closed) [13] if $p\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (7) a generalized pre regular-closed set (briefly gpr-closed)[10] if $p\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in (X, τ) .
- (8) a g^* -closed set [18] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open in (X, τ) .
- (9) a wg -closed set [16] if $\text{cl}(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (10) a g^{**} -closed set [20] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is g^* -open in (X, τ) .

Definition 2.3: A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called

- (1) g -continuous [4] if $f^{-1}(V)$ is a g -closed set of (X, τ) for every closed set V of (Y, σ) .
- (2) αg -continuous [10] if $f^{-1}(V)$ is an αg -closed set of (X, τ) for every closed set V of (Y, σ) .
- (3) gs -continuous [7] if $f^{-1}(V)$ is a gs -closed set of (X, τ) for every closed set V of (Y, σ) .
- (4) gsp -continuous [9] if $f^{-1}(V)$ is a gsp -closed set of (X, τ) for every closed set V of (Y, σ) .
- (5) rg -continuous [17] if $f^{-1}(V)$ is a rg -closed set of (X, τ) for every closed set V of (Y, σ) .
- (6) gp -continuous [2] if $f^{-1}(V)$ is a gp -closed set of (X, τ) for every closed set V of (Y, σ) .
- (7) gpr -continuous [10] if $f^{-1}(V)$ is a gpr -closed set of (X, τ) for every closed set V of (Y, σ) .
- (8) g^* -continuous [18] if $f^{-1}(V)$ is a g^* -closed set of (X, τ) for every closed set V of (Y, σ) .
- (9) g^* -irresolute[18] if $f^{-1}(V)$ is a g^* -closed set of (X, τ) for every g^* -closed set V of (Y, σ) .
- (10) wg -continuous [16] if $f^{-1}(V)$ is a wg -closed set of (X, τ) for every closed set V of (Y, σ) .
- (11) g^{**} -continuous[20] if $f^{-1}(V)$ is a g^{**} -closed set of (X, τ) for every closed set V of (Y, σ) .
- (12) g^{**} -irresolute[20] if $f^{-1}(V)$ is a g^{**} -closed set of (X, τ) for every g^{**} -closed set V of (Y, σ) .

Definition 2.4: A topological space (X, τ) is said to be

- (1) a $T_{1/2}$ -space [11] if every g -closed set in it is closed.
- (2) a T_b space [6] if every gs -closed set in it is closed.
- (3) a $_{\alpha}T_b$ -space [8] if every αg -closed set in it is closed.
- (4) a $T_{1/2}^*$ -space [18] if every g^* -closed set in it is closed.
- (5) a $T_{1/2}^{**}$ -space [20] if every g^{**} -closed set is closed.
- (6) a $^{**}T_{1/2}$ -space [20] if every g^{**} -closed set is g^* - closed.

3. Basic properties of pg**- closed sets

We introduce the following definition

Definition 3.1: A subset A of (X, τ) is said to be a pg^{**} -closed set if $p\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is g^* -open in X .

The class of pg^{**} - closed subset of (X, τ) is denoted by $PG^{**}C(X, \tau)$.

Proposition 3.2: Every closed set is pg^{**} - closed.

Proof follows from the definition.

The following example supports that a pg^{**} - closed set need not be closed in general.

Proposition 3.3: Every pre closed set is pg^{**} - closed.

Proof follows from the definition.

Proposition 3.4: Every g^{**} -closed set is pg^{**} - closed.

Proof follows from the definition.

Proposition 3.5: Every g^* -closed set is pg^{**} - closed.

Proof follows from the definition.

Proposition 3.6: Every g -closed set is pg^{**} - closed.

Proof follows from the definition.

The converse of the above propositions need not be true in general.

Example 3.7: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, c\}\}$. Let $A = \{a\}$ then A is a pg**-closed set but not a closed set and a g**-closed set of (X, τ) . So the class of pg**-closed sets properly contains the class of closed sets and the class of g**-closed sets. Also $A = \{a\}$ is not a g-closed set.

