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TRANSIENT BEHAVIOUR OF M/G/1 RETRIAL QUEUEING SYSTEM WITH CONSTANT RETRIAL POLICY BY SUPPLEMENTARY VARIABLE TECHNIQUE

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ABSTRACT

*Consider a single server retrial queueing system in which customers arrive in a Poisson process with arrival rate λ. These customers are identified as primary calls. Further assume that the service time follows a general distribution with probability density function $b(x)$ and cumulative distribution function $B(x)$. If the server is free at the time of a primary call arrival, the arriving call begins to be served immediately by the server and after completion of the service, the customer leaves the system and server becomes idle. If the server is busy, then the arriving customer goes to orbit of infinite capacity and makes a retrial at a later time. This pool of sources of repeated calls may be viewed as a sort of queue (FCFS). We assume that the access from the orbit to the service facility is governed by the constant retrial rate of policy, that is the retrial customer at the head of the retrial queue repeats its request for service with an exponentially distributed retrial times with retrial rate $(1-\delta_{0n})\sigma$, where δ_{0n} denotes Kronecker's delta. If an incoming repeated call finds the server free, it is served in the same manner and leaves the system after service completion, while the source which produced this repeated call disappears. Otherwise, the system state does not change. The input flow of primary calls, interval between repetitions and service time are mutually independent. The transient behaviour of this model is analysed by Supplementary variable technique. **S**teady state probability distributions and system performance measures have been derived for various service time distributions (Exponential and Erlang). Numerical study have been done for Analysis of Mean number of customers in the orbit, Probabilities of server free, busy for various values of system parameters.*

Keywords: *Retrial queue – Constant retrial policy – General Service time – elapsed service time - Supplementary variable technique.*

1. INTRODUCTION

Queueing systems in which arriving customers who find server and waiting positions (if any) occupied may retry for service after a period of time are called Retrial queues. Models of retrial queues are an important part of queueing theory. These models arise because of the necessity to allow for the retrial effect in various networking systems and day to day life. Therefore, much attention is paid to the analysis of such models of queues. Retrial queueing models accurately describe the operation of many telecommunication networks. So their investigation is very important. Retrial queues have been considered as an interesting problem in tele-traffic theory and telephone networks where subscribers redial after receiving a busy signal. For example, peripherals in computer systems may make retrials to receive service from a central processor. Hosts in local area networking (LAN) may make many retrials in order to access the communication medium, which is clearly indicated in the carrier sense multiple access (CSMA) protocol that controls this access. We would like to point out that the Retrial queues can be directly applied to solve many practical problems. The detailed survey of retrial queues and bibliographical information have been obtained from Kulkarni (1983), Yang and Templeton (1987), K. Farahmand (1990), Templeton (1990), Falin (1990), Kulkarni and Liang (1996), monograph by Falin and Templeton (1997) and Artalejo (1995, 1999a, 1999b, 2010). Artalejo and Falin (2002) have compared standard and retrial queueing systems. Artalejo and Gomez–corral (2008) have studied a computational approach of retrial queueing systems.

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2. DESCRIPTION OF RETRIAL QUEUEING SYSTEM

This model is studied on the basis of the following assumptions.

Assumptions

1. Primary arrivals follows a Poisson process with rate λ i.e. the time between the primary arrivals is an exponential distribution with mean $\frac{1}{\lambda}$.
2. The service time follows a General distribution with probability density function $b(x)$ and distribution function is $B(x)$. Let $\mu(x)dx$ be the conditional probability density of completion of a service during the interval $(x, x+\Delta x)$ given that the elapsed service time x, so that

$$\mu(x) = \frac{b(x)}{1-B(x)} \tag{1}$$

and ,therefore

$$b(x) = \mu(x)e^{-\int_0^x \mu(t)dt} \tag{2}$$

3. The size of the population is infinite.
4. The size of the orbit is infinite.
5. The service will be given one by one.
6. The probability of one primary arrival during the time interval (t, t + Δt) is λΔt+ O(Δt).
7. The Probability of repeated attempt during the interval (*t, t* + Δt) is given that σ Δt + O(Δt).
8. The probability of more than one primary / repeated arrival during the time interval (t, t + Δt) is 0.
9. The probability of more than one departure during the time interval (t, t + Δt) is 0.
10. The number of primary/repeated arrivals in non-overlapping intervals is statistically independent.
11. The number of departures in non-overlapping intervals are statistically independent.

Consider a single server retrial queueing system with constant retrial policy in which customers arrive in a Poisson process with arrival rate λ. These customers are identified as primary calls. Further assume that the service time follows a general distribution with probability density function $b(x)$ and cumulative distribution function $B(x)$. If the server is free at the time of a primary call arrival, the arriving call begins to be served immediately by the server and after completion of the service, the customer leaves the system and server becomes idle. If the server is busy, then the arriving customer goes to orbit of infinite capacity and makes a retrial at a later time. This pool of sources of repeated calls may be viewed as a sort of queue (FCFS). We assume that the access from the orbit to the service facility is governed by the constant retrial rate of policy, that is the retrial customer at the head of the retrial queue repeats its request for service with an exponentially distributed retrial times with retrial rate $(1-\delta_{0n})\sigma$,where δ_{0n} denotes Kronecker's delta. If an incoming repeated call finds the server free, it is served in the same manner and leaves the system after service completion, while the source which produced this repeated call disappears. Otherwise, the system state does not change. The input flow of primary calls, interval between repetitions and service time are mutually independent. The transient behaviour of this model is analysed by Supplementary variable technique. **S**teady state probability distributions and system performance measures have been derived for various service time distributions (Exponential and Erlang). Numerical study have been done for Analysis of Mean number of customers in the orbit, Probabilities of server free, busy for various values of system parameters.

