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A NOTE ON QUASI-SIMILARITY OF OPERATORS IN HILBERT SPACES

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ABSTRACT

In this paper we introduce the notion of Quasi-similarity of bounded linear operators in Hilbert Spaces. We do so by defining a quasi- affinity from one Hilbert Space H to K. Some results on quasi- affinities are also discussed. It has already been shown that on a finite dimensional Hilbert Space, quasi similarity is an equivalence relation that is; it is reflexive, symmetric and also transitive. Using the definition of commutants of two operators, we give an alternative result to show that quasi similarity is an equivalence relation on an infinite dimensional Hilbert Space. Finally, we establish the relationship between quasi similarity and almost similarity equivalence relations in Hilbert Spaces using hermitian and normal operators.

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Key words*: Quasi-similarity, Quasi affinities, Equivalence Relations, Commutants.*

1. INTRODUCTION

In this paper Hilbert spaces or subspaces will be denoted by capital letters, H and K respectively and T, A, B e.t.c. denotes bounded linear operators where an operator means a bounded linear transformation.$B(H)$ will denote the Banach algebra of bounded linear operators on H. $B(H, K)$ denotes the set of bounded linear transformations from H to K, which is equipped with the (induced uniform) norm. If $T \in B(H)$, then T^* denotes the adjoint while $Ker(T)$, $Ran(T)$, $\bar{M}$ and $M^{\perp}$ stands for the kernel of T, range of T, closure of M and orthogonal complement of a closed subspace M of H respectively. For an operator T, we also denote by $\sigma(T)$, $\| T \|$ the spectrum and norm of T respectively.

We need the following definitions:

An operator $T \in B(H)$ is said to be:

Self adjoint or Hermitian if $T^* = T$ (equivalently, if $\langle Tx, x\rangle \in \mathbb{R}, \forall \in H$);

Unitary if $T^*T = TT^* = I$; *Normal* if $T^*T = TT^*$ (equivalently, if $\| Tx \| = \| T^*x \| \ \forall x \in H$).

Let H and K be Hilbert spaces. An operator $X \in B(H, K)$ is *invertible* if it is injective (one -to- one) and surjective (onto or has dense range); equivalently if $Ker(X) = \{0\}$ and $\overline{Ran(X)} = K$. we denote the class of invertible linear operators by $\mathcal{G}\ (H, K)$.

The *commutator* of two operators A and B, denoted by $[A, B]$ is defined by $AB - BA$. The *self –commutator* of an operator A is $[A, A^*] = A^*A - AA^*$.

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Two operators$T \in B(H)$ and $S \in B(K)$ are *similar* (denoted T $\approx$ S) if there exists an operator $X \in \mathcal{G}\,(H,K)$ such that $XT = SX$ (i.e,$X^{-1}SX$ *or* $S = XTX^{-1}$) where $\mathcal{G}\,(H,K)$ is a Banach subalgebra of B (H,K) which is an invertible operator from H to K.

Linear operators $T \in B(H)$ and $S \in B(K)$ are *unitarily equivalent* (denoted $T \cong S$), if there exists a unitary operator $U \in \mathcal{G}\,(H,K)$ such that $UT = SU$ $(i.e, T = U^*SU$ or equivalently $S = UTU^*)$.

Two operators are considered the *"same"* if they are unitarily equivalent, since they have the same properties of invertibility, normality, spectral picture (norm, spectrum and spectral radius).

An operator $X \in$ B (H,K) is *quasi-invertible* or *a quasi-affinity* if it is an injective operator with dense range (i.e. $Ker\ X = \{0\}$ and $\overline{Ran(X)} = K$; equivalently, $Ker\ X = \{\bar{0}\}$ and, $Ker\ X^* = \{\bar{0}\}$. Thus $X \in B(H,K)$ is quasi-invertible if and only if $X^* \in B(K,H)$ is quasi-invertible).

An operator $T \in B(H)$is *a quasi –affine transform* of $S \in B(K)$ if there exists a quasi-invertible $X \in B(H,K)$ such that $XT = SX$ ($ie\ X\ intertwines\ T\ and\ S$). T is a *quasi-affine transform* of S if there exists a quasinvertible operator intertwining T to S.

