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RADON MEASURE ON COMPACT MEASURABLE MANIFOLD WITH SOME MEASURABLE PROPERTIES

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ABSTRACT

R*adon measure is studied on e-Hausdorff, e-second countable locally compact measurable manifold* (M, τ_1, Σ_1). *It is shown that some measurable properties like Radon e-regularity and Radon e-normality on* $(M, \tau_1, \Sigma_1, \mu_1)$ *are invariant under measurable homeomorphism and Radon measure-invariant transformations.*

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1. INTRODUCTION

The Borel sets on a measure space are approximation of open subsets which cover the topological space X and they are the smallest subsets on measure space (R^n, τ, Σ, μ). Extended topological properties which are well defined on measure space (R^n, τ, Σ, μ) are measurable and measure-invariant under measurable homeomorphism [11], [13] and [14]. Any Hausdorff second countable topological space modelled on such a measure space is a measure manifold $(M, \tau_1, \Sigma_1, \mu_1)$ which admits e-Hausdorff e-second countable property. Such a measure manifold is measurable [11], [12], [13] and [14].

In our previous paper [15], we have introduced Radon measure on e-second countable space, measurable regular space [e-regular space] and e-normal space. Now, in this paper, we study these extended topological properties on $(M, \tau_1, \Sigma_1, \mu_1)$ and show that they are invariant under measurable homeomorphism Radon measure-invariant transformations.

We observe that a measure space $(R^n, \tau_2, \Sigma_2, \mu_2)$ which is locally compact also admits partitions of unity which is a collection $\{f_i : (R^n, \tau_2, \Sigma_{2,}\mu_2) \longrightarrow (R^n, \tau, \Sigma, \mu)_{/i\in I}\}$ of measurable real valued functions with compact support, supp(f) = $\overline{p \in A \subset (R^n, \tau_2, \Sigma_{2,}\mu_2) : f(p) \neq 0\}}$ where A is a Borel subset of $(R^n, \tau_2, \Sigma_{2,}\mu_2)$ and the measure function μ exists on $(R^n, \tau_2, \Sigma_{2,}\mu_2)$, for which $\mu(A) > 0$ such that,

(i) $0 \leq f_i \leq 1$ for all i,
(ii) every $p \in A \subset (R^n, \tau_2, \Sigma_{2,}\mu_2)$ has a Borel neighbourhood $A \subset (R^n, \tau_2, \Sigma_{2,}\mu_2)$ such that $A \cap \text{supp}(f_i) \neq 0$ for which $\mu(A \cap \text{supp}(f_i)) > 0$, for all but finitely many of the f_i,
(iii) for each $p \in A \subset (R^n, \tau_2, \Sigma_{2,}\mu_2)$, $f_i(p) = 1$.

This partition of unity $\{f_i\}_i$ on $(R^n, \tau_2, \Sigma_{2,}\mu_2)$ is subordinate to a Borel open cover $\{V_{\alpha_{/\alpha\in I}}\} \subset (R^n, \tau_2, \Sigma_{2,}\mu_2)$ if and only if for each f_i there is an element of A of the Borel open cover $\{V_{\alpha_{/\alpha\in I}}\}$ such that supp$(f_i) \subset \{V_{\alpha_{/\alpha\in I}}\} \subset (R^n, \tau_2, \Sigma_{2,}\mu_2)$ for which $\mu(\text{supp}(f_i)) < \mu(A) < \mu(\{V_{\alpha_{/\alpha\in I}}\})$.

Now, we study the concept of partitions of unity on Hausdorff second countable smooth manifold M [7].

A partition of unity on Hausdorff second countable smooth manifold M is a collection $\{\varphi_\alpha : \text{M} \longrightarrow R_{/\alpha\in I}\}$ of smooth functions on M where the partition of unity is subordinate to $\mathcal{A} = \{U_\alpha\}_{\alpha\in\Lambda}$ where $\mathcal{A}$ is an atlas and U_α is chart on M [7].

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In this paper Radon e-regularity and Radon e-normality properties are studied on a locally compact Radon measure manifold (M, $\tau_1, \Sigma_1, \mu_{R_1}$) and show that they are invariant under measurable homeomorphism and Radon measure-invariant transformations.

2. PRELIMINARIES

In this section we consider some basic concepts:

We introduce all the following definitions that are valid for a general measure μ as follows:

Definition 2.1: Locally Finite measure on (R^n, τ, Σ) [16]
The measure μ is locally finite if every point p of (R^n, τ, Σ, μ) has a neighbourhood $\bar{A}$ of p for which $\mu(\bar{A}) <\infty$.

Definition 2.2: Inner regular measure on (R^n, τ, Σ) [16]
Let (R^n,τ) be a locally compact Hausdorff topological space and let Σ be a σ −algebra on (R^n, τ). Let μ be a measure on (R^n, τ, Σ). A measurable subset A of (R^n, τ, Σ) is said to be inner regular if

$\mu(\bar{A})$ = sup{ $\mu(F)$: F⊆ $\bar{A}$: F compact and measurable}.

Definition 2.3: e-Hausdorff space [15] The space (R^n, τ, Σ, μ) is called e-Hausdorff provided that if p and q are distinct members of (R^n, τ, Σ, μ) then there exist disjoint Borel open sets A and B such that $p \in A$ and $q \in B$.

Definition 2.4: Second countable measure space (e-second countable space) [13] A measure space (R^n, τ, Σ, μ) is e-second countable provided there is a countable base for all q ∈ (R^n, τ, Σ, μ) satisfying the following conditions:
(i) For each q_i ∈(R^n, τ, Σ, μ) ∃ $B_i \in \mathfrak{B}_q$ and $A_i \in \Sigma$ such that $q_i \in B \subseteq A_i$, i = 1, 2,...n
(ii) For $B_i \subseteq A_i$, $\mu(B_i) \leq \mu(A_i)$

Definition 2.5: e-normal space [11] A measure space (R^n, τ, Σ, μ) is said to be e-normal if each pair of disjoint F_σ-sets A and B in (R^n, τ, Σ, μ), ∃ a pair of disjoint G_δ-sets U and V such that A ⊂ U and B ⊂ V.

Definition 2.6: Radon measure on (R^n, τ, Σ, μ) [16] A Radon measure on a measurable topological space (R^n, τ, Σ) is a positive Borel measure μ : B⟶ [0,∞] such that
(i) μ is locally finite
(ii) μ is inner regular with respect to the compact subsets.

Then (R^n, τ, Σ, μ_R) is called a Radon measure space.

If any Hausdorff second countable topological space is modelled on a Radon measure space (R^n, τ, Σ, μ_R) if ∃ a measurable homeomorphism between U ⊂ (M, $\tau_1, \Sigma_1, \mu_{R_1}$) ⟶ ($R^n$, τ, Σ, μ_R) then (M, $\tau_1, \Sigma_1, \mu_{R_1}$) is a Radon measure manifold.

Definition 2.7: Measurable Homeomorphism [11] Let (M, τ) be a Hausdorff second-countable topological space and (R^n, τ, Σ, μ) be a measure space. Then the function φ: U ⊂ M ⟶(R^n, τ, Σ, μ) is called measurable Homeomorphism if
(i) φ is bijective and bi continuous.
(ii) φ and φ^{-1} are measurable.

In the main results we show that some extended topological properties on Radon measure manifold are invariant with respect to measurable homeomorphism and Radon measure-invariant transformation. For this we use Inverse Function Theorem for measure manifold and also we use the concept of Radon e-regular space.

Inverse Function Theorem for measure space [11] Let F: (R^n, τ, Σ, μ) ⟶ ($R^m, \tau_1, \Sigma_1, \mu_1$) be a mapping and suppose that F_{*p}:$T_p(R^n) \longrightarrow T_{f(p)}(R^m)$ is a linear isomorphism at some point p of (R^n, τ, Σ, μ). Then there exists a neighbourhood U of p in (R^n, τ, Σ, μ)such that the restriction of F to U is a diffeomorphism onto a neighbourhood V of F(p) in $(R^m, \tau_1, \Sigma_1, \mu_1)$. This implies for every function F, ∃ F^{-1}: $(R^m, \tau_1, \Sigma_1, \mu_1) \longrightarrow (R^n, \tau, \Sigma, \mu)$.