3. DEFINITIONS AND EQUATIONS GOVERNING THE SYSTEM

Let $P_{n1}(x,t)$ be the probability that there are $n \geq 0$ customers in the orbit at time t excluding the one in service and elapsed service time of this customer is *x*, that is

$$P_{n1}(x,t)dx = Prob\{N(t) = n; x < X_0(t) < x + \Delta x\}$$

where $X_0(t)$ is the elapsed service time of a customer who is in service at time t. Let $P_{n0}(t)$ be the probability that there are $n \geq 0$ customers in the orbit at time t when the server is idle in the system. $P_{n1}(t) = \int_0^\infty P_{n1}(x,t)dx$ is the probability that server is busy and there are n customers in the orbit excluding the one in service irrespective of the elapsed time *x*. If the system is in steady state, we define $P_{n1}(x) = \lim_{t\to\infty} P_{n1}(x,t)$ and $P_{n1} = \lim_{t\to\infty} P_{n1}(t)$.

The governing equations are described as follows

$$\frac{\partial}{\partial x}P_{n1}(x,t)+\frac{\partial}{\partial t}P_{n1}(x,t)+\left(\lambda+\mu(x)\right)P_{n1}(x,t)=\lambda P_{n-11}(x,t) \text{ for n = 1,2,3,...} \tag{3}$$

$$\frac{\partial}{\partial x}P_{01}(x,t)+\frac{\partial}{\partial t}P_{01}(x,t)+\left(\lambda+\mu(x)\right)P_{01}(x,t)=0 \tag{4}$$

$$\frac{d}{dt}P_{n0}(t)+(\lambda+\sigma)P_{n0}(t)=\int_0^\infty P_{n1}(x,t)\mu(x)dx \text{ for n = 1,2,3,...} \tag{5}$$

$$\frac{d}{dt}P_{00}(t)+\lambda P_{00}(t)=\int_0^\infty P_{01}(x,t)\mu(x)d\ =\int_0^\infty P_{01}(x,t)\mu(x)dx \tag{6}$$

The above equations (3) to (6) are solved subject to the following boundary conditions

$$P_{n1}(0,t)=\lambda P_{n0}(t)+\sigma P_{n+10}(t) \tag{7}$$

$$P_{01}(0,t)=\lambda P_{00}(t)+\sigma P_{10}(t) \tag{8}$$

Initial condition is

$$P_{00}(0)=1 \tag{9}$$

We define the following probability generating functions for the server is idle/busy in the transient state

$$P_0(t,z)=\sum_{n=0}^\infty P_{n0}(t)z^n \text{ and } P_1(t,z)=\sum_{n=0}^\infty P_{n1}(t)z^n \tag{10}$$

The Laplace transform of $f(t)$ is defined by

$$f^*(s)=\int_0^\infty e^{-st}f(t)dt \tag{11}$$

Apply Laplace to the equations (3) to (10), we get

$$\frac{\partial}{\partial x}P_{n1}{}^*(x,s)+\left(s+\lambda+\mu(x)\right)P_{n1}{}^*(x,s)=\lambda P^*{}_{n-11}(x,s) \text{ for n = 1,2,3,...} \tag{12}$$

$$\frac{\partial}{\partial x}P_{01}{}^*(x,s)+\left(s+\lambda+\mu(x)\right)P_{01}{}^*(x,s)=0 \tag{13}$$

$$(s+\lambda+\sigma)P_{n0}{}^*(t)=\int_0^\infty P_{n1}{}^*(x,s)\mu(x)dx \text{ for n = 1,2,3,...} \tag{14}$$

$$\left(\lambda+s\right)P_{00}{}^*(s)=1+\int_0^\infty P_{01}{}^*(x,s)\mu(x)dx \tag{15}$$