Two operators $T \in B(H)$ and $S \in B(K)$ are *quasi-similar* (denoted T$\sim$S) if they are quasi-affine transforms of each other (i.e., if there exists quasi-invertible operators $X \in B(H,K)$ and $Y \in B(K,H)$ such that $TX = XS\ and\ YS = TY$).

T is said to be *densely intertwined* to S if there exists an operator with dense range intertwining T to S.

Two operators S and T are said to be *almost similar* (denoted by $S \overset{a.s}{\sim} T$) if there exists an invertible operator N such that the following two conditions are satisfied:

$$S^*S = N^{-1}(T^*T)N$$
$$S^* + S = N^{-1}(T^* + T)N.$$

Almost similarity of operators is also an equivalence relation.

2. MAIN RESULTS

2.1. Quasi-affinities of operators

Definition 2.1.1: *The commutator* of $A \in B(H)$, $\{A\}'$is the set of all operators in B(H) that commuteswith A, i.e. $\{\mathrm{A}\}' = \{\mathrm{C} \in \mathrm{B(H)}: \mathrm{CA} = \mathrm{AC}\}$.

Proposition 2.1.2: *The commutant of an operator (is the set of all operators intertwining it to itself) intertwines itself.*

Claim: $\mathrm{C_1}{+}\mathrm{C_2} \in \{\mathrm{A}\}'$ and $\mathrm{C_1C_2} \in \{\mathrm{A}\}'$ whenever $\mathrm{C_1}, \mathrm{C_2} \in \{\mathrm{A}\}'$.

Proof: $\{\mathrm{A}\}' = \{\mathrm{C} \in \mathrm{B(H)}: \mathrm{CA} = \mathrm{AC}\}$. Now $(\mathrm{C_1}{+}\mathrm{C_2})\mathrm{A} = \mathrm{C_1A} + \mathrm{C_2A} = \mathrm{AC_1}{+}\mathrm{AC_2} = \mathrm{A}(\mathrm{C_1}{+}\mathrm{C_2})$, that is $(\mathrm{C_1}{+}\mathrm{C_2})\mathrm{A} = \mathrm{A}(\mathrm{C_1}{+}\mathrm{C_2})$ and $(\mathrm{C_1C_2})\mathrm{A} = \mathrm{C_1}(\mathrm{C_2A}) = \mathrm{C_1}(\mathrm{AC_2}) = (\mathrm{AC_2})\mathrm{C_1} = \mathrm{A}(\mathrm{C_2C_1}) = \mathrm{A}(\mathrm{C_1C_2})$ that is $(\mathrm{C_1C_2})\mathrm{A} = \mathrm{A}(\mathrm{C_1C_2})$ as required.

Actually $\{\mathrm{A}\}'$ is an operator algebra which contains the identity.

Theorem 2.1.3: *Unitary equivalence is an equivalence relation.*

Proof: See [9].

Remark 2.1.4: *It has already been proved in [9] that similarity is an equivalence relation on* $B(H)$.

The natural concept of equivalence between Hilbert space operators is unitary equivalence which is stronger than similarity.

Theorem 2.1.5 [10, Proposition 3.3]: *If* X *is a quasi-affinity from* H *to* K *and* Y *is a quasi-affinity from* K *to* L, *then*

(a) YX *is a quasi-affinity from* H *to* L *and* XY *is a quasi-affinity from* L *to* H.
(b) *If* $X \in B(H)$ $is\ a\ quasi-affinity, then$ X^**is a quasi-affinity.*

Proof: (a) Since S and T are called quasi-similar there exist quasi-affinities $X \in B(H,K)$ and $Y \in B(K,H)$ such that $XS = TX$ and $TY = YS$.

With this in mind, we draw the following diagram such that it "commutes".

$$\begin{tikzcd}
H \arrow[r, "Y"] \arrow[d, "S"] \arrow[dr, "Y"] & K \arrow[r, "X"] \arrow[d, "T"] \arrow[dr, "X"] & H \arrow[r, "Y"] \arrow[d, "S"] \arrow[dr] & K \arrow[d, "T"] \\
H \arrow[r, "Y"] & K \arrow[r, "X"] & H \arrow[r, "Y"] & K
\end{tikzcd}$$

We want to prove that XY and YX are quasi-affinities. Clearly, XY is one-to-one since it is the composition of one-to-one operators. It suffices to prove that XY has a dense range.