Example 3.8: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$. Let $A = \{a, b\}$ then A is a pg**-closed set but not a pre closed set and a g*-closed set of (X, τ) . So the class of pg**-closed sets properly contains the class of pre closed sets and the class of g*-closed sets.

Proposition 3.9: Every pg**-closed set is (1) rg-closed (2) gpr-closed (3) gsp-closed.

Proof follows from the definition.

The converse of the above propositions need not be true in general as seen in the following examples.

Example 3.10: In example (3.8), let $A = \{a\}$ is gpr-closed and rg-closed but it is not pg**-closed. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let $A = \{a\}$ then A is a gsp-closed set but not a pg**-closed set of (X, τ) . Therefore the class of pg**-closed sets is properly contained in the class of gpr-closed, rg-closed, gsp-closed sets.

Remark 3.11: pg**-closedness is independent from α -closedness, semi-closedness, sg-closedness, g α -closedness, g α *-closedness and semi-preclosedness.

Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$. Let $A = \{a, b\}$ then A is a pg**-closed set. A is neither α -closed nor semi-closed, in fact, it is not even a semi-preclosed set. Also it is not sg-closed, g α -closed and g α *-closed set.

Proposition 3.12: If A and B are pg**-closed sets, then $A \cup B$ is also a pg**-closed set.

Proof follows from the fact that $\text{pcl}(A \cup B) = \text{pcl}(A) \cup \text{pcl}(B)$.

Proposition 3.13: If A is both g*-open and pg**-closed, then A is pre closed.

Proof follows from the definition of pg**-closed sets.

Proposition 3.14: A is a pg**-closed of (X, τ) if $\text{pcl}(A) \setminus A$ does not contain any non-empty g*-closed set.

Proof: Let F be a g*-closed set of (X, τ) such that $F \subseteq \text{pcl}(A) \setminus A$. Then $A \subseteq X \setminus F$. Since A is pg**-closed and $X \setminus F$ is g*-open, $\text{pcl}(A) \subseteq X \setminus F$. This implies $F \subseteq X \setminus \text{pcl}(A)$. So, $F \subseteq (X \setminus \text{pcl}(A)) \cap (\text{pcl}(A) \setminus A) \subseteq (X \setminus \text{pcl}(A)) \cap (\text{pcl}(A)) = \emptyset$. Therefore $F = \emptyset$.

Proposition 3.15: If A is a pg**-closed set of (X, τ) such that $A \subseteq B \subseteq \text{pcl}(A)$, then B is also a pg**-closed set of (X, τ) .

Proof: Let U be a g*-open set of (X, τ) such that $B \subseteq U$. Then $A \subseteq U$, since A is pg**-closed, then $\text{pcl}(A) \subseteq U$. Now $\text{pcl}(B) \subseteq \text{pcl}(\text{pcl}(A)) = \text{pcl}(A) \subseteq U$. Therefore B is also a pg**-closed set of (X, τ) .

4. pg**-continuous and pg**-irresolute maps.

We introduce the following definitions.

Definition 4.1: A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called pg**-continuous if $f^{-1}(V)$ is a pg**-closed set of (X, τ) for every closed set of (Y, σ) .

Theorem 4.2: Every continuous map is pg**-continuous.

Proof: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be continuous and let F be any closed set of Y , then $f^{-1}(F)$ is closed in X . Since every closed set is pg**-closed, $f^{-1}(F)$ is pg**-closed. Therefore f is pg**-continuous.

The following example shows that the converse of the above theorem need not be true in general.

Example 4.3: Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\sigma = \{\emptyset, X, \{b\}\}$, $f : (X, \tau) \rightarrow (Y, \sigma)$ is defined as the identity map. The inverse image of all the closed sets of (Y, σ) are pg**-closed in (X, τ) . Therefore f is pg**-continuous but not continuous.

Thus the class of all pg**-continuous maps properly contains the class of continuous maps.

Theorem 4.4: Every pg**- continuous map is rg- continuous, gpr- continuous and gsp-continuous maps.