The above equations (12) to (15) are solved subject to the following boundary conditions

$$P_{n1}{}^*(0,s)=\lambda P_{n0}{}^*(s)+\sigma P_{n+10}{}^*(s) \tag{16}$$

$$P_{01}{}^*(0,s)=\lambda P_{00}{}^*(s)+\sigma P_{10}{}^*(s) \tag{17}$$

Theorem 1: For the M/G/1 retrial queueing system with constant retrial policy,

a. The transient solution of the number of customers in the orbit when the server is idle in the system is given by

$$P_0^*(s,z)=\frac{1+\sigma P_{00}^*(s)-\frac{\sigma}{z}P_{00}^*(s)\bar{b}(s+\lambda-\lambda z)}{(s+\lambda+\sigma)-\left(\lambda+\frac{\sigma}{z}\right)\bar{b}(s+\lambda-\lambda z)}$$

b. The transient solution of the number of customers in the orbit when the server is busy in the system is given by

$$P_1^*(s,z)=\frac{\left(\lambda+\frac{\sigma}{z}\right)\left(\frac{1+\sigma P_{00}^*(s)-\frac{\sigma}{z}P_{00}^*(s)\bar{b}(s+\lambda-\lambda z)}{(s+\lambda+\sigma)-\left(\lambda+\frac{\sigma}{z}\right)\bar{b}(s+\lambda-\lambda z)}\right)-\frac{\sigma}{z}P_{00}^*}{s+\lambda-\lambda z}\left(1-\bar{b}\left(s+\lambda-\lambda z\right)\right)$$

c. The Steady state solution of the number of customers in the orbit when the server is idle in the system is given by

$$P_0(z) = \frac{p_{00}}{1-\left(\frac{\lambda z}{\sigma}\right)\psi(z)} \text{ where } \psi(z) = \frac{1-\bar{b}}{\bar{b}-z},\ P_{00} = 1-\rho_1 \text{ and } \rho_1 = \frac{\lambda(\lambda+\sigma)}{\sigma}E(X)$$

d. The Steady state solution of the number of customers in the orbit when the server is busy in the system is given by

$$P_1(z) = P_{00}\left(\frac{\psi(z)}{1-\left(\frac{\lambda z}{\sigma}\right)\psi(z)}\right)$$

Proof: We define the generating functions

$$w(x,t,z) = \sum_{n=0}^{\infty} P_{n1}(x,t)z^n \tag{18}$$

$$P_0(t,z) = \sum_{n=0}^{\infty} P_{n0}(t)z^n \tag{19}$$

Apply the Laplace transform to the equations (18) and (19), we get

$$w^*(x,s,z) = \sum_{n=0}^{\infty} P^*{}_{n1}(x,s)z^n \tag{20}$$

$$P^*{}_0(s,z) = \sum_{n=0}^{\infty} P^*{}_{n0}(s)z^n \tag{21}$$

Partially differentiate (20) with respect to *x*

$$\frac{\partial}{\partial x}w^*(x,s,z) = \sum_{n=0}^{\infty}\frac{\partial}{\partial x}P^*{}_{n1}(x,s)z^n \tag{22}$$

Substituting (12) and (13) in (20), we get

$$\frac{\partial}{\partial x}w^*(x,s,z) + \left(s+\lambda-\lambda z+\mu(x)\right)w^*(x,s,z) = 0 \tag{23}$$

Solving the above differential equation, we get

$$w^*(x,s,z) = w^*(0,s,z)e^{-(s+\lambda-\lambda z)x}e^{-\int_0^x \mu(x)dx} \tag{24}$$

Where

$$w^*(0,s,z) = \sum_{n=0}^{\infty} P^*{}_{n1}(0,s)z^n \tag{25}$$

Substituting (16) and (17) in (25), we get

$$w^*(0,s,z) = \left(\lambda + \frac{\sigma}{z}\right)P_0^*(s,z) - \frac{\sigma}{z}P_{00}^* \tag{26}$$

Multiplying the equation (24) by $\mu(x)$ and integrating with respect to x between 0 and ∞

$$\int_0^{\infty} w^*(x,s,z)\mu(x)dx = w^*(0,s,z)\int_0^{\infty} e^{-(s+\lambda-\lambda z)x}e^{-\int_0^x \mu(x)dx}\mu(x)dx \tag{27}$$

$$\int_0^{\infty} w^*(x,s,z)\mu(x)dx = w^*(0,s,z)\int_0^{\infty} e^{-(s+\lambda-\lambda z)x}b(x)dx \tag{28}$$

$$\int_0^{\infty} w^*(x,s,z)\mu(x)dx = w^*(0,s,z)\bar{b}(s+\lambda-\lambda z) \tag{29}$$

where $\bar{b}(s)=\int_0^\infty e^{-sx}b(x)dx$ is the Laplace transform of the service time dustribution

Integrating the equation (24) with respect to x between 0 and ∞, we get

$$P_1^*(s,z)=\frac{w^*(0,s,z)}{s+\lambda-\lambda z}\left(1-\bar{b}\left(s+\lambda-\lambda z\right)\right) \tag{30}$$

Substituting the equation (26) in (30), we get

$$P_1^*(s,z)=\frac{\left(\lambda+\frac{\sigma}{z}\right)P_0^*(s,z)-\frac{\sigma}{z}P_{00}^*}{s+\lambda-\lambda z}\left(1-\bar{b}\left(s+\lambda-\lambda z\right)\right) \tag{31}$$

Multiplying the equation (14) by z^n on both sides and summing over n = 1 to ∞, we get

$$(s+\lambda+\sigma)\sum_1^\infty P_{n0}{}^*(t)z^n=\int_0^\infty\left(\sum_1^\infty P_{n1}{}^*(x,s)z^n\right)\mu(x)dx \tag{32}$$

$$(s+\lambda+\sigma)\left(P_0^*\left(s,z\right)-P_{00}^*(s)\right)=\int_0^\infty\left(w^*(x,s,z)-P_{01}^*(x,s)\right)\mu(x)d\ x \tag{33}$$