Note that $(XY) \subseteq H$. It follows that $\overline{XYH} = \overline{X(\overline{YH})} = \overline{X(K)} = H$. Therefore $\overline{Ran(XY)} = H$. This proves that XY has dense range.

Similarly, YX is one-to- one (since it is the composition of one-to-one operators). To show that it has dense range, note that $(YX) \subseteq K$. It follows that $\overline{YXK} = \overline{Y(\overline{XK})} = \overline{Y(H)} = K$. Therefore $\overline{Ran(YX)} = K$.

Now $S(XY) = XTY = (XY)S$, which shows that XY is a quasi-affinity in $\{S\}'$, the commutant of S.

Also, $(YX)T = Y(XT) = YSX = T(YX)$, that is, YX is a quasi-affinity in $\{T\}'$, the commutant of T.

(b) Since $X \in B(H)$ is a quasi-affinity, $KerX = \{0\}, \overline{Ran(X)} = H$. We recall that

$$Ker\ X = Ran\ (X^*) \tag{1}$$

$$Ker\ (X^*) = Ran\ (X) \tag{2}$$

$$\overline{Ran(X)} = Ker\ (X^*)^{\perp} \tag{3}$$

$$\overline{Ran(X^*)} = Ker\ \ (X) \tag{4}$$

Therefore, since $Ker\ X = \{0\}$, we have $Ker\ (X)^{\perp} = H = \overline{Ran(X^*)}$ by (4) which implies that X^* has a dense range. X^* is one-to-one (since $Ker\ (X^*) = 0$). X^* is therefore a quasi-affinity.

Note: The proof of the following Theorem follows from Theorem 2.1.5.

Theorem 2.1.6 [10, Proposition 3.4]: *If A is a quasi-affine transform of B and B is a quasi-affine transform of C, then*
(a) A is a quasi-affine transform of C.
(b) B^* is a quasi-affine transform of A^*.

Proposition 2.1.7[10]: *If X is a quasi-affinity from H to K, then $|X| = \sqrt{X^*X}$ is a quasi-affinity on H (i.e. fromKto H). Moreover, $X|X|^{-1}$ extends by continuity to a unitary transformation U from H to K.*

Lemma 2.1.8 [3, Lemma 2.6]: *Let $X \in B(H, K)$ and $Y \in B(K, L)$ be quasi-affinities where H, K and L are finite dimensional Hilbert spaces. Then the inverse $(YX)^{-1} \in B(L, H)$ of the composite YX exists and $(YX)^{-1} = X^{-1}Y^{-1}$.*

Proof: The operator $YK \in B(L, K)$ is bijective, so that YX exists. We thus have $(YX)(YX)^{-1} = I_L$ is the identity operator on L. Applying Y^{-1} and using $Y^{-1}Y = I_K$, we obtain $Y^{-1}YX(YX)^{-1} = X(YX)^{-1} = Y^{-1}I_L = Y^{-1}$. Applying X^{-1} and using $X^{-1}X = I_H$ we obtain $X^{-1}X(YX)^{-1} = (YX)^{-1} = X^{-1}Y^{-1}$.

Proposition 2.1.9 [10, Proposition 3.4]: *If a unitary operator A on a Hilbert space H is the quasi-affine transform of a unitary operator B on a Hilbert space K then A and B are unitarily equivalent.*

Proof: Let $X \in B(H, K)$ be a quasi-affinity. Then

$$XA = BX \tag{1}$$

implies that $X = B^{-1}X = XA^{-1} = XA^*$ (2)

From (1) and (2) we obtain
$X|^2A = X^*XA = X^*BX = AX^*X = A|X|^2$ and by iteration $|X|^{2n}A = A|X|^{2n}(n = 0,1,\dots)$; hence $p(|X|^2)A = A\,p(|X|^2)$ for every polynomial $p(x)$. Let $\{p_n(\mathrm{x})\}$ be a sequence of polynomials tending to $|X|^{\frac{1}{2}}$ uniformly on the interval $0\leq x \leq \| X \|^{\frac{1}{2}}$. Then $p_n(|X|^2)$ converges (in the operator norm) to $|X|$ so that we obtain a limit relation
$|X|A = A|X|$ (3)

From (1) and (3) it follows that $BU|X| = BX = XA = U|X|A = UA|X|$; because $|X|H$ is dense in H, it results that $BU = UA$. By Proposition 2.1.3 above U is unitary and hence A and B are unitarily equivalent.