Proof: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a pg**- continuous map. Let V be a closed set of (Y, σ) . Since f is pg**- continuous, then $f^{-1}(V)$ is pg**- closed set in (X, τ) . By proposition (3.9) $f^{-1}(V)$ is rg-closed, gpr-closed and gsp-closed set of (X, τ) .

The converse of the above theorem need not be true as seen in the following example.

Example 4.5: Let $X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a\}\}, \sigma = \{\phi, X, \{b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then $f^{-1}(\{a\}) = \{a\}$ is not pg**- closed in (X, τ) . But $\{a\}$ is rg-closed and gpr-closed. Therefore f is rg- continuous and gpr- continuous but f is not pg**- continuous.

Example 4.6: Let $X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}, \sigma = \{\phi, X, \{b, c\}\}$.

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then $f^{-1}(\{a\}) = \{a\}$ is not pg**- closed in (X, τ) . But $\{a\}$ is gsp-closed. Therefore f is gsp- continuous but f is not pg**- continuous.

Thus the class of all pg**-continuous maps is properly contained in the classes of rg-continuous, gpr- continuous and gsp-continuous maps.

The following example shows that the compositions of two pg**- continuous maps need not be a pg**- continuous map.

Example 4.7: Let $X = Y = Z = \{a, b, c\}$ and let $f : (X, \tau) \rightarrow (Y, \sigma), g : (Y, \sigma) \rightarrow (Z, \eta)$, be the identity maps. $\tau = \{\phi, X, \{a\}, \{a, c\}\}, \sigma = \{\phi, X, \{a\}\}, \eta = \{\phi, X, \{b\}\}$. $(f \circ g)^{-1}(\{a, c\}) = f^{-1}(g^{-1}(\{a, c\})) = f^{-1}(\{a, c\}) = \{a, c\}$ is not pg**- closed in (X, τ) . But f and g are pg**- continuous maps.

Theorem 4.8: Every g*- continuous map is pg**- continuous map.

Proof: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be g*- continuous and let V be a closed set of Y . Then $f^{-1}(V)$ is g*- closed and hence by proposition (3.5), it is pg**- closed. Hence f is pg**- continuous map.

The following example shows that the converse of the above theorem is not true in general.

Example 4.9: Let $X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a\}\}, \sigma = \{\phi, X, \{b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then $A = \{a, c\}$ is closed in (Y, σ) and is pg**- closed in (X, τ) but not g*- closed in (X, τ) . Therefore f is pg**- continuous but not g*- continuous.

Theorem 4.10: Every g - continuous map is pg**- continuous map.

Proof: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be g - continuous and let V be a closed set of Y . Then $f^{-1}(V)$ is g- closed and hence by proposition (3.6), it is pg**- closed. Hence f is pg**- continuous map.

The following example shows that the converse of the above theorem is not true in general.

Example 4.11: Let $X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{a, c\}\}, \sigma = \{\phi, X, \{a, b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then $A = \{c\}$ is closed in (Y, σ) and is pg**- closed in (X, τ) but not g - closed in (X, τ) . Therefore f is pg**- continuous but not g - continuous.

Theorem 4.12: Every g**- continuous map is pg**- continuous map.

Proof: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be g**- continuous and let V be a closed set of Y . Then $f^{-1}(V)$ is g**- closed and hence by proposition (3.4), it is pg**- closed. Hence f is pg**- continuous map.

The following example shows that the converse of the above theorem is not true in general.

Example 4.13: Let $X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a, c\}\}, \sigma = \{\phi, X, \{b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then $A = \{a\}$ is closed in (Y, σ) and is pg**- closed in (X, τ) but not g**- closed in (X, τ) . Therefore f is pg**- continuous but not g**- continuous.

Definition 4.14: A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called pg**- irresolute if $f^{-1}(V)$ is a pg**- closed set of (X, τ) for every pg**- closed set V of (Y, σ) .

Definition 4.15: Let (X, τ) and (Y, σ) be two topological spaces and $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be pg**-resolute if $f(U)$ is pg**-open in Y whenever U is pg**-open in X .