Substituting the equations (15) and (29) in (33), we get

$$(s+\lambda+\sigma)P_0^*\left(s,z\right)=1+\sigma P_{00}^*(s)+w^*(0,s,z)\bar{b}(s+\lambda-\lambda z) \tag{34}$$

Substituting the equation (26) in (34), we get

$$(s+\lambda+\sigma)P_0^*\left(s,z\right)=1+\sigma P_{00}^*(s)+\left[\left(\lambda+\frac{\sigma}{z}\right)P_0^*(s,z)-\frac{\sigma}{z}P_{00}^*(s)\right]\bar{b}(s+\lambda-\lambda z) \tag{35}$$

$$P_0^*\left(s,z\right)=\frac{1+\sigma P_{00}^*(s)-\frac{\sigma}{z}P_{00}^*(s)\bar{b}(s+\lambda-\lambda z)}{(s+\lambda+\sigma)-\left(\lambda+\frac{\sigma}{z}\right)\bar{b}(s+\lambda-\lambda z)} \tag{36}$$

The equation (36) represents Transient solutions of the number of customers in the orbit when the server is idle in the system.

$$P_1^*(s,z)=\frac{\left(\lambda+\frac{\sigma}{z}\right)\left(\frac{1+\sigma P_{00}^*(s)-\frac{\sigma}{z}P_{00}^*(s)\bar{b}(s+\lambda-\lambda z)}{(s+\lambda+\sigma)-\left(\lambda+\frac{\sigma}{z}\right)\bar{b}(s+\lambda-\lambda z)}\right)-\frac{\sigma}{z}P_{00}^*}{s+\lambda-\lambda z}\left(1-\bar{b}\left(s+\lambda-\lambda z\right)\right) \tag{37}$$

The equation (37) represents Transient solutions of the number of customers in the orbit when the server is busy in the system.

Apply the Tauberian theorem of the Laplace transform to the equation (35), we get

$$(\lambda+\sigma)P_0\left(z\right)=\sigma P_{00}+\left[\left(\lambda+\frac{\sigma}{z}\right)P_0(z)-\frac{\sigma}{z}P_{00}\right]\bar{b}(\lambda-\lambda z) \tag{38}$$

Simplify the equation (38), we get

$$P_0(z)=\frac{\sigma p_{00}\left(z-\bar{b}\right)}{z(\lambda+\sigma)-\left(\sigma+\lambda z\right)\bar{b}} \tag{39}$$

$$P_0(z)=\frac{p_{00}}{1-\left(\frac{\lambda z}{\sigma}\right)\psi(z)}\ \text{ where }\ \psi(z)=\frac{1-\bar{b}}{\bar{b}-z} \tag{40}$$

The equation (40) represents the steady state probability generating function of the number of customers in the orbit when the server is idle in the system.

Apply the Tauberian theorem of the Laplace transform to the equation (31), we get

$$P_1(z)=\frac{\left(\lambda+\frac{\sigma}{z}\right)P_0(z)-\frac{\sigma}{z}P_{00}}{\lambda-\lambda z}\left(1-\bar{b}\left(\lambda-\lambda z\right)\right) \tag{41}$$

Substituting the equation (40) in (41), we get

$$P_1(z)=P_{00}\left(\frac{\psi(z)}{1-\left(\frac{\lambda z}{\sigma}\right)\psi(z)}\right) \tag{42}$$

The equation (42) represents the steady state probability generating function of the number of customers in the orbit when the server is busy in the system.

The normalised condition is

$$P_0(1)+P_1(1)=1 \tag{43}$$

$$\psi(1)=\frac{\lambda E(X)}{1-\lambda E(X)}=\frac{\rho}{1-\rho} \text{ where } \rho=\lambda E(X) \tag{44}$$

$$\psi'(1)=\frac{\lambda^2 E(X^2)}{2\left(1-\rho\right)^2} \tag{45}$$

$$P_0(1)=\lim_{z\to 1}P_0(z)=\lim_{z\to 1}\frac{p_{00}}{1-\left(\frac{\lambda z}{\sigma}\right)\psi(z)}=P_{00}\left(\frac{1-\rho}{1-\rho_1}\right) \tag{46}$$

$$P_1(1)=\lim_{z\to 1}P_1(z)=\lim_{z\to 1}P_{00}\left(\frac{\psi(z)}{1-\left(\frac{\lambda z}{\sigma}\right)\psi(z)}\right)=P_{00}\left(\frac{\rho}{1-\rho_1}\right) \tag{47}$$