Theorem 2.1.10: *Quasi-similarity is an equivalence relation on the class of all operators.*

Proof: Let $A \in B(H), B \in B(K), C \in B(L)$ respectively. First we show $A{\sim}A$.

Then $XA = AX$ and $AY = YA$ where X and Y are quasi-affinities. Choosing $X = Y = I$ (without loss of generality) we have that $A{\sim}A$. This proves reflexivity.

Now suppose that $A{\sim}B$. We show that $B{\sim}A$. Since $A{\sim}B$ there exist quasi-affinities $X \in B(H,K)$ and $Y \in B(K,H)$ such that $XA = BX$ and $BY = YA$. By symmetry of compositions, it is true that $BX = XA$ and $YA = BY$. Hence $B{\sim}A$. This proves symmetry.

Suppose $A{\sim}B$ and $B{\sim}C$. Then we show that $A{\sim}C$.

There exist quasi- affinities $X \in B(H,K)$, $Y \in B(K,H)$ and $Z \in B(K,L)$, $R \in B(L,K)$ respectively such that
$XA = BX, \quad AY = YB$ (1)
and $ZB = CZ, \quad CR = RB$ (2)

$RZYX$ is a quasi-affinity; it is one-to-one since it is a composition of one-to-one operators.
$RZYXA = RZAYX$, since $YX \in \{A\}'$
$= RZYBX$, since $AY = YB$
$= RBZYX$, since $ZY \in \{B\}'$
$= CRZYX$, since $RB = CR$

Which is clearly a quasi-affinity and $AYXZR = YXAZR$, since $YX \in \{A\}'$
$= YBXZR$, since $XA = BX$
$= YXZBR$, since $XZ \in \{B\}'$
$= YXZRC$, since $ZR \in \{C\}'$.

Therefore $A{\sim}C$ .This proves that quasisimilarity is an equivalence relation.

Theorem 2.1.11: *If $T \in B(H)$ and $S \in B(K)$ are similar operators, then they are quasi-similar.*

Proof: There exist a quasi-invertible operator $X \in B(H,K)$ such that $XT = SX$.

This implies that $X^{-1}S = TX^{-1}$, where $X^{-1} \in B(K,H)$. $\Rightarrow S{\sim}T$.

2.2. RELATIONSHIP BETWEEN UNITARY EQUIVALENCE, QUASISIMILARITY AND ALMOST SIMILARITY

Proposition 2.2.1 [8, Proposition 1.2]: *If $A, B \in B(H)$ such that A and B are unitarily equivalent, then $A \overset{a.s}{\sim} B$.*

Proof: By assumption, there exists a unitary operator U such that $A = U^*BU$ which implies that $A^* = U^*B^*U$. Thus, $A^*A = U^*B^*UU^*BU = U^*B^*BU = U^{-1}B^*BU$, and $A^* + A = U^*B^*U + U^*BU = U^*(B^* + B)U = U^{-1}(B^* + B)U$.

Proposition 2.2.2 [8, Proposition 1.3]: *If $A, B \in B(H)$ such that $A \overset{a.s}{\sim} B$, and if A is hermitian, then A and B are unitarily equivalent.*

Proof: By assumption, there exists an invertible operator N such that $A^* + A = N^{-1}(B^{*+}B)N$. Since A is hermitian and $A \overset{a.s}{\sim} B$ by Proposition 4.1.8 [7],B is hermitian. Thus we have $2A = N^{-1}2BN$ which implies that $A = N^{-1}BN$. This implies that A and B are similar ($i.e.$ $A{\sim}B$) and since both operators are normal (both A and B are hermitian), they are unitarily equivalent.