Definition 4.16: A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called pg**-homeomorphism if

- (i) f is one – one and onto.
- (ii) f is pg**-irresolute and pg**-resolute.

Theorem 4.17: Every pg**-irresolute function is pg**-continuous.

Proof follows from the definition.

Theorem 4.18: Every g -irresolute function is pg**-continuous.

Proof follows from the definition.

Theorem 4.19: Every g^* -irresolute function is pg**-continuous.

Proof follows from the definition.

Theorem 4.20: Every g^{**} -irresolute function is pg**-continuous.

Proof follows from the definition.

Converse of the above theorems need not be true in general as seen in the following example.

Example 4.21: Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, c\}\}$, $\sigma = \{\emptyset, X, \{a\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = b$, $f(b) = a$, $f(c) = c$, $\{b, c\}$ is the only closed set of Y . $f^{-1}(\{b, c\}) = \{a, c\}$ is pg**-closed in (X, τ) . Therefore f is pg**-continuous. $\{b, c\}$ is g -closed, g^* -closed and g^{**} -closed set of Y but $f^{-1}(\{b, c\}) = \{a, c\}$ is not g -closed, g^* -closed and g^{**} -closed set in X .

Therefore f is not g -irresolute, g^* -irresolute and g^{**} -irresolute. Therefore f is pg**-continuous but not g -irresolute, g^* -irresolute and g^{**} -irresolute. Also $\{b, c\}$ is a pg**-closed set in Y but $f^{-1}(\{b, c\}) = \{a, c\}$ is not pg**-closed in (X, τ) . Therefore f is not a pg**-irresolute. Hence f is pg**-continuous but not pg**-irresolute.

Theorem 4.22: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ and $g: (Y, \sigma) \rightarrow (Z, \eta)$, be any two functions then,

- (i) $g \circ f$ is pg**-continuous if g is continuous and f is pg**-continuous.
- (ii) $g \circ f$ is pg**-irresolute if both f and g are pg**-irresolute.
- (iii) $g \circ f$ is pg**-continuous if g is pg**-continuous and f is pg**-irresolute.

5. Applications of pg**- closed sets

As applications of pg**-closed sets, new spaces, namely, $_pT_{1/2}^{**}$ -space, $_{ap}T_c^*$ -space, $_pT_{1/2}^*$ -space, $^{**}T_{1/2}$ -space and $_pT_c^*$ -space are introduced.

We introduce the following definition.

Definition 5.1: A space (X, τ) is called a $_pT_{1/2}^{**}$ space if every pg**-closed set is closed.

Theorem 5.2: Every $_pT_{1/2}^{**}$ space is $T_{1/2}$ space.

Proof follows from the definition.

Theorem 5.3: Every $_pT_{1/2}^{**}$ space is $T_{1/2}^*$ space.

Proof follows from the definition.

The converse need not be true in general as seen in the following example.

Example 5.4: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, $G^*C(X, \tau) = \{\emptyset, X, \{b, c\}\} = C(X, \tau)$. Therefore (X, τ) is a $T_{1/2}^*$ space but not $_pT_{1/2}^{**}$ space since $\{a, b\}$ is a pg**-closed set but not a closed set of (X, τ) .

Theorem 5.5: Every T_b space is $_pT_{1/2}^{**}$ space.

Proof follows from the definition.

The converse need not be true in general as seen in the following example.

Example 5.6: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. (X, τ) is a $_pT_{1/2}^{**}$ space but not a T_b space since $\{a\}$ is g_s -closed but not closed.

Remark 5.7: T_d -ness is independent of $_pT_{1/2}^{**}$ -ness as it can be seen from the following example.

Example 5.8: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. (X, τ) is a $_pT_{1/2}^{**}$ space but not a T_d space since $\{a\}$ is g_s -closed but not g -closed.

Example 5.9: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. (X, τ) is a T_d space but not a $_pT_{1/2}^{**}$ space since $\{c\}$ is pg^{**} -closed but not closed.