Substituting the equations (46) and (47) in (43), we get

$$P_{00}=1-\rho_1 \tag{48}$$

Theorem 2: The system performance measures of M/G/1 retrial queueing system with constant retrial policy are given below

a. Probability that the server is idle in the system $= P_0 = 1-\rho$

b. Probability that the server is busy in the system $= P_1 = \rho$

c. Average number of customers in the orbit = $L_q =$ $\mathrm{L}_q = \frac{1}{\left(1-\rho_1\right)}\left(\frac{\lambda^2 E(X^2)}{2}\left(1+\frac{\lambda}{\sigma}\right)+\frac{\lambda\rho}{\sigma}\right)$

d. Average number of customers in the system = $L_s = L_q + \rho$

Proof

When the steady state prevails, the equation (10) becomes

$$P_0(z)=\sum_{n=0}^{\infty}P_{n0}z^n \text{ and } P_1(z)=\sum_{n=0}^{\infty}P_{n1}z^n \tag{49}$$

Probability that the server is idle in the system

$$= P_0(1)=\sum_{n=0}^{\infty}P_{n0}=P_{00}\left(\frac{1-\rho}{1-\rho_1}\right)=1-\rho \tag{50}$$

Probability that the server is busy in the system

$$= P_1(1)=\sum_{n=0}^{\infty}P_{n1}=P_{00}\left(\frac{\rho}{1-\rho_1}\right)=\rho \tag{51}$$

Differentiate (40) with respect to z, we get

$$P_0^{'}(z) = \frac{P_{00}\left(\frac{\lambda}{\sigma}\right)\left(z\psi^{'}(z)+\psi(z)\right)}{\left(1-\frac{\lambda z}{\sigma}\psi(z)\right)^2} \tag{52}$$

$$P_0^{'}(1) = \frac{P_{00}\left(\frac{\lambda}{\sigma}\right)\left(\psi^{'}(1)+\psi(1)\right)}{\left(1-\frac{\lambda}{\sigma}\psi(1)\right)^2} = \frac{\left(\frac{\lambda}{\sigma}\right)\left(\psi^{'}(1)+\psi(1)\right)(1-\rho)^2}{(1-\rho_1)} \tag{53}$$

$$P_0^{'}(1) = \frac{\left(\frac{\lambda}{\sigma}\right)\left(\frac{\lambda^2 E(X^2)}{2}+\rho(1-\rho)\right)}{(1-\rho_1)} \tag{54}$$

Differentiate (42) with respect to z, we get

$$P_1^{'}(z) = P_0(z)\psi^{'}(z) + P_0^{'}(z)\psi(z) \tag{55}$$

$$P_1^{'}(1) = \frac{\left(\frac{\lambda^2 E(X^2)}{2}+\frac{\lambda}{\sigma}\rho^2\right)}{(1-\rho_1)} \tag{56}$$

Average number of customers in the orbit = $\frac{d}{dz}\left(P_0(z)+P_1(z)\right)\Big|_{z=1}$

$$\frac{d}{dz}\left(P_0(z)+P_1(z)\right)\Big|_{z=1} = \frac{\left(\frac{\lambda}{\sigma}\right)\left(\frac{\lambda^2 E(X^2)}{2}+\rho(1-\rho)\right)}{(1-\rho_1)} + \frac{\left(\frac{\lambda^2 E(X^2)}{2}+\frac{\lambda}{\sigma}\rho^2\right)}{(1-\rho_1)} \tag{57}$$

$$= \frac{1}{(1-\rho_1)}\left(\frac{\lambda^2 E(X^2)}{2}\left(1+\frac{\lambda}{\sigma}\right)+\frac{\lambda\rho}{\sigma}\right)$$

$$\mathrm{L}_q = \frac{1}{(1-\rho_1)}\left(\frac{\lambda^2 E(X^2)}{2}\left(1+\frac{\lambda}{\sigma}\right)+\frac{\lambda\rho}{\sigma}\right) \tag{58}$$

Average number of customers in the system = $\frac{d}{dz}\left(P_0(z)+zP_1(z)\right)\Big|_{z=1} = L_q + P_1(1) = L_q + \rho$

4. STABILITY CONDITION

The stability analysis is very important for every queueing system. Single server retrial queueing system with constant retrial policy is stable provided

$$\left(1+\frac{\lambda}{\sigma}\right)\lambda E(X) < 1$$

- If the service time follows an exponential distribution then the stability of the system is $\left(1+\frac{\lambda}{\sigma}\right)\frac{\lambda}{\mu} < 1$
- If the service time follows an Erlang distribution with k stages then the stability of the system is $\left(1+\frac{\lambda}{\sigma}\right)\frac{k\lambda}{\mu} < 1$

5. SPECIAL CASES

We note that many particular cases of this work can be derived for various Service time distributions

Case-1: The service time distribution follows an exponential distribution

This model becomes M/M/1 single server retrial queueing system with constant retrial policy which was discussed by Fayolle in 1986, where the retrial queue was investigated for a telephone exchange model where the customers in the retrial group form a queue and only the customer at the head of the orbit can request service after an exponentially distributed retrial time.

