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LORENTZ SPACES

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ABSTRACT

In mathematical analysis, Lorentz spaces, introduced by George Lorentz in the 1950s,[1][2] are generalisations of the more familiar L^p spaces.

The Lorentz spaces are denoted by $L^{p,q}$. Like the L^p spaces, they are characterized by a norm (technically a quasinorm) that encodes information about the "size" of a function, just as the L^p norm does. The two basic qualitative notions of "size" of a function are: how tall is graph of the function, and how spread out is it. The Lorentz norms provide tighter control over both qualities than the L^p norms, by exponentially rescaling the measure in both the range (p) and the domain (q). The Lorentz norms, like the L^p norms, are invariant under arbitrary rearrangements of the values of a function.

1. INTRODUCTION

Functional Analysis is considered as a powerful tool when applied to Mathematical problems related to physical situations. Operator Theory is used to study transformations between the vector spaces concerned with Functional Analysis; such as differential operators or self - adjoint operators. The analysis might study the spectrum of an individual operator or the semigroup structure of a collection of them. In the first text book on Operator Theory, “Théorie des Opẽrations Linéaires”, published in Warsaw in 1932, Stefan Banach states that the subject of the book is the study of functions on spaces of infinite dimensions, especially spaces of type B, otherwise known as Banach Spaces.

The most thorough history of Operator Theory was given by Jean Dieudonne’s in History of Functional Analysis. The concepts whose origins include: linearity, spaces of infinite dimension, matrices and the spectrum.

2. BANACH FUNCTION SPACES

Definition 2.1: Let Ω be a non – empty set. A σ - algebra Σ over Ω is a collection of subsets of Ω satisfying the following conditions:
(i) $\Omega \in \Sigma$
(ii) If $A \in \Sigma,$ then $A^c \in \Sigma,$ where A^c is the complement of *A* relative to Ω.
(iii) If $A = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \in \Sigma$ for $n = 1, 2, \ldots$ then $A \in \Sigma$

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A function $\mu : \Sigma \to [0,\infty]$ where $[0,\infty]$ is the extended interval is said to be a measure with respect to the σ - algebra Σ over Ω if it satisfies the following conditions:

(i) The empty set has measure zero : $\mu(\phi) = 0$

(ii) Countable additivity or σ - additivity if $\{E_1, E_2, E_3, \ldots\}$ is a countable sequence of pair wise disjoint sets in $\sum$, the measure of the union of all the E_i's is equal to the sum of the measures of each E_i

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

$\left(\Omega, \sum, \mu\right)$ or (Ω, μ) is then called a measure space. The elements of $\sum$ are called measurable sets.

Definition 2.2: Let (Ω, μ) be a measure space. Let $L(\mu)$ denote the collection of all extended scalar – valued μ - measurable functions on Ω, M^+ the collection μ - measurable functions on Ω whose values lie in $[0,\infty]$, and M_0 the class of functions in $L(\mu)$ that are finite $\mu - a \cdot e.$ Here *f* is said to be finite $\mu - a \cdot e.$ If its value lies in $(-\infty, \infty)$. M^+ and M_0 can also be written as $M^+(\Omega, \mu)$ and $M_0(\Omega, \mu)$, indicating their underlying measure spaces and their measure. Singularly, M_0^+ and $M_0^+(\Omega, \mu)$ denotes the class of functions in $L(\mu)$ whose values lie in $[0,\infty)$.

As usual, we say $f, g \in L(\mu)$ are equal $\mu - a \cdot e$ if $\mu\left(\{x \in \Omega : f(x) \neq g(x)\}\right) = 0.$ Any two measurable functions that are equal $\mu - a \cdot e \cdot$ will be considered as the same. Similarly, we say $f \leq g \; \mu - a \cdot e \cdot$ for $f, g \in L(\mu)$, if $\mu\left(\{x \in \Omega : f(x) \leq g(x)\}\right) = 0$.