Inverse Function Theorem for measure manifold [11] Let F: (M, τ, Σ, μ)⟶ ($M_1, \tau_1, \Sigma_1, \mu_1$) be a C^∞ homeomorphism, measurable and measure-invariant mapping of measure manifolds and suppose that F_{*p}:$T_p(M) \longrightarrow T_{f(p)}(M_1)$ is a linear isomorphism at some point p of M. Then there exists a measure chart (U, φ) of p in M such that the restriction of F to (U, φ) is a diffeomorphism onto a measure chart (V, ψ) of F(p) in M_1. This implies for every C^∞function F which is homeomorphism, measurable and measure-invariant has a C^∞function F^{-1}: ($M_1, \tau_1, \Sigma_1, \mu_1$) ⟶(M, τ, Σ, μ) which is also homeomorphism, measurable and measure-invariant.

To measure the whole compact measure manifold (M, $\tau_1, \Sigma_{1,}\mu_1$), we study the following concept of partitions of unity:

Definition 2.8: Partitions of unity $\{\varphi_\alpha\}$ [7] A partition of unity on a smooth manifold M is a collection $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ of smooth functions on M such that

(i) $0 \le \varphi_\alpha \le 1$ for all α,
(ii) The collection of supports $\{\text{support}(\varphi_\alpha)\}_{\alpha\in\Lambda}$ is locally finite; that is, each point p of M has a neighbourhood W_p such that $W_p \cap \text{support}(\varphi_\alpha) = \emptyset$ for all but a finite number of $\alpha \in \Lambda$,
(iii) $\sum_{\alpha\in\Lambda} \varphi_\alpha(p) = 1$ for all $p \in U \subset$ M (this sum has only finitely many non zero terms by (ii)).

If O = $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ is an open cover of M and supp$(\varphi_\alpha) \subset O_\alpha$ if each $\alpha \in \Lambda$, then we say that $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ is a partition of unity subordinate to O = $\{\varphi_\alpha\}_{\alpha\in\Lambda}$.

In [5], S. C. P. Halakatti have extended the following regularity property on measure space.

Definition 2.9: Radon measure on locally compact measurable regular space / Radon e-regular space [15]
Let (R^n, τ, Σ) be a locally compact measurable regular space. A Radon measure μ_R on (R^n, τ, Σ) is defined as follows:
$\forall\, p \in \bar{A}$ and Borel closed set F in (R^n, τ, Σ), ∃ Borel open set A ⊂ $\bar{A}$ and B ⊂ $\bar{B} \in \Sigma$, ∋
$p \in \bar{A}$, F ⊂ $\bar{B}$, p ∉ F and $\bar{A} \cap \bar{B} = \emptyset$ satifying the Radon measure conditions:

I. For p ∈ $\bar{A}$,
   (i) ∀ p ∈ $\bar{A} \subset (R^n, \tau, \Sigma)$, $\mu_R(\overline{A}) < \infty$;
   (ii) For any Borel compact subset $\bar{A} \subset (R^n, \tau, \Sigma)$,$\mu_R(\overline{A})$ = sup$\{\mu(E_i),\ i \in I : E_i \subseteq \bar{A} : E_i$ compact and measurable}
II. For F ⊂ $\bar{B}$,
   (i) ∀ q ∈ F⊂ $\bar{B} \subset (R^n, \tau, \Sigma)$, $\mu_R(\overline{B}) < \infty$;
   (ii) For any Borel compact subset $\bar{B} \subset (R^n, \tau, \Sigma)$, $\mu_R(\overline{B})$ = sup$\{\mu(F_i),\ i \in I : F_i \subseteq \bar{B} : F_i$ compact and measurable}

Radon e-regularity property: A locally compact measurable regular space is said to satisfy a Radon e-regularity property, if it admits a Radon measure.

Theorem 2.1: [10] If regularity is a topological invariant under homeomorphism then it is invariant under measure transformation.

Theorem 2.2 [15]: A second countable measure space is e-normal with Radon measure.

Theorem 2.3[8]: Any connected compact 1 dimensional manifold with non-empty boundary is homeomorphic to the unit interval [0, 1].

Theorem 2.4 [7]: Let M be a (normal) smooth manifold and $\{U_\alpha\}_{\alpha\in\Lambda}$ be a locally finite cover of M. If each U_α has compact closure, then there is a partition of unity $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ subordinate to $\{U_\alpha\}_{\alpha\in\Lambda}$.

Lemma 2.1 [8]: If a topological space X can be represented as the union of a non-decreasing sequence of open subsets, all homeomorphic to R, then X is homeomorphic to R.

3. MAIN RESULTS

In this section, we study Radon e-regularity and Radon e-normality properties on a locally compact measure manifold (M, $\tau_1, \Sigma_1, \mu_{R_1}$) and show that these properties are invariant under measurable homeomorphism and under Radon measure-invariant transformations by using measurable homeomorphism concept [11] and Inverse Function Theorem for measure manifold [11]. A smooth manifold M admits partitions of unity [7]. By using partitions of unity we show that a locally compact measure manifold becomes compact measure manifold.

Since ($R^n, \tau_2, \Sigma_{2,}\mu_2$) admits partitions of unity, measure manifold (M, τ_1, Σ_1, μ_1) also admits partitions of unity. One can observe that, by doing so, locally compact measure manifold becomes a compact measure manifold:

Proposition 3.1: A compact measure manifold (M, τ_1, Σ_1, μ_1) admits partitions of unity.

Proof: Let (M, τ_1, Σ_1, μ_1) be a locally compact measure manifold.

We prove that a compact measure manifold (M, τ_1, Σ_1, μ_1) admits partitions of unity.

Let (R, τ, Σ, μ) be a locally compact measure space of dimension 1, then there exists a partitions of unity φ_i: $\{(M, \tau_1, \Sigma_1, \mu_1) \longrightarrow (R, \tau, \Sigma, \mu)\ _{/i \in I}\}$ of measurable real valued functions with compact support, supp(φ_i) = $\overline{q \in (U, \tau_{1_{/U}}, \Sigma_{1_{/U}}, \mu_{1_{/U}}) \subset (M, \tau_1, \Sigma_1, \mu_1)\ : \varphi_i(q) \neq 0\}}$ on $(M, \tau_1, \Sigma_1, \mu_1)$ where $(U, \tau_{1_{/U}}, \Sigma_{1_{/U}}, \mu_{1_{/U}})$ is a measure chart on $(M, \tau_1, \Sigma_1, \mu_1)$ and the measure function μ exists on $(M, \tau_1, \Sigma_1, \mu_1)$, for which $\mu(U, \tau_{1_{/U}}, \Sigma_{1_{/U}}, \mu_{1_{/U}}) > 0$ such that $(M, \tau_1, \Sigma_1, \mu_1)$ is a locally compact measure manifold which satisfies the following conditions:

(i) $0 \leq \varphi_i \leq 1$ for all i ,
(ii) every $q \in (U, \tau_{1_{/U}}, \Sigma_{1_{/U}}, \mu_{1_{/U}}) \subset (M, \tau_1, \Sigma_1, \mu_1)$ has a Borel neighbourhood $(U, \tau_{1_{/U}}, \Sigma_{1_{/U}}, \mu_{1_{/U}})$ such that $(U, \tau_{1_{/U}}, \Sigma_{1_{/U}}, \mu_{1_{/U}}) \cap$ supp(φ_i) = φ for all but a finite number of $i \in I$, for which $\mu((U, \tau_{1_{/U}}, \Sigma_{1_{/U}}, \mu_{1_{/U}}) \cap$ supp(φ_i)) = 0.
(iii) $\sum_{i \in I} \varphi_i(q) = 1$ for all $q \in (U, \tau_{1_{/U}}, \Sigma_{1_{/U}}, \mu_{1_{/U}}) \subset (M, \tau_1, \Sigma_1, \mu_1)$.