The probability density function of exponential distribution is

$$b(x) = \mu e^{-\mu x}, x > 0 \quad (59)$$

For an exponential distribution, $E(X) = \frac{1}{\mu}$ and $E(X^2) = \frac{2}{\mu^2}$ (60)

The Laplace transform of $b(x)$ is $\bar{b}(s) = \frac{\mu}{s+\mu}$ (61)

$$\bar{b}(\lambda - \lambda z) = \frac{\mu}{\lambda - \lambda z + \mu} = \frac{1}{1+\rho-\rho z} \quad (65)$$

$$\psi(z) = \frac{1-\bar{b}}{\bar{b}-z} = \frac{1-\frac{1}{1+\rho-\rho z}}{\frac{1}{1+\rho-\rho z}-z} = \frac{\rho}{1-\rho z} \quad (66)$$

$$P_0(z) = \frac{p_{00}}{1-\left(\frac{\lambda z}{\sigma}\right)\psi(z)} = \frac{p_{00}}{1-\left(\frac{\lambda z}{\sigma}\right)\left(\frac{\rho}{1-\rho z}\right)} \quad (67)$$

After the simplification of equation (67), we get

$$P_0(z) = \frac{P_{00}\left(1-\rho z\right)}{1-\rho_1 z} \text{ where } \rho = \frac{\lambda}{\mu} \text{ and } \rho_1 = \frac{\lambda\left(\lambda+\sigma\right)}{\sigma\mu} \quad (68)$$

The equation (68) represents the steady state probability generating function of number of customers in the orbit when the server is idle in the system.

$$P_1(z) = P_{00}\left(\frac{\psi(z)}{1-\left(\frac{\lambda z}{\sigma}\right)\psi(z)}\right) = P_{00}\left(\frac{\frac{\rho}{1-\rho z}}{1-\left(\frac{\lambda z}{\sigma}\right)\left(\frac{\rho}{1-\rho z}\right)}\right) \quad (69)$$

After the simplification of equation (69), we get

$$P_1(z) = P_{00}\left(\frac{\rho}{1-\rho_1 z}\right) \quad (70)$$

The equation (70) represents the steady state probability generating function of number of customers in the orbit when the server is busy in the system.

$$L_q = \frac{P_{00}}{\left(1-\rho_1\right)^2}\left(\frac{\lambda^2 E(X^2)}{2}\left(1+\frac{\lambda}{\sigma}\right)+\frac{\lambda\rho}{\sigma}\right) = \frac{P_{00}}{\left(1-\rho_1\right)^2}\left(\frac{\lambda^2\left(\frac{2}{\mu^2}\right)}{2}\left(1+\frac{\lambda}{\sigma}\right)+\frac{\lambda\rho}{\sigma}\right) \quad (71)$$

After the simplification of equation (71), we get

$$L_q = \frac{\rho_1}{\left(1-\rho_1\right)}\left(\rho+\frac{\lambda}{\lambda+\sigma}\right) \quad (72)$$

The equation (72) represents the average number of customers in the orbit.

$$L_s = L_q + \rho = \frac{\rho_1}{\left(1-\rho_1\right)}\left(\rho+\frac{\lambda}{\lambda+\sigma}\right)+\rho \quad (73)$$

The equation (73) represents the average number of customers in the system.

Case-2: The service time distribution follows an Erlang distribution with k phases

This model becomes $M/E_k/1$ single server retrial queueing system with constant retrial policy

The probability density function of Erlang distribution is

$$b(x)=\frac{\mu^k x^{k-1}e^{-\mu x}}{(k-1)!}\mu e^{-\mu x}, x>0$$

For an Erlang distribution, $E(X)=\frac{k}{\mu}$ and $E(X^2)=\frac{k(k+1)}{\mu^2}$

The Laplace transform of $b(x)$ is $\bar{b}(s)=\left(\frac{\mu}{s+\mu}\right)^k$

$$\bar{b}(\lambda-\lambda z)=\left(\frac{\mu}{\lambda-\lambda z+\mu}\right)^k=\left(\frac{k}{k+\rho-\rho z}\right)^k \tag{74}$$

$$\psi(z)=\frac{1-\bar{b}}{\bar{b}-z}=\frac{1-\left(\frac{k}{k+\rho-\rho z}\right)^k}{\left(\frac{k}{k+\rho-\rho z}\right)^k-z}=\frac{(k+\rho-\rho z)^k-k^k}{k^k-z(k+\rho-\rho z)^k} \tag{75}$$

$$P_0(z)=\frac{p_{00}}{1-\left(\frac{\lambda z}{\sigma}\right)\psi(z)}=\frac{p_{00}}{1-\left(\frac{\lambda z}{\sigma}\right)\left(\frac{(k+\rho-\rho z)^k-k^k}{k^k-z(k+\rho-\rho z)^k}\right)} \tag{76}$$

After the simplification of equation (76), we get

$$P_0(z)=\frac{P_{00}\left(1-z\left(1+\frac{1}{k}(\rho-\rho z)\right)^k\right)}{1+z\left(\frac{\lambda}{\sigma}-\frac{\rho_1}{\rho}\left(1+\frac{1}{k}(\rho-\rho z)\right)^k\right)} \text{ where } \rho=\lambda E(X) \text{ and } \rho_1=\frac{\lambda(\lambda+\sigma)}{\sigma}E(X) \tag{77}$$