Definition 2.3: A mapping $\rho : M^+ \to [0,\infty]$ is called a function norm if, for all $f, g, f_n, (n = 1, 2, 3, \ldots)$ in M^+, for all constants $a \geq 0$, and for all μ - measurable subsets *E* of Ω, the following properties hold:

(i) $\rho(f) = 0 \Leftrightarrow f = 0 \quad \mu - a \cdot e$

$\rho(af) = a\rho(f)$

$\rho(f + g) \leq \rho(f) + \rho(g)$

(ii) $0 \leq g \leq f \, \mu - a \cdot e \Rightarrow \rho(g) \leq \rho(f)$

(iii) $0 \leq f_n \uparrow f \, \mu - a \cdot e \Rightarrow \rho(f_n) \uparrow \rho(f)$

(iv) $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty$

(v) $\mu(E) < \infty \Rightarrow \int_E f d\mu \leq C_{E\rho}(f)$

for some constant C_E, $0 < C_E < \infty$, depending on *E* and ρ but independent of *f*.

Definition 2.4: Let ρ be a function norm the collection $\Omega = \Omega(\rho)$ of all functions *f* in $L(\mu)$ for which $\rho(f) < \infty$ is called a Banach function space. For each $f \in \Omega$ define

$$\|f\|_{\Omega} = \rho(|f|) \tag{1}$$

Definition 2.5: Let (X, Ω) be a measurable space. Let μ a finite measure on X, that is, μ takes the value in $[0,\infty)$. A set *A* in Ω is called an atom if $\mu(A) > 0,$ and for any measure subsets *B* of *A* with $\mu(A) > \mu(B),$ one has $\mu(B) = 0$ if X consists of only atoms, then μ is said to be purely atomic.

3. REARRANGEMENTS OF DISTRIBUTION FUNCTIONS

Let M_0 be the class of all functions in $L(\mu)$ that are finite $\mu - a \cdot e \cdot$ For $f \in M_0,$ define the distribution function $\mu_f : (0,\infty) \to [0,\infty]$ of $|f|$ by

$$\mu_f(\mu) = \mu\left(\left\{x \in \Omega : |f(x)| > \lambda\right\}\right) \tag{2}$$

and the non – increasing rearrangement

$f^* : [0,\infty) \to R$ of f by

$$f^*(t) = \inf\left\{\lambda > 0 : \mu_f(\lambda) \le t\right\} = \sup\left\{\lambda > 0 : \mu_f(\lambda) > t\right\} \tag{3}$$

Intuitively, the rearrangement of f rearranges $|f|$ in a decreasing order.

Example 3.1: Suppose $f(x) = \sum_{j=1}^{3} a_j \chi_{E_j}(x)$

where $a_1 > a_2 > a_3$ and $\{E_j\}$ are disjoint sets.

Now $\mu_f(\lambda) = \begin{cases} \mu(E_1) + \mu(E_2) + \mu(E_3), & 0 \le \lambda < a_3 \\ \mu(E_1) + \mu(E_2) & a_3 \le \lambda < a_2 \\ \mu(E_2) & a_1 \le \lambda < a_2 \end{cases}$