If $(\mathcal{A}, \tau_{1_{/\mathcal{A}}}, \Sigma_{1_{/\mathcal{A},}}, \mu_{1_{/\mathcal{A}}}) = (U, \tau_{1_{/U}}, \Sigma_{1_{/U}}, \mu_{1_{/U}})$ is an open cover of $(\mathcal{A}, \tau_{1_{/\mathcal{A}}}, \Sigma_{1_{/\mathcal{A},}}, \mu_{1_{/\mathcal{A}}})$ which is a measurable atlas of $(M, \tau_1, \Sigma_1, \mu_1)$ and supp(φ_i) $\subset ((U, \tau_{1_{/U}}, \Sigma_{1_{/U}}, \mu_{1_{/U}}), \varphi_i) \subset (M, \tau_1, \Sigma_1, \mu_1)$ for which μ(supp (φ_i)) $< \mu(U, \tau_{1_{/U}}, \Sigma_{1_{/U}}, \mu_{1_{/U}}) < \mu(M, \tau_1, \Sigma_1, \mu_1)$ for each $i \in I$, then we say that $\{\varphi_i\}_{i \in I}$ is a partition of unity $(\mathcal{A}, \tau_{1_{/\mathcal{A}}}, \Sigma_{1_{/\mathcal{A},}}, \mu_{1_{/\mathcal{A}}}) = (U, \tau_{1_{/U}}, \Sigma_{1_{/U}}, \mu_{1_{/U}})\ _{i \in I}$.

Since $(M, \tau_1, \Sigma_1, \mu_1)$ is e-Hausdorff, e-second countable, , it is covered by a countable union of Borel open charts $((U, \tau_{1_{/U}}, \Sigma_{1_{/U}}, \mu_{1_{/U}}), \varphi_i) \subset (M, \tau_1, \Sigma_1, \mu_1)$ such that

(i) $\overline{(U_i, \tau_{1_{/U_i}}, \Sigma_{1_{/U_i}}, \mu_{1_{/U_i}})}$ is compact Borel subset of $(M, \tau_1, \Sigma_1, \mu_1)$ for which $\mu\, \overline{(U_i, \tau_{1_{/U_i}}, \Sigma_{1_{/U_i}}, \mu_{1_{/U_i}})} > 0$,
(ii) $\overline{(U_i, \tau_{1_{/U_i}}, \Sigma_{1_{/U_i}}, \mu_{1_{/U_i}})} \subset \overline{(U_{i+1}, \tau_{1_{/U_{i+1}}}, \Sigma_{1_{/U_{i+1}}}, \mu_{1_{/U_{i+1}}})}$ for which $\mu\, \overline{(U_i, \tau_{1_{/U_i}}, \Sigma_{1_{/U_i}}, \mu_{1_{/U_i}})} < \mu\, \overline{(U_{i+1}, \tau_{1_{/U_{i+1}}}, \Sigma_{1_{/U_{i+1}}}, \mu_{1_{/U_{i+1}}})}$ ... sub-additivity property
(iii) $\bigcup_{i=1}^{\infty}(U_i, \tau_{1_{/U_i}}, \Sigma_{1_{/U_i}}) = (M, \tau_1, \Sigma_1)$ such that $\mu(\bigcup_{i=1}^{\infty}(U_i, \tau_{1_{/U_i}}, \Sigma_{1_{/U_i}})) = \sum_{i=1}^{\infty} \mu\,(U_i, \tau_{1_{/U_i}}, \Sigma_{1_{/U_i}})$ ... σ-additivity property

Therefore, $(M, \tau_1, \Sigma_1, \mu_1)$ is a compact measure manifold. Now, we can measure the compact measure manifold $(M, \tau_1, \Sigma_1, \mu_1)$ by Radon measure.

Now, we introduce the concept of Radon e-normality property and prove some results:

Definition 3.1: Locally compact measurable normal topological space The space (R^n, τ, Σ) is called a locally compact measurable normal topological space if given any pair of disjoint F_σ-sets E and F ∈ Σ in (R^n, τ, Σ), ∃ a pair of disjoint G_δ-sets $\bar{A}$ and $\bar{B}$ such that E ⊂ $\bar{A}$ and F ⊂ $\bar{B}$.

Definition 3.2: Radon measure on locally compact measurable normal topological space Let (R^n, τ, Σ) be a locally compact measurable normal topological space. A Radon measure μ_R on (R^n, τ, Σ) is defined as follows:

For any pair of disjoint Borel closed sets $\bar{E} \supset E$ and $\bar{F} \supset F$in (R^n, τ, Σ), ∃ Borel open sets A ⊂ $\bar{A}$ and B ⊂ $\bar{B}$ ∈ Σ, ∋ $E \subset \bar{A}$ and F ⊂ $\bar{B}$ and $\bar{A} \cap \bar{B} = \emptyset$ satisfying the Radon measure conditions:

1. For $E \subset \bar{A}$ in (R^n, τ, Σ),
   (i) $\mu_R\,(\overline{A}) < \infty$,
   (ii) $\mu_R(\overline{A})$ = sup{ $\mu(E_i)$; $i \in I$: $E_i \subseteq \bar{A}$: E_i compact and measurable}
2. For F ⊂ $\bar{B}$ in (R^n, τ, Σ),
   (i) $\mu_R(\overline{B}) <\infty$;
   (ii) For any Borel compact subset $\bar{B}$ in (R^n, τ, Σ), $\mu_R(\overline{B})$ = sup{$\mu(F_i)$; $i \in I$: $F_i \subseteq \bar{B}$: F_i compact and measurable}.

Definition 3.3: Radon e-normality property A locally compact measurable normal topological space is said to satisfy a Radon e-normality property, if it admits a Radon measure.

Now, by using Radon e-regularity property and theorem 2.1 [10], we prove the following results:

Theorem 3.1: If (R^m, τ_2, Σ_2,μ_{R_2}) is a locally compact Radon e-regular space then Radon e-regularity property is invariant under measurable homeomorphism and Radon measure invariant transformation
T: (R^n, τ_1, Σ_1,μ_{R_1}) ⟶ (R^m, τ_2, Σ_2,μ_{R_2}).

Proof: Let (R^m, τ_2, Σ_2,μ_{R_2}) be a locally compact Radon e-regular space.

We show that $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ is a locally compact Radon e-regular space. That is, we show that Radon e-regularity is invariant under measurable homeomorphism and Radon measure-invariant transformations by using measurable homeomorphism concept [11] and Inverse Function Theorem for measure space [11].

Now, we show that T: $(R^n, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (R^m, \tau_2, \Sigma_2, \mu_{R_2})$ and T^{-1}: $(R^m, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ are C^∞ measurable homeomorphisms and measure invariant transformations with Radon measures μ_{R_1} and μ_{R_2} on R^n and R^m respectively.

In step I, we show that Radon e-regularity property is invariant under measurable homeomorphism.

Step-I: Let T and T^{-1} be C^∞ measurable homeomorphisms.

Since $(R^m, \tau_2, \Sigma_2, \mu_{R_2})$ is a Radon e-regular space. That is, for every point p$\in R^m$ and Borel compact subset F not containing p in R^m, $\exists$ Borel open subsets A $\subset \bar{A}$, B $\subset \bar{B} \in \Sigma_2$, where $\bar{A}$ and $\bar{B}$ are Borel compact subsets of (R^m, τ_2, Σ_2), $\ni$ p $\in$ A $\subset \bar{A}$, F $\subset$ B $\subset \bar{B}$ and $\mu_{R_2}(\bar{A} \cap \bar{B})$ = 0 satisfying the Radon measure conditions:

1. For p $\in$ A $\subset \bar{A} \subset (R^m, \tau_2, \Sigma_2)$,
   (i) $\forall$ p $\in \bar{A} \subset (R^m, \tau_2, \Sigma_2)$; $\mu_{R_2}(\bar{A}) < \infty$;
   (ii) For any Borel compact subset $\bar{A} \subset (R^m, \tau_2, \Sigma_2)$, $\mu_{R_2}(\bar{A})$ = sup$\{\mu(E_i);\ i \in I\colon E_i \subseteq \bar{A}\colon E_i$ compact and measurable$\}$
2. For F $\subset$ B $\subset \bar{B} \subset (R^m, \tau_2, \Sigma_2)$,
   (i) $\forall$ q $\in \bar{B} \subset (R^m, \tau_2, \Sigma_2)$, $\mu_{R_2}(\bar{B}) < \infty$;
   (ii) For any Borel compact subset $\bar{B} \subset (R^m, \tau_2, \Sigma_2)$, $\mu_{R_2}(\bar{B})$ = sup$\{\mu(F_i);\ i \in I\colon F_i \subseteq \bar{B}\colon F_i$ compact and measurable$\}$ and $\mu_{R_2}(\bar{A} \cap \bar{B})$ = 0. (i)