The equation (77) represents the steady state probability generating function of number of customers in the orbit when the server is idle in the system.

$$P_1(z)=P_{00}\left(\frac{\psi(z)}{1-\left(\frac{\lambda z}{\sigma}\right)\psi(z)}\right)=P_{00}\left(\frac{\frac{(k+\rho-\rho z)^k-k^k}{k^k-z(k+\rho-\rho z)^k}}{1-\left(\frac{\lambda z}{\sigma}\right)\left(\frac{(k+\rho-\rho z)^k-k^k}{k^k-z(k+\rho-\rho z)^k}\right)}\right) \tag{78}$$

After the simplification of equation (78), we get

$$P_1(z)=P_{00}\left(\frac{\left(1+\frac{1}{k}(\rho-\rho z)\right)^k-1}{1+z\left(\frac{\lambda}{\sigma}-\frac{\rho_1}{\rho}\left(1+\frac{1}{k}(\rho-\rho z)\right)^k\right)}\right) \tag{79}$$

The equation (79) represents the steady state probability generating function of number of customers in the orbit when the server is busy in the system.

$$L_q=\frac{1}{(1-\rho_1)}\left(\frac{\lambda^2E(X^2)}{2}\left(1+\frac{\lambda}{\sigma}\right)+\frac{\lambda\rho}{\sigma}\right)=\frac{1}{(1-\rho_1)}\left(\frac{\lambda^2\left(\frac{k(k+1)}{\mu^2}\right)}{2}\left(1+\frac{\lambda}{\sigma}\right)+\frac{\lambda\rho}{\sigma}\right) \tag{80}$$

After the simplification of equation (80), we get

$$L_q = \frac{1}{(1-\rho_1)}\left(\frac{\lambda^2\left(\frac{k(k+1)}{\mu^2}\right)}{2}\left(1+\frac{\lambda}{\sigma}\right)+\frac{\lambda\rho}{\sigma}\right) \tag{81}$$

The equation (81) represents the average number of customers in the orbit.

$$L_s = L_q + \rho = \frac{1}{(1-\rho_1)}\left(\frac{\lambda^2\left(\frac{k(k+1)}{\mu^2}\right)}{2}\left(1+\frac{\lambda}{\sigma}\right)+\frac{\lambda\rho}{\sigma}\right)+\rho \tag{82}$$

The equation (82) represents the average number of customers in the system.

Case-3: As σ→∞, this model becomes M/G/1 queueing model

The probability generating function of the number of customers in the system is given by

$$P(z) = P_0(z) + zP_1(z) = \frac{p_{00}}{1-\left(\frac{\lambda z}{\sigma}\right)\psi(z)} + zP_{00}\left(\frac{\psi(z)}{1-\left(\frac{\lambda z}{\sigma}\right)\psi(z)}\right) \tag{83}$$

As σ→∞, the equation (83) becomes

$$P(z) = P_0(z) + zP_1(z) = P_{00}\left(1+z\psi(z)\right) = P_{00}\left(1+z\left(\frac{1-\bar{b}}{\bar{b}-z}\right)\right)P_{00}\left(1+z\left(\frac{1-\bar{b}}{\bar{b}-z}\right)\right) = P_{00}\frac{\bar{b}(1-z)}{\bar{b}-z}$$

$$P_{00} = 1-\frac{\lambda(\lambda+\sigma)}{\sigma}E(X) = 1-\lambda E(X) \tag{84}$$

$$P(z) = \frac{\left(1-\lambda E(X)\right)(1-z)\bar{b}(\lambda-\lambda z)}{\bar{b}(\lambda-\lambda z)-z} \tag{85}$$

The Equation (85) represents the **Pollaczek – Khinchine formula** for M/G/1 queueing system

$$L_q = \frac{P_{00}}{(1-\rho_1)^2}\left(\frac{\lambda^2 E(X^2)}{2}\left(1+\frac{\lambda}{\sigma}\right)+\frac{\lambda\rho}{\sigma}\right) = \frac{\lambda^2 E(X^2)}{2(1-\rho)} \tag{86}$$

The equation (86) represents the average number of customers in the queue.

6. NUMERICAL STUDY

The values of parameters λ, μ and σ will be chosen so that it satisfies the stability condition which is discussed in section 4. The System performance measures of this model have been done and expressed in the form of tables which are shown below for various service distributions.

Tables 1and 2 show the impact of σ over Mean number of customers in the orbit if the **service distribution follows an exponential distribution**. Further, we infer the following

- Mean number of customers in the orbit decreases as retrial rate σ increases.
- Mean number of customers in the orbit increases as arrival rate increases.
- P_0 and P_1 are independent of retrial rate σ.
- This model becomes standard single server queueing model if σ is large.

Tables 3 and 4 show the impact of σ over Mean number of customers in the orbit if the **service distribution follows an Erlang distribution.** Further, we infer the following

- Mean number of customers in the orbit decreases as retrial rate σ increases.
- Mean number of customers in the orbit increases as arrival rate increases.
- P_0 and P_1 are independent of retrial rate σ.
- This model becomes standard single server queueing model with Erlang type service if σ is large.