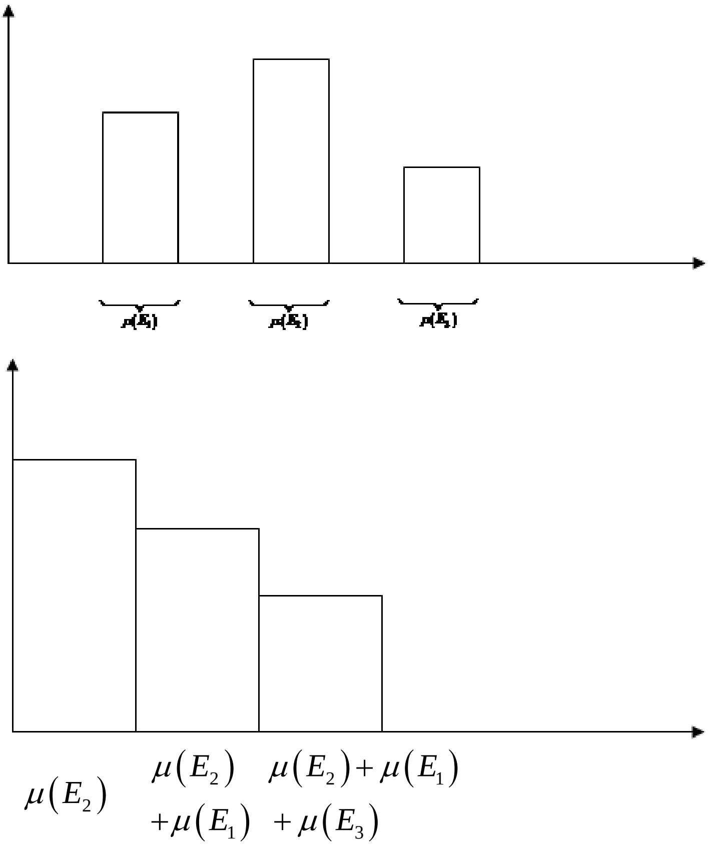

Figure-1: Graphs of f and f^*

$$f^*(t) = \begin{cases} a_2, 0 \le t < \mu(E_2) \\ a_1, \mu(E_2) \le t < \mu(E_1) + \mu(E_2) \\ a_3, \mu(E_1) + \mu(E_2) \le t < \mu(E_1) + \mu(E_2) + \mu(E_3) \end{cases}$$

The graphs of f and f^* are shown in the figure 1

Example 3.2: Suppose $g(x) = \arctan(x),\ x \geq 0$. In the context of Lebesgue measure M on $(0,\infty)$, the distribution function M_g is infinite for $0 \leq \lambda < \pi/2$ and zero for all $\lambda \geq \pi/2$. Hence $g^*(t) = \pi/2$ for all $t \geq 0$.

Figure 2 shows the graph of *g* and g^*.

The distribution function and the decreasing rearrangement f^* of *f* possesses some elementary but important properties that will be addressed frequently in the latter part of this thesis.