Theorem 5.10: The following conditions are equivalent in topological space (X, τ) .

- (i) (X, τ) is a $_pT_{1/2}^{**}$ space.
- (ii) Every singleton of X is either g^* -closed or open.

Proof:

(i) \Rightarrow (ii): Let (X, τ) be a $_pT_{1/2}^{**}$ space. Let $x \in X$ and suppose $\{x\}$ is not g^* -closed. Then $X \setminus \{x\}$ is not g^* -open. This implies that X is the only g^* -open set containing $X \setminus \{x\}$. Therefore $X \setminus \{x\}$ is closed since (X, τ) is a $_pT_{1/2}^{**}$ space. Therefore $\{x\}$ is open in (X, τ) .

(ii) \Rightarrow (i): Let A be a pg^{**} -closed set of (X, τ) $A \subseteq pcl(A) \subseteq cl(A)$ and let $x \in pcl(A)$ this implies $x \in cl(A)$. By (ii) $\{x\}$ is g^* -closed or open.

Case-(i): Let $\{x\}$ be g^* -closed. If $x \notin A$, then $pcl(A) \setminus A$ contains a non-empty g^* -closed set $\{x\}$. But it is not possible by proposition (3.14). Therefore $x \in A$.

Case-(ii): Let $\{x\}$ be open. Now $x \in cl(A)$, then $\{x\} \cap A \neq \emptyset$. Therefore $x \in A$ and so $cl(A) \subseteq A$ and hence $A = cl(A)$ or A is closed. Therefore (X, τ) is a $_pT_{1/2}^{**}$ space.

We introduce the following definition.

Definition 5.11: A space (X, τ) is called an $_{ap}T_c^*$ -space if every αg -closed set of (X, τ) is pg^{**} -closed.

Theorem 5.12: Every $_{\alpha}T_b$ -space is an $_{ap}T_c^*$ -space but not conversely.

Example 5.13: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$. (X, τ) is an $_{ap}T_c^*$ -space but not $_{\alpha}T_b$ -space since $\{a, c\}$ is αg -closed but not closed.

Definition 5.14: A subset A of a space (X, τ) is called a pg^{**} -open set if its complement is a pg^{**} -closed set of (X, τ) .

Theorem 5.15: If (X, τ) is an $_{ap}T_c^*$ -space for each $x \in X$, $\{x\}$ is either αg -closed or pg^{**} -open.

Proof: Let $x \in X$ suppose that $\{x\}$ is not an αg -closed set of (X, τ) . Then $\{x\}$ is not a closed set since every closed set is an αg -closed set. Therefore $X \setminus \{x\}$ is not open. Therefore $X \setminus \{x\}$ is an αg -closed set since X is the only open set which contains $X \setminus \{x\}$. Since (X, τ) is an $_{ap}T_c^*$ -space, $X \setminus \{x\}$ is a pg^{**} -closed set or $\{x\}$ is pg^{**} -open.

Remark 5.16: The converse of the above theorem is not true as it can be seen from the following example.

Example 5.17: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, c\}\}$. (X, τ) is not a $_{ap}T_c^*$ -space but $\{b\}$ αg -closed and $\{a\}$ and $\{c\}$ are pg^{**} -open.

We introduce the following definition.

Definition 5.18: A space (X, τ) is called a $^{**}_pT_{1/2}$ -space if every pg^{**} -closed set of (X, τ) is a g^* -closed set.

Theorem 5.19: Every ${}_pT_{1/2}^{**}$ space is ${}_pT_{1/2}^{**}$ -space.

Proof: Let (X, τ) be a ${}_pT_{1/2}^{**}$ space. Let A be a pg^{**} - closed set of (X, τ) . Since (X, τ) is a ${}_pT_{1/2}^{**}$ -space, A is closed. But since every closed set is g^* - closed, A is g^* - closed. Therefore (X, τ) is a ${}_pT_{1/2}^{**}$ -space.

Theorem 5.20: Every T_b -space is a ${}_pT_{1/2}^{**}$ -space.