Table-1: System performance measures for $\lambda = 5$ and $\mu = 10$ for various values of σ.

σ	P_0	P_1	L_q	L_s	W_q	W_s
10	0.5	0.5	2.5000	3.0000	0.5000	0.6000
30	0.5	0.5	0.9000	1.4000	0.1800	0.2800
50	0.5	0.5	0.7222	1.2222	0.1444	0.2444
70	0.5	0.5	0.6538	1.1538	0.1308	0.2308
90	0.5	0.5	0.6176	1.1176	0.1235	0.2235
100	0.5	0.5	0.6053	1.1053	0.1211	0.2211
300	0.5	0.5	0.5339	1.0339	0.1068	0.2068
500	0.5	0.5	0.5202	1.0202	0.1040	0.2040
700	0.5	0.5	0.5144	1.0144	0.1029	0.2029
900	0.5	0.5	0.5112	1.0112	0.1022	0.2022
1000	0.5	0.5	0.5101	1.0101	0.1020	0.2020
3000	0.5	0.5	0.5033	1.0033	0.1007	0.2007
5000	0.5	0.5	0.5020	1.0020	0.1004	0.2004

Table-2: System performance measures for $\lambda = 8$ and $\mu = 15$ for various values of σ.

σ	P_0	P_1	L_q	L_s	W_q	W_s
10	0.466667	0.53333	23.4667	24.0000	2.9333	3.0000
30	0.466667	0.53333	1.5489	2.0822	0.1936	0.2603
50	0.466667	0.53333	1.0890	1.6224	0.1361	0.2028
70	0.466667	0.53333	0.9315	1.4648	0.1164	0.1831
90	0.466667	0.53333	0.8518	1.3852	0.1065	0.1731
100	0.466667	0.53333	0.8252	1.3585	0.1031	0.1698
300	0.466667	0.53333	0.6769	1.2102	0.0846	0.1513
500	0.466667	0.53333	0.6494	1.1828	0.0812	0.1478
700	0.466667	0.53333	0.6379	1.1712	0.0797	0.1464
900	0.466667	0.53333	0.6315	1.1648	0.0789	0.1456
1000	0.466667	0.53333	0.6293	1.1626	0.0787	0.1453
3000	0.466667	0.53333	0.6161	1.1494	0.0770	0.1437
5000	0.466667	0.53333	0.6134	1.1468	0.0767	0.1433

Table-3: System performance measures for $\lambda = 5$, $\mu = 50$ and k = 3 for various values of σ.

σ	P_0	P_1	L_q	L_s	W_q	W_s
10	0.7	0.3	0.4364	0.7364	0.0873	0.1073
30	0.7	0.3	0.1846	0.4846	0.0369	0.0569
50	0.7	0.3	0.1433	0.4433	0.0287	0.0487
70	0.7	0.3	0.1263	0.4263	0.0253	0.0453
80	0.7	0.3	0.1211	0.4211	0.0242	0.0442
90	0.7	0.3	0.1171	0.4171	0.0234	0.0434
100	0.7	0.3	0.1139	0.4139	0.0228	0.0428
300	0.7	0.3	0.0950	0.3950	0.0190	0.0390
500	0.7	0.3	0.0912	0.3912	0.0182	0.0382
700	0.7	0.3	0.0897	0.3897	0.0179	0.0379
900	0.7	0.3	0.0888	0.3888	0.0178	0.0378
1000	0.7	0.3	0.0885	0.3885	0.0177	0.0377
3000	0.7	0.3	0.0866	0.3866	0.0173	0.0373
5000	0.7	0.3	0.0863	0.3863	0.0173	0.0373

Table-4: System performance measures for $\lambda = 8$, $\mu = 50$ and k = 3 for various values of σ.

σ	P_0	P_1	L_q	L_s	W_q	W_s
10	0.52	0.48	4.8565	5.3365	0.6071	0.6271
30	0.52	0.48	0.8229	1.3029	0.1029	0.1229
50	0.52	0.48	0.5753	1.0553	0.0719	0.0919
70	0.52	0.48	0.4859	0.9659	0.0607	0.0807
90	0.52	0.48	0.4398	0.9198	0.0550	0.0750
100	0.52	0.48	0.4242	0.9042	0.0530	0.0730
300	0.52	0.48	0.3362	0.8162	0.0420	0.0620
500	0.52	0.48	0.3196	0.7996	0.0400	0.0600
700	0.52	0.48	0.3126	0.7926	0.0391	0.0591
900	0.52	0.48	0.3087	0.7887	0.0386	0.0586
1000	0.52	0.48	0.3074	0.7874	0.0384	0.0584
3000	0.52	0.48	0.2994	0.7794	0.0374	0.0574
5000	0.52	0.48	0.2978	0.7778	0.0372	0.0572

7. CONCLUSION

Single server retrial queueing system with general service for constant retrial policy is discussed and derived the probability generating functions for number of customers in the orbit when the server is idle / busy. Various special cases have been verified for different service time distributions. Numerical studies have been done in elaborate manner.

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