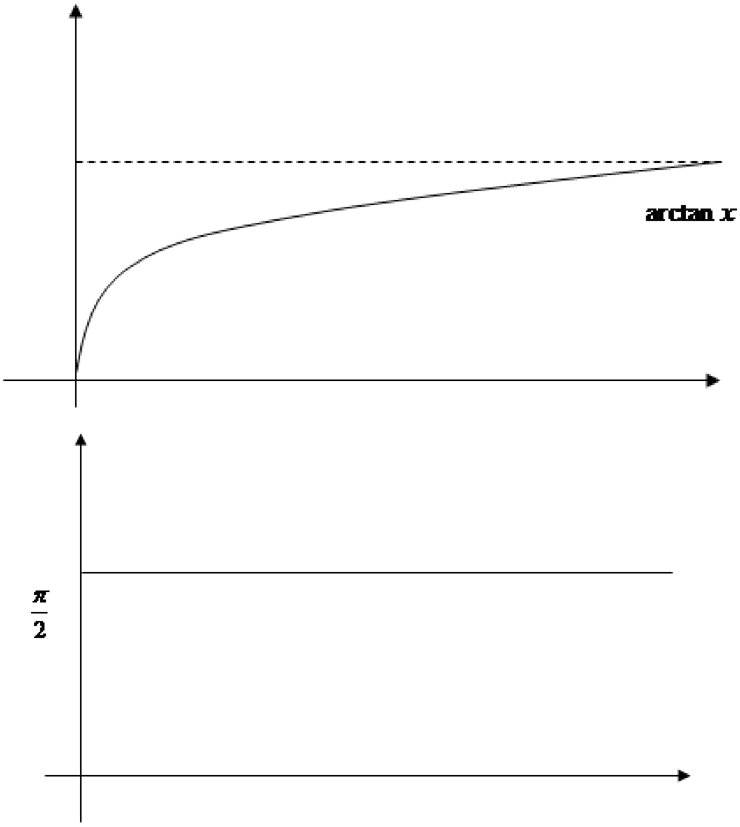

Figure-2: Graphs of *g* and g^*

Proposition 3.3: Suppose $f, g, f_n, (n = 1,2,3,...)$ belong to $M_0(\Omega, \mu)$ and let a be any non zero scalar. The distribution function μ_f is non negative, decreasing and right – continuous on $[0,\infty)$. Further more,

$$|g| \leq |f|\, \mu - a \cdot e \Rightarrow \mu_g \leq \mu_f \tag{4}$$

$$\mu_{af}(\lambda) = \mu_f(\lambda/|a|), (\lambda > 0, a \neq 0) \tag{5}$$

$$\mu_{f+g}(\lambda_1 + \lambda_2) \leq \mu_f(\lambda_1) + \mu_g(\lambda_2),\ \lambda_1\lambda_2 \geq 0 \tag{6}$$

$$|f| \leq \lim\inf_{n\to\infty} |f_n|\, \mu - a \cdot e \Rightarrow \mu_g \leq \lim\inf_{n\to\infty} \mu f_n \tag{7}$$

In particular,

$$|f_n| \uparrow |f|\, \mu - a \cdot e \Rightarrow \mu f_n \uparrow \mu_f \tag{8}$$

Proof: Non negativity and decreasing are obvious.

To establish right – continuity, let

$$E(\lambda) = \{x : |f(x)| > \lambda\}, \lambda > 0, \text{ and fix } \lambda_0 \geq 0.$$

The set $E(\lambda)$ increases as λ decreases, and

$$E(\lambda_0)=\bigcup_{n=1}^{\infty}E\left(\lambda_0+\frac{1}{n}\right).$$

Hence by the monotone convergence theorem

$$\mu_f\left(\lambda_0+\frac{1}{n}\right)=\mu\left(E\left(\lambda_0+\frac{1}{n}\right)\uparrow\mu\left(E(\lambda_0)\right)=\mu_f(\lambda_0)\right)$$

and this establishes the right – continuity

(i) If $|g|<|f|$, then for any $\lambda>0$,

$$\{x\in\Omega:|g(x)|>\lambda\}\le\{x\in\Omega:|f(x)|>\lambda\}.$$

Hence,

$$\mu_g(\lambda)=\mu\left(\{x\in\Omega:|g(x)|>\lambda\}\right)$$
$$\le\mu\left(\{x\in\Omega:|f(x)|>\lambda\}\right)=\mu f.$$

(ii) For any *a* and $\lambda>0$

$$\mu_{af}(\lambda)=\{x\in\Omega:|a\cdot f(x)|>\lambda\}$$
$$=\left\{x\in\Omega:|f(x)|>\frac{\lambda}{|a|}=\mu_f(\lambda/|a|)\right.$$

(iii) For $|f(x)+g(x)|>\lambda_1+\lambda_2$, either $|f(x)|>\lambda_1$ or $|g(x)|>\lambda_2$. So $\{x\in\Omega:|f(x)+g(x)|>\lambda\}$

$$\le\{x\in\Omega:|f(x)|>\lambda_1\}\cup\{x\in\Omega:|g(x)|>\lambda_2\}$$

which implies $\mu_{f+g}(\lambda)\le\mu_f(\lambda_1)+\mu_g$.

(iv) Fix $\lambda>0$ and let

$$E=\{x:|f(x)|>\lambda\},\ E_n=\{x:|f_n(x)|>\lambda\},\ n=1,2,3,\ldots$$

Then $E\subset\bigcup_{m=1}^{\infty}\bigcap_{n>m}E_n$.

$$\mu\left(\bigcap_{n>m}E_n\right)\le\inf_{n>m}\mu(E_n)\le\sup_k\inf_{n>k}\mu(E_n)$$
$$=\lim_{n\to\infty}\inf\mu(E_n)$$

for each $m=1,2,\ldots$

But $\bigcap_{n>m}E_n$ increases with m.