Proof: Let (X, τ) be a T_b - space. Then by theorem (4.5), it is a ${}_pT_{1/2}^{**}$ space. Therefore by theorem (4.19), it is ${}_pT_{1/2}^{**}$ -space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.21: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, (X, τ) is a ${}_pT_{1/2}^{**}$ -space but not a T_b - space since $A = \{a\}$ is gs -closed but not closed.

Theorem 5.22: Every ${}_pT_{1/2}^{**}$ -space is a ${}^*T_{1/2}$ -space.

Proof: Let (X, τ) be a ${}_pT_{1/2}^{**}$ - space. Let A be a g - closed set of (X, τ) . Then by proposition (3.6), A is pg^{**} - closed. Since (X, τ) is an ${}_pT_{1/2}^{**}$ - space, A is g^* - closed. Therefore it is a ${}^*T_{1/2}$ -space .

The converse of the above theorem need not be true as seen in the following example.

Example 5.23: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$, (X, τ) is a ${}^*T_{1/2}$ -space but not a ${}_pT_{1/2}^{**}$ - space since $A = \{c\}$ is pg^{**} - closed but not g^* - closed.

Theorem 5.24: Every ${}_pT_{1/2}^{**}$ -space is a ${}_pT_{1/2}^{**}$ -space.

Proof: Let (X, τ) be a ${}_pT_{1/2}^{**}$ - space. Let A be a g^{**} - closed set of (X, τ) . Then by proposition (3.4), A is pg^{**} - closed. Since (X, τ) is a ${}_pT_{1/2}^{**}$ - space, A is g^* - closed. Therefore it is a ${}_pT_{1/2}^{**}$ -space .

The converse of the above theorem need not be true as seen in the following example.

Example 5.25: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, c\}\}$, (X, τ) is a ${}_pT_{1/2}^{**}$ -space but not a ${}_pT_{1/2}^{**}$ - space since $A = \{c\}$ is pg^{**} - closed but not g^* - closed.

Theorem 5.26: If (X, τ) is a ${}_pT_{1/2}^{**}$ - space, then for each $x \in X$, $\{x\}$ is either closed or g^* -open.

Proof: Suppose (X, τ) is a ${}_pT_{1/2}^{**}$ - space. Let $x \in X$ and let $\{x\}$ not be closed. Then $X \setminus \{x\}$ is not open set. Therefore $X \setminus \{x\}$ is a g -closed set since X is the only open set which contains $X \setminus \{x\}$. By theorem (3.6) $X \setminus \{x\}$ is a pg^{**} - closed set. Since (X, τ) is a ${}_pT_{1/2}^{**}$ - space, $X \setminus \{x\}$ is g^* -closed set. Therefore $\{x\}$ is g^* -open.

Definition 5.27: A space (X, τ) is called an ${}_pT_{1/2}^*$ -space if every pg^{**} -closed set of (X, τ) is g -closed.

Theorem 5.28: Every ${}_pT_{1/2}^{**}$ -space is a ${}_pT_{1/2}^*$ -space.

Proof: Let (X, τ) be a ${}_pT_{1/2}^{**}$ - space. Let A be a pg^{**} - closed set of (X, τ) . Then A is closed since (X, τ) is a ${}_pT_{1/2}^{**}$ -space. But every closed set is g -closed set, Therefore A is g - closed. Therefore (X, τ) is a ${}_pT_{1/2}^*$ -space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.29: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, (X, τ) is a ${}_pT_{1/2}^*$ -space but not a ${}_pT_{1/2}^{**}$ -space since $A = \{c\}$ is pg^{**} - closed but not closed.

Theorem 5.30: The space (X, τ) is a ${}_pT_{1/2}^{**}$ -space if and only if it is a ${}_pT_{1/2}^*$ -space and a $T_{1/2}$ space.

Proof: Necessity: Let (X, τ) be a ${}_pT_{1/2}^{**}$ -space. Let A be a g -closed set of (X, τ) . Then by theorem (3.6) A is pg^{**} - closed. Also since (X, τ) is a ${}_pT_{1/2}^{**}$ -space, A is a closed set. Therefore (X, τ) is a $T_{1/2}$ space. By theorem (4.24) (X, τ) is a ${}_pT_{1/2}^*$ -space.