   By using the Inverse Function Theorem for measure spaces [11], we write,$\forall$ Borel open subsets A $\subset \bar{A}$, B $\subset \bar{B} \in \Sigma_2$, $\exists$ T^{-1}(A) $\subset T^{-1}(\bar{A})$, T^{-1} (B) $\subset T^{-1}(\bar{B}) \in \Sigma_1$, where T^{-1} $(\bar{A})$ and $T^{-1}(\bar{B})$ are Borel compact subsets of (R^n, τ_1, Σ_1), $\ni T^{-1}$(p) $\in T^{-1}(\bar{A})$, T^{-1} (E) $\subset T^{-1}(\bar{B})$ and $\mu_{R_1}(T^{-1}(\bar{A}) \cap T^{-1}(\bar{B}))$ = 0 satisfying the Radon measure conditions:

3. For T^{-1} (p) $\in T^{-1}(\bar{A}) \subset (R^n, \tau_1, \Sigma_1)$,
   (i) $\forall\ T^{-1}$(p) $\in T^{-1}(\bar{A}) \subset (R^n, \tau_1, \Sigma_1)$, $\mu_{R_1}(T^{-1}(\bar{A})) < \infty$;
   (ii) For any Borel compact subset T^{-1} $(\bar{A}) \subset (R^n, \tau_1, \Sigma_1)$, $\mu_{R_1}(T^{-1}(\bar{A}))$= sup$\{\mu(T^{-1}(E_i)\ ; i \in I\colon T^{-1}(E_i) \subseteq T^{-1}(\bar{A})\colon T^{-1}(E_i)$ compact and measurable$\}$
4. For T^{-1} (F) $\subset T^{-1}(\bar{B})$,
   (i) $\forall\ T^{-1}$(q) $\in T^{-1}(\bar{B}) \subset (R^n, \tau_1, \Sigma_1)$, $\mu_{R_1}(T^{-1}(\bar{B})) < \infty$;
   (ii) For any Borel compact subset $T^{-1}(\bar{B}) \subset (R^n, \tau_1, \Sigma_1)$, $\mu_{R_1}(T^{-1}(\bar{B}))$=sup$\{\mu(T^{-1}(F_i)\ ; i \in I\colon T^{-1}(F_i) \subseteq T^{-1}(\bar{B})\colon T^{-1}(F_i)$ compact and measurable$\}$ and $\mu_{R_1}(T^{-1}(\bar{A}) \cap T^{-1}(\bar{B}))$ = 0. (ii)

   Therefore, $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ is a Radon e-regular space.
   ⇒Radon e-regularity property is invariant under measurable homeomorphism.

Step-II: In step II, we show that T preserves the Radon e-regularity property under Radon measure- invariant transformation.

Since for every Borel open subsets A$\subset \bar{A}$, B $\subset \bar{B} \in \Sigma_2$, $\exists$ Borel subsets T^{-1} (A) $\subset T^{-1}(\bar{A})$, T^{-1} (B) $\subset T^{-1}(\bar{B}) \in \Sigma_1$, such that, $\mu_{R_1}(T^{-1}(\bar{A})) = \mu_{R_2}(\bar{A})$
and $\mu_{R_1}(T^{-1}(\bar{B})) = \mu_{R_2}(\bar{B})$ (iii)

Therefore, from (i), (ii) and (iii), we confirm that Radon e-regularity property is preserved under the measurable homeomorphism and Radon measure- invariant transformation T.

In step III, we show that $\exists$ a measurable transformation T^{-1} which is Radon measure-invariant.

Step-III: Consider the measurable homeomorphism T^{-1}: $(R^m, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (R^n, \tau_1, \Sigma_1, \mu_{R_1})$.

By using the Inverse Function Theorem for measure spaces [11], we conclude that $\exists$ a measurable transformation T^{-1} which is Radon measure- invariant.

Therefore, from steps I, II and III, we come to the conclusion that Radon e-regularity property is invariant under measurable homeomorphism and Radon measure-invariant transformations T and T^{-1}.

Now, we show that Radon e-normality property is invariant under measurable homeomorphism and Radon measure invariant transformation.

Theorem 3.2: If $(R^m, \tau_2, \Sigma_2, \mu_{R_2})$ is a locally compact Radon e-normal space then Radon e-normality property is invariant under measurable homeomorphism and Radon measure invariant transformation
$T: (R^n, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (R^m, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof: Let $(R^m, \tau_2, \Sigma_2, \mu_{R_2})$ be a locally compact Radon e-normal space.

We show that $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ is a locally compact Radon e-normal space. That is, we show that Radon e-normality property is invariant under measurable homeomorphism and Radon measure-invariant transformations by using measurable homeomorphism concept [11] and Inverse Function Theorem for measure spaces [11].

In step I, by using measurable homeomorphism concept [11], we show that Radon e-normality property is invariant under measurable homeomorphism.

Step-I: Now, we shall show that $T: (R^n, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (R^m, \tau_2, \Sigma_2, \mu_{R_2})$ and $T^{-1}: (R^m, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (R^n, \tau_1, \Sigma_1, \mu_{R_1})$ are C^∞ measurable homeomorphisms.

Since $(R^m, \tau_2, \Sigma_2, \mu_{R_2})$ is a Radon e-normal space i.e., for any disjoint Borel compact subsets E and F in R^m, $\exists$ Borel open subsets A $\subset \bar{A}$, B $\subset \bar{B} \in \Sigma_2$, where $\bar{A}$ and $\bar{B}$ are Borel compact subsets of $(R^m, \tau_2, \Sigma_2, \mu_{R_2})$, such that E $\subset \bar{A}$, F $\subset \bar{B}$ and $\mu_{R_2}(\bar{A} \cap \bar{B}) = 0$ satisfying the Radon measure conditions:

1. For E $\subset$ A$\subset \bar{A} \subset (R^m, \tau_2, \Sigma_2)$,
   (i) $\mu_{R_2}(\bar{A}) < \infty$;
   (ii) For any Borel compact subset $\bar{A} \subset (R^m, \tau_2, \Sigma_2)$, $\mu_{R_2}(\bar{A}) = \sup\{\mu(E_i)\ ;\ i \in I : E_i \subseteq \bar{A} : E_i$ compact and measurable$\}$
2. For F $\subset$ B $\subset \bar{B} \subset (R^m, \tau_2, \Sigma_2)$,
   (i) $\mu_{R_2}(\bar{B}) < \infty$;
   (ii) For any Borel compact subset $\bar{B} \subset (R^m, \tau_2, \Sigma_2)$, $\mu_{R_2}(\bar{B}) = \sup\{\mu(F_i)\ ;\ i \in I : F_i \subseteq \bar{B} : F_i$ compact and measurable$\}$ and $\mu_{R_2}(\bar{A} \cap \bar{B}) = 0$. (i)

By using the Inverse Function Theorem for measure spaces [11], we write, $\forall$ Borel open subsets A $\subset \bar{A}$, B $\subset \bar{B} \in \Sigma_2$, $\exists$ Borel subsets T^{-1}(A) $\subset T^{-1}(\bar{A})$, T^{-1}(B) $\subset T^{-1}(\bar{B}) \in \Sigma_1$, where $T^{-1}(\bar{A})$ and $T^{-1}(\bar{B})$ are Borel compact subsets of (R^n, τ_1, Σ_1), $\ni T^{-1}$(E) $\subset T^{-1}(\bar{A})$, T^{-1}(F) $\subset T^{-1}(\bar{B})$ and $\mu_{R_1}(T^{-1}(\bar{A}) \cap T^{-1}(\bar{B})) = 0$ satisfying the Radon measure conditions:

3. (i) For T^{-1}(E) $\subset T^{-1}(\bar{A}) \subset (R^n, \tau_1, \Sigma_1)$, $\mu_{R_1}(T^{-1}(\bar{A})) < \infty$;
   (ii) For any Borel compact subset $T^{-1}(\bar{A}) \subset (R^n, \tau_1, \Sigma_1)$, $\mu_{R_1}(T^{-1}(\bar{A})) = \sup\{\mu(T^{-1}(E_i)$;
   $i \in I$:$T^{-1}(E_i) \subseteq T^{-1}(\bar{A})$:$T^{-1}(E_i)$ compact and measurable$\}$.
4. For T^{-1}(F) $\subset T^{-1}(\bar{B}) \subset (R^n, \tau_1, \Sigma_1)$,
   (i) $\mu_{R_1}(T^{-1}(\bar{B})) < \infty$;
   (ii) For any Borel compact subset $T^{-1}(\bar{B}) \subset (R^n, \tau_1, \Sigma_1)$,$\mu_{R_1}(T^{-1}(\bar{B}))$ =sup$\{\mu(T^{-1}(F_i)$;
   $i \in I$:$T^{-1}(F_i) \subseteq T^{-1}(\bar{B}) : T^{-1}(F_i)$ compact and measurable$\}$ and $\mu_{R_1}(T^{-1}(\bar{A}) \cap T^{-1}(\bar{B})) = 0$. (ii)

Therefore, $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$ is a Radon e-normal space.
$\Rightarrow$ Radon e-normality property is invariant under measurable homeomorphism.