Write $F_m=\bigcup_{n>m}E_n$, then $F_1\le F_2<F_3\le\ldots$

Apply the monotone convergence theorem, one can obtain

$$\mu_f(\lambda)=\mu(E)$$
$$\le\mu\left(\bigcup_{m=1}^{\infty}\bigcap_{n>m}E_n\right)$$
$$=\mu\left(\bigcup_{m=1}^{\infty}F_m\right)$$

$$= \lim_{m\to\infty} \mu\left(F_m\right)$$
$$= \lim_{m\to\infty} \mu\left(\bigcap_{n>m} E_n\right)$$
$$\le \liminf_{n\to\infty} \mu\left(E_n\right)$$
$$= \liminf_{n\to\infty} \mu_{f_n}\left(\lambda\right)$$

which is (7) and then (8).

Proposition 3.4: Suppose f, g and $f_n, n = 1,2,3,...$ belong to $M_0\left(\Omega,\mu\right)$ and let a be a scalar. The non – increasing rearrangement f^* is a non negative, decreasing, right – continuous function on $\left[0,\infty\right)$. Furthermore,

$$|g| \le |f| \mu - a\cdot e \Rightarrow g^* \le f^* : \tag{9}$$
$$(af)* = |a| f^*. \tag{10}$$
$$\left(f+g\right)^*\left(t_1+t_2\right) \le f^*\left(t_1\right) + g^*\left(t_2\right),\ t_1, t_2 \ge 0 \tag{11}$$
$$|f| \le \liminf_{n\to\infty} |f_n| \mu - a\cdot e \Rightarrow f^* \le \liminf_{n\to\infty} f_n^* \tag{12}$$

In particular,

$$|f_n| \uparrow |f| \mu - a\cdot e \Rightarrow f_n^* \uparrow f^* \tag{13}$$
$$f^*\left(\mu_f\left(\lambda\right)\right) \le \left(\mu_f\left(\lambda\right) < \infty\right);$$
$$\mu_f\left(f^*\left(f\right)\right) \le t, \left(f^*\left(t\right) < \infty\right) \tag{14}$$

f and f^* are equimeasurable

$$\left(|f|^p\right)^* = \left(f^*\right)^p,\ 0 < 1 < \infty \tag{15}$$

Here, we say two measurable functions are equimeasurable if they have the same distribution functions.

Proof: Non negativity and decreasing are obvious and the right – continuity of f^* is a direct consequences of the right – continuity μ_f .. Properties (9) to (13) are immediate consequences of proposition 3.3.

For any λ_0, let $t_0 = \mu\left(\lambda_0\right)$

$$f^*\left(\mu_f\left(\lambda_0\right)\right) = f^*\left(t_0\right) = \inf\left\{\lambda > 0 : \mu_f\left(\lambda\right) \le t_0\right\}$$
$$= \inf\left\{\lambda > 0 : \mu_f\left(\lambda\right) \le \mu_f\left(\lambda\right)\right\} \le \lambda_0$$

Fix $t \ge 0$ and suppose $\lambda = f^*\left(t\right)$ infinite.

By the definition of f^*, there is sequence $\lambda_n \downarrow \lambda$ with $\mu_f\left(\lambda_n\right) \le t$, so the right – continuity of μ_f gives

$$\mu_f\left(f^*\left(t\right)\right) = \mu_f\left(\lambda\right) = \lim_{n\to\infty} \mu_f\left(\lambda_n\right) \le t.$$

These two equations give (14)

From (12) in many situations we can consider the cases of simple functions only, and then apply this property to obtain the result of general cases by taking the limit of those simple functions.