Sufficiency: Let (X, τ) be a $T_{1/2}$ space and a ${}_pT_{1/2}^*$ -space. Let A be a pg^{**} - closed set. Then A is g -closed since (X, τ) is a ${}_pT_{1/2}^*$ -space. Also since (X, τ) is a $T_{1/2}$ -space, A is a closed set. Therefore (X, τ) is a ${}_pT_{1/2}^{**}$ -space.

Theorem 5.31: Every ${}^{**}T_{1/2}$ -space is a ${}_pT_{1/2}^*$ -space.

Let (X, τ) be a ${}^{**}T_{1/2}$ -space. Let A be a pg^{**} - closed set. Then A is g^* -closed since (X, τ) is a ${}^{**}T_{1/2}$ -space. But every g^* -closed set is g -closed, and hence A is a g -closed set. Therefore (X, τ) is a ${}_pT_{1/2}^*$ -space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.32: Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}\}$, (X, τ) is a ${}_pT_{1/2}^*$ -space but not a ${}^{**}T_{1/2}$ - space since $A = \{c\}$ is pg^{**} - closed but not g^* - closed.

We introduce the following definition

Definition 5.33: A space (X, τ) is called a ${}_pT_c^*$ -space if every gs - closed set of (X, τ) is a pg^{**} - closed set.

Theorem 5.34: Every T_c -space is a ${}_pT_c^*$ -space.

Proof: Let (X, τ) be a T_c - space. Let A be a gs - closed set of (X, τ) . Then A is g^* -closed since (X, τ) is a T_c -space. But by proposition (3.5) A is pg^{**} - closed set. Therefore (X, τ) is a ${}_pT_c^*$ -space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.35: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$, (X, τ) is a ${}_pT_c^*$ -space but not a T_c - space since $A = \{c\}$ is gs - closed but not g^* - closed.

Theorem 5.36: Every T_b -space is a ${}_pT_c^*$ -space.

Proof: Let (X, τ) be a T_b - space. Let A be a gs - closed set of (X, τ) . Then A is closed since (X, τ) is a T_b -space. But by proposition (3.2) A is pg^{**} - closed set. Therefore (X, τ) is a ${}_pT_c^*$ -space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.37: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$, (X, τ) is a ${}_pT_c^*$ -space but not a T_b - space since $A = \{c\}$ is gs - closed but not a closed set.

Theorem 5.38: If (X, τ) is a ${}_pT_c^*$ -space and a ${}_pT_{1/2}^*$ -space, then it is a ${}_aT_d$ -space.

Proof: Let (X, τ) be a ${}_pT_c^*$ -space and a ${}_pT_{1/2}^*$ -space. Let A be a αg - closed set of (X, τ) . Then A is also gs -closed. Since (X, τ) is a ${}_pT_c^*$ -space, A is pg^{**} - closed set. Also since (X, τ) is a ${}_pT_{1/2}^*$ -space, A is a g -closed set. Therefore (X, τ) is a ${}_aT_d$ -space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.39: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, c\}\}$, (X, τ) is a ${}_aT_d$ -space but not a ${}_pT_{1/2}^*$ - space since $A = \{c\}$ is pg^{**} - closed but not ag -closed set.

Theorem 5.40: If (X, τ) is a ${}_pT_c^*$ -space and a ${}_pT_{1/2}^{**}$ -space, then it is a ${}_aT_b$ -space.

Proof: Let (X, τ) be a ${}_pT_c^*$ -space and a ${}_pT_{1/2}^{**}$ -space. Let A be a αg - closed set of (X, τ) . Then A is also gs -closed. Since (X, τ) is a ${}_pT_c^*$ -space, A is pg^{**} - closed set. But every pg^{**} - closed set is closed since (X, τ) is a ${}_pT_{1/2}^{**}$ -space, A is a closed set. Therefore (X, τ) is a ${}_aT_b$ -space.