Step-II: In step II, we show that T preserves the Radon e-normality property under measurable homeomorphism and Radon measure- invariant transformation.

Since for every Borel open subsets A$\subset \bar{A}$, B $\subset \bar{B} \in \Sigma_2$, where $\bar{A}$ and $\bar{B}$ are Borel compact subsets of (R^m, τ_2, Σ_2), $\exists$ Borel open subsets T^{-1}(A) $\subset T^{-1}(\bar{A})$, T^{-1}(B) $\subset T^{-1}(\bar{B}) \in \Sigma_1$ such that,
$\mu_{R_1}(T^{-1}(\bar{A})) = \mu_{R_2}(\bar{A})$ and $\mu_{R_1}(T^{-1}(\bar{B})) = \mu_{R_2}(\bar{B})$. (iii)

Therefore, from (i), (ii) and (iii), we confirm that Radon e-normality property is preserved under the measurable homeomorphism and under Radon measure- invariant transformation.

In step III, we show that $\exists$ a measurable transformation T^{-1} which is Radon measure-invariant.

Step-III: Consider the measurable homeomorphism T^{-1}: $(R^m, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (R^n, \tau_1, \Sigma_1, \mu_{R_1})$.

By using the Inverse Function Theorem for measure spaces, we conclude that $\exists$ a measurable transformation T^{-1} which is Radon measure- invariant.

Therefore, from steps I, II and III, we come to the conclusion that Radon e-normality property is invariant under measurable homeomorphism and Radon measure-invariant transformations T and T^{-1}.

Now, we extend Radon e-regularity property and Radon e-normality property on locally compact measurable manifold $(M, \tau_1, \Sigma_1, \mu_1)$. i.e., we introduce Radon measure on compact measurable chart, compact measurable atlas and compact measurable manifold and designate these concepts as Radon measure chart, Radon measure atlas and Radon measure manifold as follows:

Definition 3.4: Radon measure chart Compact measurable chart $((U, \tau_{1/U}, \Sigma_{1/U}), \varphi)$ of locally compact measurable manifold (M, , τ_1, Σ_1) equipped with a Radon measure $\mu_{R_{1/U}}$ is called a Radon measure chart denoted by $((U, \tau_{1/U}, \Sigma_{1/U}, \mu_{R_{1/U}}), \varphi)$ satisfying the following conditions:

a) φ is measurable homeomorphism, if for every Borel measurable subset $V \subset \bar{V} \in (R^n, \tau, \Sigma)$, $\varphi^{-1}(\bar{V}) = (U, \varphi) \in (M, \tau_1, \Sigma_1)$ is also Borel measurable.
b) φ is Radon measure-invariant. i.e., $\mu_{R_1}(\varphi^{-1}(\bar{V})) = \mu_R(\bar{V})$ where $\varphi^{-1}(\bar{V}) = (U, \varphi)$ a Borel measurable chart which satisfies the Radon measure conditions:

1. (i) For $p \in V \subset \bar{V} \in \Sigma$; $\mu_R(\bar{V}) < \infty$;
   (ii) For any Borel compact subset $\bar{V} \subset (R^n, \tau, \Sigma)$, $\mu_R(\bar{V}) = \sup\{\mu_R(E_i); i \in I : E_i \subseteq \bar{V} : E_i$ compact and measurable$\}$
2. (i) For $\varphi^{-1}(\bar{V}) \in \Sigma_1$, where Σ_1 a σ-algebra induced on second countable Hausdorff topological space, $\mu_{R_1}(\varphi^{-1}(\bar{V})) < \infty$; where $\varphi^{-1}(\bar{V}) = (U, \varphi)$ is a measurable chart
   (ii) For any Borel subset $\varphi^{-1}(\bar{V}) \subset (M, \tau_1, \Sigma_1)$, $\mu_{R_1}(\varphi^{-1}(\bar{V})) = \sup\{\mu_{R_1}(\varphi^{-1}(E_i))$; $i \in I : \varphi^{-1}(E_i) \subseteq \varphi^{-1}(\bar{V})$: $\varphi^{-1}(E_i)$ compact and measurable$\}$.

A locally compact measure manifold $(M, \tau_1, \Sigma_1, \mu_1)$ admits partitions of unity [3.1]. By using this proposition, $(M, \tau_1, \Sigma_1, \mu_1)$ becomes a compact measure manifold. Hence the whole of measure manifold $(M, \tau_1, \Sigma_1, \mu_1)$ is measurable.

Since $\bigcup_{i=1}^{\infty}(U_i, \tau_{1/U_i}, \Sigma_{1/U_i}, \mu_{1/U_i}) = (M, \tau_1, \Sigma_1, \mu_1)$, we can measure compact measure manifold $(M, \tau_1, \Sigma_1, \mu_1)$ by Radon measure.

Now, we introduce the concept of Radon measure atlas as follows:

Definition 3.5: Radon measure atlas By an R^n-Radon measure atlas of class C^k ($k \geq 1$) on a locally compact measurable manifold (M, τ_1, Σ_1), we mean a countable collection $(\mathcal{A}, \tau_{1/\mathcal{A}}, \Sigma_{1/\mathcal{A}}, \mu_{R_{1/\mathcal{A}}})$ of n-dimensional Radon measure charts $((U_i, \tau_{1/U_i}, \Sigma_{1/U_i}, \mu_{R_{1/U_i}}), \varphi_i)$ for all $i \in N$ on $(M, \tau_1, \Sigma_1, \mu_{R_1})$ satisfying the following conditions:

(a_1) $\bigcup_{i \in I}((U_i, \tau_{1/U_i}, \Sigma_{1/U_i}, \mu_{R_{1/U_i}}) = (M, \tau_1, \Sigma_1, \mu_{R_1})$ i.e., the countable union of all Radon measure charts in $(\mathcal{A}, \tau_{1/\mathcal{A}}, \Sigma_{1/\mathcal{A}}, \mu_{R_{1/\mathcal{A}}})$ cover $(M, \tau_1, \Sigma_1, \mu_{R_1})$.

(a_2) For any pair of Radon measure charts $((U_i, \tau_{1/U_i}, \Sigma_{1/U_i}, \mu_{R_{1/U_i}}), \varphi_i)$ and $((U_j, \tau_{1/U_j}, \Sigma_{1/U_j}, \mu_{R_{1/U_j}}), \varphi_j)$ in $(\mathcal{A}, \tau_{1/\mathcal{A}}, \Sigma_{1/\mathcal{A}}, \mu_{R_{1/\mathcal{A}}})$, the transition maps $\varphi_i \circ \varphi_j^{-1}$ and $\varphi_j \circ \varphi_i^{-1}$ are :

(1) differentiable maps of class C^k ($k \geq 1$)
i.e., $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j) \subseteq (R^n, \tau_2, \Sigma_2, \mu_{R_2})$ and
$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j) \subseteq (R^n, \tau_2, \Sigma_2, \mu_{R_2})$ are differentiable maps of class C^k ($k \geq 1$).