4. MAXIMAL OPERATORS

Let *f* belong to $M_0(\Omega,\mu)$. Then f^* denotes the maximal function of *f*, defined by

$$f^{**}(t)=\frac{1}{t}\int_0^t f^*(s)\,ds,\ t>0$$

A maximal operator is the mapping taking *f* to its maximal function f^{**}. The following proposition provides us some basic properties of the maximal operators.

Proposition 4.1: Suppose *f*, *g* and $f_n, n=1,2,3...$ belong to M_0, and let a be any scalar. Then f^{**} is non negative, decreasing and continuous on $(0,\infty)$. Furthermore, the following properties holds.

$$f^{**}=0 \Leftrightarrow f=0\ \mu-a\cdot e \tag{17}$$

$$f^{*}\le f^{**} \tag{18}$$

$$|g|\le|f|\,\mu-a\cdot e\Rightarrow g^{**}\le f^{**} \tag{19}$$

$$(af)=|a|\,f^{**} \tag{20}$$

$$|f_n|\uparrow|f|\,\mu-a\cdot e\Rightarrow f_n^{**}\uparrow f^{**} \tag{21}$$

Proof: Properties (19), (20) and (21) are direct consequences of proposition 4.1. For (18), consider

$$f^{**}(t)=\frac{1}{t}\int_0^t f^*(s)\,ds\ge f^*(t)\frac{1}{t}\int_0^t ds=f^*(t).$$

So we have (18)

To establish (17), if $f=0\mu-a\cdot e,$ then $f^*=0$ and hence $f^{**}(t)=0.$ On the other hand if $f^{**}(t)=0$ by (18), for every $t\in R,$ $f^*(t)\le f^{**}(t)$. So $f^*\equiv 0,$ which implied ess

$\sup f=\inf\left\{a\in R:\mu\left(\left\{x:\left|f(x)\right|>a\right\}\right)=0\right\}=0$ that is $f=0\ \mu-a\cdot e\cdot$ This establishes (17).

5. IMPORTANT INEQUALITIES

There are several inequalities that contribute to the construction of rearrangement – invariant spaces as well as Lorentz spaces. Hardy's inequality (Lemma 5.3) will be used to prove another important Lemma. Lemma 5.1 which gives the equivalence of the two definitions of Lorentz spaces.

Lemma 5.1: Let *g* be a non negative simple function on (Ω,μ) and let *E* be an arbitrary $\mu-$measurable subset of Ω. Then

$$\int_E g\,d\mu\le\int_0^{\mu(E)} g^*(s)\,ds \tag{22}$$

Proof: Express *g* in the form of

$$g(x)=\sum_{j=1}^{n} b_j\chi_{F_j}(x)$$

where $F_1\subseteq F_2\subseteq...\subseteq F_n$ and $b_j>0,\ j=1,2,...,n.$

Then $g^*(x)=\sum_{j=1}^{n} b_j\chi_{(0,\mu(F_i))}(x)$

By a simple computation, we obtain

$$\int_E g\,d\mu=\sum_{j=1}^{n} b_j\mu\left(E\bigcap F_j\right)$$

$$\leq \sum_{j=1}^{n} b_j \min\left(\mu(E), \mu(F_j)\right)$$

$$= \sum_{j=1}^{n} b_j \int_0^{\mu(E)} \chi_{(0,\mu(F_j))}(s)\,ds$$

$$= \int_0^{\mu(E)} g^*(s)\,ds$$

Theorem 5.2: If *f* and *g* belong to $M_0 = M_0(\Omega, \mu)$, then

$$\int_\Omega |fg|\,d\mu \leq \int_0^\infty f^*(s) g^*(s)\,ds \tag{23}$$

Proof: By (12) is proposition 4.2 and the definition of f^*, it suffices to consider simple and positive f and g. Write

$$f(x) = \sum_{j=1}^{m} a_j \chi_{F_j}(x)$$

where $E_1 \leq E_2 \leq \ldots \leq E_m$ and $a_j > 0$, $j = 1, 2, \ldots, m$. Then