Theorem 5.41: If (X, τ) is a ${}_pT_c^*$ -space and a ${}_pT_{1/2}^*$ -space, then it is a T_d -space.

Proof: Let (X, τ) be a ${}_pT_c^*$ -space and a ${}_pT_{1/2}^*$ -space. Let A be a gs -closed set of (X, τ) . Since (X, τ) is a ${}_pT_c^*$ -space, A is pg^{**} - closed set. Also since (X, τ) is a ${}_pT_{1/2}^*$ -space, A is a g -closed set. Therefore (X, τ) is a T_d -space.

The converse of the above theorem need not be true as seen in the following example.

Example 5.42: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, c\}\}$, (X, τ) is a T_d -space but not a ${}_pT_{1/2}^*$ - space since $A = \{c\}$ is pg^{**} - closed but not ag -closed set.

Theorem 5.43: If (X, τ) is a $_pT_c^*$ -space, then for each $x \in X$, $\{x\}$ is either semi-closed or pg^{**} -open in (X, τ) .

Proof: Suppose (X, τ) is a $_pT_c^*$ -space. Let $x \in X$ and let $\{x\}$ not be semi-closed. Then $X \setminus \{x\}$ is sg-closed. Also $X \setminus \{x\}$ is gs-closed. Since (X, τ) is a $_pT_c^*$ -space, $X \setminus \{x\}$ is pg^{**} -closed set. Therefore $\{x\}$ is pg^{**} -open.

Theorem 5.44: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a pg^{**} -continuous map. If (X, τ) is $_pT_{1/2}^{**}$ -space then f is continuous.

Theorem 5.45: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a pg^{**} -continuous map. If (X, τ) is $^{**}T_{1/2}$ -space then f is g^* -continuous.

Theorem 5.46: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a pg^{**} -continuous map. If (X, τ) is $_pT_{1/2}^*$ -space then f is g -continuous.

Theorem 5.47: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a gs -continuous map. If (X, τ) is $_pT_c^*$ -space then f is pg^{**} -continuous.

Theorem 5.48: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a g^* -irresolute map and a pre-closed map. Then $f(A)$ is a pg^{**} -closed set of (Y, σ) for every pg^{**} -closed set A of (X, τ) .

Proof: Let A be a pg^{**} -closed set of (X, τ) . Let U be a g^* -open set of (Y, σ) such that $f(A) \subseteq U$. Since f is g^* -irresolute, $f^{-1}(U)$ is g^* -open in (X, τ) . Now $f^{-1}(U)$ is g^* -open and A is pg^{**} -closed set of (X, τ) , then $pcl(A) \subseteq f^{-1}(U)$. Then $f(pcl(A)) = pcl(f(pcl(A)))$. Therefore $pcl[f(A)] \subseteq pcl[f(pcl(A))] = f(pcl(A)) \subseteq U$. Therefore $f(A)$ is a pg^{**} -closed set of (Y, σ) .

Theorem 5.49: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be onto, pg^{**} -irresolute and closed. If (X, τ) is $_pT_{1/2}^{**}$ then (Y, σ) is also a $_pT_{1/2}^{**}$ -space.

Definition 5.50: A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called a pg^{**} -closed map if $f(A)$ is a pg^{**} -closed set of (Y, σ) for every pg^{**} -closed set A of (X, τ) .

Theorem 5.51: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be onto, pg^{**} -irresolute and pre- g^* -closed. If (X, τ) is $^{**}T_{1/2}$, then (Y, σ) is also a $^{**}T_{1/2}$ -space.

Theorem 5.52: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be onto, gs -irresolute and pg^{**} -closed map. If (X, τ) is $_pT_c^*$, then (Y, σ) is also a $_pT_c^*$ -space.

Theorem 5.53: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be onto, pg^{**} -irresolute and g -closed map. If (X, τ) is $_pT_{1/2}^*$, then (Y, σ) is also a $_pT_{1/2}^*$ -space.

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