(2) measurable :
Transition maps $\varphi_i \circ \varphi_j^{-1}$ and $\varphi_j \circ \varphi_i^{-1}$ are measurable functions if,
a) If any Borel subset $K \subseteq \varphi_i(U_i \cap U_j)$ is measurable in $(R^n, \tau_2, \Sigma_2, \mu_{R_2})$ then, $(\varphi_i \circ \varphi_j^{-1})^{-1}(K) \in \varphi_j(U_i \cap U_j)$ is also measurable.
b) $\varphi_j \circ \varphi_i^{-1}$ is measurable if $S \subseteq \varphi_j(U_i \cap U_j)$ is measurable in $(R^n, \tau_2, \Sigma_2, \mu_{R_2})$, then $(\varphi_j \circ \varphi_i^{-1})^{-1}(S) \in \varphi_i(U_i \cap U_j)$ is also measurable.

c) For any two Radon measure atlases $(\mathcal{A}_1,\tau_{1/\mathcal{A}_1},\Sigma_{1/\mathcal{A}_1},\mu_{R_1/\mathcal{A}_1})$ and $(\mathcal{A}_2,\tau_{1/\mathcal{A}_2},\Sigma_{1/\mathcal{A}_2},\mu_{R_1/\mathcal{A}_2})$, we say that a mapping T: $\mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is measurable if T^{-1}(E) is measurable for every measurable chart A = $(U,\tau_{1/U},\Sigma_{1/U},\mu_{R_1/U}) \subset (\mathcal{A}_2,\tau_{1/\mathcal{A}_2},\Sigma_{1/\mathcal{A}_2},\mu_{R_1/\mathcal{A}_2})$ and the mapping is Radon measure preserving if $\mu_{R_1/\mathcal{A}_1} = \mu_{R_1/\mathcal{A}_2}$ satisfying the Radon measure conditions:

1. (i) $\forall$ p $\in$ A $\subset \mathcal{A}_2$, $\mu_{R_1/\mathcal{A}_2}$(A) <∞; where A is a Borel compact subset of $\mathcal{A}_2$.
(ii) $\mu_{R_1/\mathcal{A}_2}$(A) = sup{ $\mu_{/\mathcal{A}_2}(E_i)$; $i \in$ I : $E_i \subseteq A$: E_i compact and measurable}.
T^{-1}: $\mathcal{A}_2 \longrightarrow \mathcal{A}_1$ is Radon measure preserving transformation if $\mu_{R_1/\mathcal{A}_2}$(E) = $\mu_{R_1/\mathcal{A}_1}(T^{-1}$(E)) where $\mathcal{A}_1 \sim \mathcal{A}_2$ and $\mu_{R_1/\mathcal{A}_2} = \mu_{R_1/\mathcal{A}_1}$ satisfying the Radon measure conditions:
2. (i) $\forall$ T^{-1}(p) $\in T^{-1}$(A) $\subset \mathcal{A}_1$; $\mu_{R_1/\mathcal{A}_1}(T^{-1}$(A)) <∞;
(ii) $\mu_{R_1/\mathcal{A}_1}(T^{-1}$(A)) = sup{ $\mu_{/\mathcal{A}_2}(T^{-1}(E_i))$; $i \in$I: $T^{-1}(E_i)) \subseteq T^{-1}$(A): $T^{-1}(E_i)$ compact and measurable}

Then, we call T a Radon-measure preserving transformation.

(a_4) If a measurable transformation T: $\mathcal{A} \longrightarrow \mathcal{A}$ preserves a Radon measure μ_{R_1}, then we say that μ_{R_1} is T- invariant.

If T is invariant and if both T and T^{-1} are measurable and Radon measure preserving then we call T an invertible Radon measure preserving transformation.

Let A_m^k(M) denotes the set of all Radon measure atlases of class C^k on (M, τ_1, Σ_1, μ_{R_1}).

Two Radon measure atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ in A_m^k(M) are said to be equivalent if $(\mathcal{A}_1 \cup \mathcal{A}_2) \in A_m^k$(M). In order to have that $\mathcal{A}_1 \cup \mathcal{A}_2$ to be a member of A_m^k(M) we require that for every Radon measure chart $(U_i,\varphi_i) \in \mathcal{A}_1$ and $(V_j,\varphi_j) \in \mathcal{A}_2$, the set of $\varphi_i(U_i \cap V_j)$ and $\varphi_j(U_i \cap V_j)$ are Borel subsets in $(R^n,\tau_2, \Sigma_2,\mu_{R_2})$ and maps $\varphi_i \circ \varphi_j^{-1}$ and $\varphi_j \circ \varphi_i^{-1}$ are of class C^k and are measurable. The relation introduced is an equivalence relation in A_m^k(M) and hence partitions A_m^k(M) into disjoint equivalence classes. Each of these equivalence classes forms a differentiable structure of class C^k on (M, τ_1, Σ_1, μ_{R_1}).

3.6: Radon measure manifold

A Radon measure space (M, τ_1, Σ_1, μ_{R_1}) together with this differentiable structure of class C^k forms a Radon measure differentiable manifold of class C^k.

We study e-regularity property on Radon measure manifold (M, τ_1, Σ_1, μ_{R_1}).

By using the theorem 3.2, which states that "If $(R^m, \tau_2, \Sigma_2,\mu_{R_2})$ is a locally compact Radon e-regular space then Radon e-regularity property is invariant under measurable homeomorphism and under Radon measure invariant transformation", we prove that Radon e-regularity property on compact Radon measure manifold (M, τ_1, Σ_1, μ_{R_1}) is invariant under measurable homeomorphism and under Radon measure-invariant transformation.

Theorem 3.3: If $(R^n,\tau_2, \Sigma_2,\mu_{R_2})$ is a locally compact Radon e-regular space then Radon e-regularity property is invariant under measurable homeomorphism and under Radon measure-invariant transformation
φ:U$\subset$(M, τ_1, Σ_1, μ_{R_1}) $\longrightarrow(R^n,\tau_2, \Sigma_2,\mu_{R_2})$.

Proof: Let $(R^n,\tau_2, \Sigma_2, \mu_{R_2})$ be a locally compact Radon e-regular space.

We show that a Radon measure manifold (M,τ_1, Σ_1, μ_{R_1}) is a Radon e-regular space. That is, we show that Radon e-regularity property is invariant under measurable homeomorphism and Radon measure-invariant transformations by using measurable homeomorphism concept [11] and Inverse Function Theorem for measure space [11].

In step I, we show that Radon e-regularity property is invariant under measurable homeomorphism.

Step-I: We shall show that φ: U $\subset$ (M, τ_1, Σ_1, μ_{R_1}) $\longrightarrow(R^n,\tau_2, \Sigma_2, \mu_{R_2})$ and φ^{-1}: V $\subset (R^n,\tau_2,\Sigma_2,\mu_{R_2}) \longrightarrow$ (M,τ_1, Σ_1, μ_{R_1}) are C^∞measurable homeomorphisms where μ_{R_1} and $\mu_{R_2}$are Radon measures on M and R^n respectively.

Since $(R^n, \tau_2, \Sigma_2, \mu_{R_2})$ is a Radon e-regular space, i.e., for every point p $\in (R^n, \tau_2, \Sigma_2)$ and Borel compact subset E not containing p in (R^n, τ_2, Σ_2), $\exists$ Borel open subsets A $\subset \bar{A}$, B $\subset \bar{B} \in \Sigma_2$, where $\bar{A}$ and $\bar{B}$ are Borel compact subsets of $(R^n, \tau_2, \Sigma_2, \mu_{R_2})$, $\ni$ p $\in$ A $\subset \bar{A}$, E $\subset$ B $\subset \bar{B}$ and $\mu_{R_2}(\bar{A} \cap \bar{B}) = 0$ satisfying the Radon measure conditions:

1. For p $\in$ A $\subset \bar{A} \subset (R^n, \tau_2, \Sigma_2)$,
   (i) $\forall$ p $\in \bar{A} \subset (R^n, \tau_2, \Sigma_2)$; $\mu_{R_2}(\bar{A}) < \infty$;
   (ii) For any Borel compact subset $\bar{A} \subset (R^n, \tau_2, \Sigma_2)$, $\mu_{R_2}(\bar{A})$ = sup{$\mu(E_i)$, i $\in I$: $E_i \subseteq \bar{A}$: E_i compact and measurable}
2. For F $\subset$ B $\subset \bar{B} \subset (R^n, \tau_2, \Sigma_2)$,
   (i) $\forall$ q $\in$ B $\subset \bar{B} \subset (R^n, \tau_2, \Sigma_2)$, $\mu_{R_2}(\bar{B}) < \infty$;
   (ii) For any Borel compact subset $\bar{B} \subset (R^n, \tau_2, \Sigma_2)$, $\mu_{R_2}(\bar{B})$ = sup{$\mu(F_i)$, i $\in I$: $F_i \subseteq \bar{B}$: F_i compact and measurable} and $\mu_{R_2}(\bar{A} \cap \bar{B}) = 0$. (i)