$$f^*(t) = \sum_{j=1}^{m} a_j \chi_{(0,\mu(F_j))}(t)$$

By lemma 4.1

$$\int |fg|\,d\mu = \sum_{j=1}^{m} a_j \int_{E_j} g\,d\mu$$

$$\leq \sum_{j=1}^{m} a_j \int_0^{\mu(E_j)} g^*(s)\,ds$$

$$= \int_0^\infty \sum_{j=1}^{m} a_j \chi_{[0,\mu(E_j))}(s) g^*(s)\,ds$$

$$= \int_0^\infty f^*(s) g^*(s)\,ds.$$

Lemma 5.3: Let ψ be a non negative measurable function on $(0,\infty)$ and suppose $-\infty < \lambda < 1$ and $1 \leq q \leq \infty$. Then

$$\left\{\int_0^\infty \left(t^\lambda \frac{1}{t}\int_0^t \psi(s)\,ds\right)^q \frac{dt}{t}\right\}^{1/q} \leq \frac{1}{1-\lambda}\left\{\int_0^\infty t^\lambda \psi(t)^q\right\}^{1/q} \tag{2.24}$$

and

$$\left\{\int_0^\infty \left(t^{1-\lambda} \frac{1}{t}\int_t^\infty \psi(s)\,ds\right)^q \frac{dt}{t}\right\}^{1/q} \leq \frac{1}{1-\lambda}\left\{\int_0^\infty \left(t^{1-\lambda} \psi(t)\right)^q\right\}^{1/q} \tag{2.25}$$

With the obvious modification if $q = \infty$.

Proof: Writing $\psi(s) = s^{-\lambda/q'} s^{\lambda/q'} \psi(s)$ and applying Holder's in equality, we obtain

$$\frac{1}{t}\int_0^t \psi(s)\,ds \leq \left(\frac{1}{t}\int_0^\infty s^{-\lambda}\,ds\right)^{1/q'} \left(\frac{1}{t}\int_0^t s^{\lambda q/q'} \psi(s)^q\,ds\right)^{1/q}$$

$$= (1-\lambda)^{-1/q'} t^{-\lambda/q' - 1/q} \left(\int_0^t S^{\lambda(q-1)} \psi(S)^q\,ds\right)^{1/q} \tag{26}$$

Let $q' = q/q-1$ in (26) and we obtain

$$\int_0^\infty \left(t^{\lambda} \int_0^t \psi(s)\,ds \right)^q \frac{dt}{t} \le (1-\lambda)^{1-q} \int_0^\infty t^{\lambda-2} \int_0^t S^{\lambda(q-1)} \psi(S)^q\, dsdt$$

$$= (1-\lambda)^{1-q} \int_0^\infty S^{\lambda(q-1)} \psi(S)^q \int_0^\infty t^{\lambda-1} dtds$$

by changing the order of integration. Integrating over t and taking q^{th} roots, we obtain (24). The proof of (25) is similar.

6. REARRANGEMENT – INVARIANT SPACES

Definition 6.1: A totally σ - finite measure space (Ω, μ) is called resonant if for each *f* and *g* in $M_0(\Omega, \mu)$, the following identity holds.

$$\int_0^\infty f^*(t)\, g^*(t)\, dt = \sup \int_\Omega \left| f\,\overline{g} \right| d\mu \tag{27}$$

Where the supremum is taken over all functions $\overline{g}$ on Ω equimeasurable with *g*.

Similarly (Ω, μ) is said to be strongly resonant if for each pair of functions f and g in $M_0(\Omega, \mu)$, there exists a function $\overline{g}$ on Ω equimeasurable with g such that

$$\int_0^\infty f^*(t)\, g^*(t)\, dt = \int_\Omega \left| f\,\overline{g} \right| d\mu \tag{28}$$

Note: Totally $\sigma-$ finite means $\sigma-$ finite in the old times.