By using the Inverse Function Theorem for measure manifolds [11], we can write, $\forall$ Borel open subsets A $\subset \bar{A}$, B $\subset \bar{B} \in \Sigma_2$, $\exists$ Borel compact charts $\varphi^{-1}(\bar{A}) = ((U_i, \tau_{1/U_i}, \Sigma_{1/U_i}), \varphi_i)$ and $\varphi^{-1}(\bar{B}) = ((U_j, \tau_{1/U_j}, \Sigma_{1/U_j}), \varphi_j) \in \Sigma_1 \ni \varphi^{-1}$(p) $\in \varphi^{-1}(\bar{A}) = ((U_i, \tau_{1/U_i}, \Sigma_{1/U_i}), \varphi_i)$, φ^{-1}(E) $\subset ((U_j, \tau_{1/U_j}, \Sigma_{1/U_j}), \varphi_j)$ and $\mu_{R_1}(U_i \cap U_j) = 0$ satisfying the Radon measure conditions:

3. For φ^{-1}(p) $\in (U_i, \varphi_i) \subset$(M, τ_1, Σ_1,μ_{R_1}),
   (i) $\forall \varphi^{-1}$(p) $\in \varphi^{-1}(\bar{A}) \subset$(M, τ_1, Σ_1,μ_{R_1}), $\mu_{R_1}(U_i) < \infty$;
   (ii) For any measurable compact chart (U_i, φ_i), $\mu_{R_1}(U_i)$ = sup{$\mu(\varphi^{-1}(E_i)$;i$\in I$:$\varphi^{-1}(E_i) \subseteq U_i$: $\varphi^{-1}(E_i)$ compact and measurable}
4. For φ^{-1}(F) $\subset \varphi^{-1}(\bar{B})$,
   (i) $\forall \varphi^{-1}$(q) $\in \varphi^{-1}(\bar{B}) \subset$(M, τ_1, Σ_1,μ_{R_1}), $\mu_{R_1}(U_j) < \infty$;
   (ii) For any measurable compact chart $\varphi^{-1}(\bar{B}) = (U_j, \varphi_j)$
   $\mu_{R_1}(U_j)$ = sup{$\mu(\varphi^{-1}(F_i)$;i$\in I$:$\varphi^{-1}(F_i) \subseteq U_j$: $\varphi^{-1}(F_i)$ compact and measurable}
   and $\mu_{R_1}(\varphi^{-1}(\bar{A}) \cap \varphi^{-1}(\bar{B})) = 0$.
   i.e., $\mu_{R_1}(U_i \cap U_j) = 0$ (ii)

Therefore, Radon measure manifold (M, τ_1, Σ_1,μ_{R_1}) is a Radon e-regular space.

$\Rightarrow$Radon e-regularity property on compact Radon measure manifold (M, τ_1, Σ_1,μ_{R_1}) is invariant under measurable homeomorphism.

Step-II: In step II, we show that φ preserves the Radon e-regularity property under Radon measure- invariant transformation.

Since, for every Borel open subsets A$\subset \bar{A}$, B$\subset \bar{B} \in \Sigma_2$, $\exists$ measurable compact charts
$\varphi^{-1}(\bar{A}) = (U_i, \varphi_i) \subset$(M, τ_1, Σ_1,μ_{R_1}) and $\varphi^{-1}(\bar{B}) = (U_j, \varphi_j) \subset$(M,$\tau_1$, Σ_1,μ_{R_1}) such that, $\mu_{R_1}(\varphi^{-1}(\bar{A})) = \mu_{R_2}(\bar{A})$
$\Rightarrow \mu_{R_1}(U_i) = \mu_{R_2}(\bar{A})$
and $\mu_{R_1}(\varphi^{-1}(\bar{B})) = \mu_{R_2}(\bar{B})$
$\Rightarrow \mu_{R_1}(U_j) = \mu_{R_2}(\bar{B})$ (iii)

Therefore, from (i), (ii) and (iii), we confirm that Radon e-regularity property is preserved under the measurable homeomorphism and Radon measure-invariant transformationφ and φ^{-1}.

Therefore, from steps I and II, we conclude that Radon e-regularity property on compact Radon measure manifold (M, τ_1, Σ_1,μ_{R_1}) is invariant under measurable homeomorphism and Radon measure-invariant transformation.

Now, the Radon e-normality property on compact measurable space which was proved in theorem 3.3, which states that "If $(R^m, \tau_2, \Sigma_2, \mu_{R_2})$ is a locally compact Radon e-normal space then Radon e-normality property is invariant under measurable homeomorphism and Radon measure invariant transformation" is extended on compact Radon measure manifold (M, τ_1,Σ_1, μ_{R_1}) as follows:

Theorem 3.4: If $(R^n, \tau_2, \Sigma_2, \mu_{R_2})$ is a locally compact Radon e-normal space then Radon e-normality property is invariant under measurable homeomorphism and under Radon measure-invariant transformation
φ : U $\subset$ (M, τ_1,Σ_1, μ_{R_1}) $\longrightarrow (R^n, \tau_2, \Sigma_2, \mu_{R_2})$.

Proof: Let $(R^n, \tau_2, \Sigma_2, \mu_{R_2})$ be a locally compact Radon e-normal space.

We show that a Radon measure manifold (M,τ_1,Σ_1, μ_{R_1}) is also a Radon e-normal space by using measurable homeomorphism concept [11] and Inverse Function Theorem for measure manifolds [11].

We show that φ: U $\subset$(M, τ_1,Σ_1, μ_{R_1}) $\longrightarrow$ (R^n,τ_2, Σ_2,μ_{R_2}) and φ^{-1}: V $\subset$ (R^n,τ_2,Σ_2,μ_{R_2})$\longrightarrow$ U $\subset$ (M,τ_1,Σ_1, μ_{R_1}) are C^∞ measurable homeomorphisms where μ_{R_1} and μ_{R_2} are Radon measures on M and R^n respectively.

In step I, we show that Radon e-normality property is invariant under measurable homeomorphism.

Step I: Since (R^n,τ_2, Σ_2,μ_{R_2}) is a locally compact Radon e-normal space. i.e., for any disjoint Borel compact subsets E and F in R^n, $\exists$ Borel open subsets A $\subset \bar{A}$, B $\subset \bar{B} \in \Sigma_2$, where $\bar{A}$ and $\bar{B}$ are Borel compact subsets of (R^n, τ_2, Σ_2), such that E $\subset \bar{A}$, F $\subset \bar{B}$ and $\mu_{R_2}(\bar{A} \cap \bar{B})$ = 0 satisfying the Radon measure conditions:

1. For E $\subset$ A$\subset \bar{A} \subset$(R^n,τ_2, Σ_2,μ_{R_2}),
   (i) $\mu_{R_2}(\bar{A}) < \infty$;
   (ii) For any Borel compact subset $\bar{A} \subset$(R^n,τ_2, Σ_2,μ_{R_2}), $\mu_{R_2}(\overline{A})$ = sup{$\mu(E_i)$, $i \in I$: $E_i \subseteq \bar{A}$: E_i compact and measurable}
2. For F $\subset$ B $\subset \bar{B} \subset$(R^n,τ_2, Σ_2,μ_{R_2}),
   (i) $\mu_{R_2}(\bar{B}) < \infty$;
   (ii) For any Borel compact subset $\bar{B} \subset$(R^n,τ_2, Σ_2,μ_{R_2}), $\mu_{R_2}(\bar{B})$ = sup{$\mu(F_i)$, $i \in I$: $F_i \subseteq \bar{B}$: F_i compact and measurable} and $\mu_{R_2}(\bar{A} \cap \bar{B})$ = 0. (i)

Again, by using theorem 3.3 and Inverse Function Theorem for measure manifolds [11] , we can write, $\forall$ Borel open subsets A $\subset \bar{A}$, B $\subset \bar{B} \in \Sigma_2$, $\exists$ measurable compact charts $\varphi^{-1}(\bar{A}) = ((U_i, \tau_{1/U_i}, \Sigma_{1/U_i}), \varphi_i)$ and
$\varphi^{-1}(\bar{B}) = ((U_j, \tau_{1/U_j}, \Sigma_{1/U_j}), \varphi_j) \in \Sigma_1$, where $\varphi^{-1}(\bar{A})$ and $\varphi^{-1}(\bar{B})$ are Borel compact charts of
(M, τ_1, Σ_1,μ_{R_1}), $\ni \varphi^{-1}$(E) $\subset \varphi^{-1}(\bar{A}) = ((U_i, \tau_{1/U_i}, \Sigma_{1/U_i}), \varphi_i)$, φ^{-1}(F) $\subset \varphi^{-1}(\bar{B}) = ((U_j, \tau_{1/U_j}, \Sigma_{1/U_j}), \varphi_j)$ and
$\mu_{R_1}(\varphi^{-1}(\bar{A}) \cap \varphi^{-1}(\bar{B}))$ = 0

$\Rightarrow \mu_{R_1}(U_i \cap U_j)$ = 0 satisfying the Radon measure conditions:

3. For φ^{-1}(E) $\subset \varphi^{-1}(\bar{A}) = (U_i, \varphi_i) \subset$ (M, τ_1, Σ_1,μ_{R_1}),
   (i) $\mu_{R_1}(U_i) < \infty$;
   (ii) For any measurable compact chart (U_i, φ_i), $\mu_{R_1}(U_i)$ = sup{$\mu(\varphi^{-1}(E_i)$; i$\in I$:$\varphi^{-1}(E_i) \subseteq U_i$: $\varphi^{-1}(E_i)$ compact and measurable}
4. For φ^{-1}(F) $\subset \varphi^{-1}(\bar{B}) = (U_j, \varphi_j) \subset$(M, τ_1, Σ_1,μ_{R_1}),
   (i) $\mu_{R_1}(U_j) < \infty$;
   (ii) For any measurable compact chart(U_j, φ_j)
   $\mu_{R_1}(U_j)$=sup{$\mu(\varphi^{-1}(F_i)$; i$\in I$:$\varphi^{-1}(F_i) \subseteq U_j$: $\varphi^{-1}(F_i)$ compact and measurable}and
   $\mu_{R_1}(\varphi^{-1}(\bar{A}) \cap \varphi^{-1}(\bar{B}))$ = 0. i.e., $\mu_{R_1}(U_i \cap U_j)$ = 0 (ii)

Therefore, (M, τ_1, Σ_1,μ_{R_1}) is a compact Radon e-normal space.

$\Rightarrow$Radon e-normality property on compact Radon measure manifold is invariant under measurable homeomorphism.

In step II, we show that φ preserves the Radon e-normality property under Radon measure- invariant transformation.

Step II: Since, for every Borel open subsets A$\subset \bar{A}$, B $\subset \bar{B} \in \Sigma_2$, $\exists$ measurable compact charts
$\varphi^{-1}(\bar{A}) = (U_i, \varphi_i) \subset$(M, τ_1, Σ_1,μ_{R_1}) and $\varphi^{-1}(\bar{B}) = (U_j, \varphi_j) \subset$(M,$\tau_1$, Σ_1,μ_{R_1}) such that,
$\mu_{R_1}(\varphi^{-1}(\bar{A})) = \mu_{R_2}(\bar{A}) \Rightarrow \mu_{R_1}(U_i) = \mu_{R_2}(\bar{A})$ and
$\mu_{R_1}(\varphi^{-1}(\bar{B})) = \mu_{R_2}(\bar{B}) \Rightarrow \mu_{R_1}(U_j) = \mu_{R_2}(\bar{B})$ (iii)

Therefore, from (i), (ii) and (iii), we confirm that Radon e-normality property is preserved under measurable homeomorphism and Radon measure-invariant transformation φ and φ^{-1}.

Therefore, from steps I and II, we conclude that Radon e-normality property on compact Radon measure manifold (M, τ_1,Σ_1, μ_{R_1}) is invariant under measurable homeomorphism and Radon measure-invariant transformation.

Example for a Radon measure manifold of dimension 1 is as follows:

Example 3.1: Measurable closed unit interval I = [0, 1] ⊂ (R^1, τ, Σ) is a Radon measure manifold of dimension 1.

Solution: Let (M,τ_1, Σ_1) = [0, 1] be measurable closed interval. The closed unit interval I = [0, 1] ⊂ R^1 is a 1-dimensional manifold whose boundary consists of the two points 0 and 1. A 1-dimensional measurable manifold (M,τ_1, Σ_1) = [0, 1] is homeomorphic to a measurable closed unit interval I = [0, 1] ⊂ (R^1, τ, Σ). Measurable closed unit interval I has a non-empty boundary consisting of the two points {0} and {1}.

We know that, any connected compact 1-dimensional manifold with non-empty boundary is homeomorphic to I = [0, 1] ⊂ R^1. For connected 1-dimensional manifolds, two invariants called compactness and presence of boundary form a complete system of topological invariants [8].

Also we know that, every subset of the unit interval has a supremum and infimum.

Now, let φ : (M,τ_1, Σ_1) ⟶ [0, 1] ⊂ (R^1, τ, Σ) be a measurable homeomorphism.

By lemma 2.1 [8], since I = [0, 1] can be represented as the union of a monotonically increasing sequence of Borel open subsets, all homeomorphic to (R^1, τ, Σ), that is, I = [0, 1] is homeomorphic to (R^1, τ, Σ). The closed unit interval [0, 1] satisfies the Radon measure conditions:

(i) $\mu_R([0, 1]) < \infty$;
(ii) For some $\varepsilon > 0$, μ_R([0, 1]) = sup{μ([s- ε, s+ ε]) : [s- ε, s+ ε] ⊆ [0, 1] : [s- ε, s+ ε] compact measurable charts}

Since the closed unit interval has supremum, the length of [0, 1] is 1.

Therefore, μ_R([0, 1]) = 1.

Therefore, Radon measure on a locally compact 1-dimensional measurable manifold (M,τ_1,Σ_1,μ_{R_1}) is 1.

Note: Lebesgue measure on measurable space (R^n, τ, Σ) is a Radon measure.

CONCLUSION

Measurable properties like Radon e-regularity and Radon e-normality are studied on locally compact Radon measure space and extended this study on Radon measure manifold. This study has applications in the field of engineering science and neural network.

REFERENCES

1. Bogachev V. I. “Measures on Topological Spaces”, Journal of Mathematical Sciences, Volume 91, No. 4(1998).
2. Bogachev V. I. “Measure theory”, Volume II, Springer (2006).
3. Dorlas T. C. “Remainder Notes for the Course on Measure on Topological Spaces”, Dublin Institute for Advanced Studies, School of theoretical Physics, 10, Dublin 4, Ireland (2010).
4. D. H. Fremlin, “Measure Theory”, Vol. 4, University of Essex (2007).
5. Hunter John K., “Measure Theory, Lecture notes”, University of California at Davis (2011).
6. Jean-Marie Morvan, “Generalized Curvatures”, Springer.
7. Jeffrey M Lee, “Manifolds and Differential Geometry”, American Mathematical Society, Volume 107(2011).
8. Oleg Viro, “1-manifolds”, Bulletin of the Manifold Atlas (2013).
9. P. Cannarsa and T. D.’ Aprile, “Lecture Notes on Measure Theory and Functional Analysis”, Dipartimento di Matematica, Universita di Roma, “Tor Vergata”, (2006/07).
10. S. C. P Halakatti and Akshata Kengangutti, “Introducing Atomic Separation Axioms on Atomic Measure Space- An Advanced Study”, International Organization of Scientific Research Journal of Mathematics (IOSR-JM), Vol. 10, Issue 5, Ver. III, pp 18-29(2014).
11. S. C. P Halakatti, Akshata Kengangutti and Soubhagya Baddi, “Generating a Measure Manifold”, International Journal of Mathematical Archive (IJMA), 6(3),pp1-8(2015).
12. S. C. P. Halakatti and H. G. Haloli, “Introducing the Concept of Measure Manifold (M,τ_1 ,Σ_1 , μ_1)”, International Organization of Scientific Research Journal of Mathematics (IOSR-JM), Vol. 10, Issue 3 Ver. II, 01-11(2014).
13. S .C. P. Halakatti and H.G. Haloli, “Topological Properties on Measure Space with Measure Conditions”, International Journal of Advancements in Research and Technology Volume3, Issue 9, pp- 08-18(2014).
14. S.C.P.Halakatti and H. G. Haloli, “Extended topological properties on measure manifold”, International Journal of Mathematical Archive (IJMA), 5(11) pp-1-8(2014).

15. S. C. P Halakatti and Soubhagya Baddi, "Radon measure on compact topological measurable space", International Organization of Scientific Research Journal of Mathematics (IOSR-JM), Vol. 11, Issue 2 Ver. III pp-10-14(2015).
16. William Arveson, "Notes on Measure and Integration in Locally Compact Spaces", Department of Mathematics, University of California, Berkeley (1996).

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