Definition 6.2: Let ρ be a function norm over a totally $\sigma-$finite measure space (Ω, μ). Then ρ is said to be rearrangement – invariant is $\rho(f) = \rho(g)$ for every pair of equi measurable functions *f* and *g* in $M_0^+(\Omega, \mu)$.

In that case, the Banach function space $\Omega = \Omega(\rho)$ generated by ρ is called a rearrangement – invariant space.

Definition 6.3: Let Ω be a rearrangement-invariant Banach function space over a resonant measure space (Ω, μ). For each finite value of $t \ge 0$. Let *E* be a subset of Ω with $\mu(E) = t$ and let

$$\psi_\Omega(t) = \|\psi_E\|_\Omega \tag{29}$$

The function ψ_Ω so defined is called the fundamental function of Ω.

Note that if $\mu(E) = \mu(F) = t$, then ψ_E are equimeasurable and hence $\|\psi_E\|_\Omega = \|\psi_F\|_\Omega$.

It follows that φ_Ω is well defined.

7. LORENTZ SPACES

Definition 7.1 (Lorentz spaces $L^{p,q}$): Let (Ω, μ) be a totally $\sigma-$ finite measurable space and suppose $0 < p, q \le \infty$. Let $f \in M_0(\Omega, \mu)$

We write,

$$\|f\|_{p,q} = \begin{cases} \left\{ \int_0^\infty \left[t^{1/p} f^*(t) \right]^q \frac{dt}{t} \right\}, & 0 < q < \infty \\ \sup\limits_{0<t<\infty} \left\{ t^{1/p} f^*(t) \right\}, & q = \infty \end{cases} \tag{30}$$

The Lorentz space $L^{p,q} = L^{p,q}(\Omega,\mu)$ is defined to be the norm space $\left\{ f \in M_0(\Omega,\mu) : \|f\|_{p,q} < \infty \right\}$ whose norm is taken to be $\|\cdot\|_{p,q}$.

Definition 7.2 (Lorentz spaces L^{pq}): Let (Ω,μ) be a totally $\sigma-$ finite measure space and suppose $1 < p \le \infty$ and $0 < q \le \infty$. Let $f \in M_0(\Omega,\mu)$, we write

$$\|f\|_{p,q} = \begin{cases} \left\{ \int_0^\infty \left[t^{1/p} f^{**}(t) \right]^q \frac{dt}{t} \right\}^{1/q}, & 0 < q < \infty \\ \sup\limits_{0<t<\infty} \left\{ t^{1/p} f^{**}(t) \right\}, & q = \infty \end{cases} \tag{31}$$

The Lorentz space $L^{p,q} = L^{pq}(\Omega,\mu)$ is defined to be the norm space $\left\{ f \in M_0(\Omega,\mu) : \|f\|_{p,q} < \infty \right\}$ whose norm is taken to be $\|.\|_{pq}$

Lemma 7.3: If $1 < p \le \infty$ and $1 \le q \le \infty$, then $\|f\|_{p,q} \le \|f\|_{p,q} \le p'\|f\|_{p,q}$ (32)

For all $f \in M_0(\Omega,\mu)$, where $p' = 1/(p-1)$. In particular, $L^{p,q}$ consists of all *f* for which $\|f\|_{p,q}$ is finite.

Proof: The first inequality in (32) follows immediately from (18), i.e., $f^* \le f^{**}$, and the definitions 7.1 and 7.2. For the second inequality, Let $\lambda = 1/p$ in (24).

Compute,

$$\|f\|_{pq} = \left\{ \int_0^\infty \left[t^{1/p} f^{**}(t) \right]^q \frac{dt}{t} \right\}^{1/q}$$

$$\le \frac{1}{1-1/p} \left\{ \int_0^\infty \left[t^{1/p} f^{*}(t) \right]^q \frac{dt}{t} \right\}^{1/q}$$

$$= \frac{p}{p-1} \|f\|_{p,q}$$

This is the desired inequality.

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