Remark 2.2.3: *The Proposition 2.2.2 gives a condition under which almost similarity of operators implies similarity.*

Theorem 2.2.4: *If A is a normal operator and $B \in B(H)$ is unitarily equivalent to A, then B is normal.*

Proof: Suppose $B = U^*AU$ where U is unitary and A is normal. Then
$B^*B = (U^*A^*U)(U^*AU) = U^*A^*AU = U^*AA^*U = B\,U^*A^*U = B\,U^*UB^* = BB^*$

which proves the claim.

Corollary 2.2.5: *If $A, B \in B(H)$ are normal where H is an infinite dimensional Hilbert space such that A and B are Quasi-similar, then $A \overset{a.s}{\sim} B$.*

Proof: Since $A, B \in B(H)$ are quasi-similar, there exists quasi-affinities $X \in B(H, K)$ and $Y \in B(K, H)$ such that

$$XA = BX \text{ and } BY = YA \tag{1}$$

X and Y are both invertible and so XY, YX are both invertible. Without loss of generality, let $N = XY$ or YX. Then $XY \in \{A\}'$ and $YX \in \{B\}'$, $i.e.$ $AXY = XYA \Rightarrow A = XYA(XY)^{-1}$
and $YXB = BYX \Rightarrow B = (YX)^{-1}BYX$ (2)

Since XY is invertible, $(XY)^* = Y^*X^*$ and $(XY)^{-1^*} = ((XY)^*)^{-1} = (Y^*X^*)^{-1} = X^{*^{-1}}Y^{*^{-1}}$.

Now, $A^*A = \left(X^{*^{-1}}Y^{*^{-1}}A^*Y^*X^*\right)XYA(XY)^{-1} = \left(X^{*^{-1}}Y^{*^{-1}}Y^*BX^*\right)XBYY^{-1}X^{-1}$
$= \left(X^{*^{-1}}BX^*\right)(XBX^{-1})$.

Since A and B are similar normal operators, they are unitarily equivalent by Proposition 2.2.2 so that

$$A^*A = \left(X^{*^{-1}}BX^*\right)XBX^{-1} = XB^*BX^{-1} \tag{3}$$

Also, $A^* + A = \left(X^{*^{-1}}BX^*\right) + (XBX^{-1}) = XB^*X^{-1} + XBX^{-1} = X(B^* + B)X^{-1}$ (4)

that is,
$A^*A = N^{-1}B^*BN$ and $A^* + A = N^{-1}B^* + BN$ where $N = X^{-1}$ is an invertible operator.

Remark 2.2.6: *Corollary 2.2.5 gives a condition under which similarity implies quasi similarity which in turn implies almost similarity.*

The following Theorem enables us obtain an example of quasi-similar operators:

Theorem 2.2.7[8, Theorem 2.5]: *Suppose that for each α in some index set A, there are Hilbert spaces H_α and K_α and operators $T_\alpha \in B(H_\alpha)$ and $S_\alpha \in B(K_\alpha)$ respectively which are quasi-similar. Let T be the operator $T = \sum_{\alpha\in A} \oplus\, T_\alpha$ acting on the Hilbert space which is the direct sum of the spaces H_α and $S = \sum_{\alpha\in A} \oplus\, S_\alpha \in B(K)$ where $K = \sum_{\alpha\in A} \oplus\, K_\alpha$. Then T is quasi-similar to S.*

Proof: Suppose X_α and Y_α are the quasi-invertible operators such that $X_\alpha T_\alpha = S_\alpha X_\alpha$ and $T_\alpha Y_\alpha = Y_\alpha S_\alpha$. If $X = \sum_{\alpha\in A} \oplus\, X_\alpha / \| X \|$ and $Y = \sum_{\alpha\in A} \oplus\, Y_\alpha / \| Y \|$, then X and Y are the quasi-invertibles and satisfy the desired equations.

Example 2.2.8: *Let A_n and B_n be unilateral shift operators with weights 1 and $\frac{1}{n}$ respectively on n-dimensional Hilbert space H. Then A is the Jordan canonical form for B_n and so A and B_n are similar. If $A = \sum_{n=0}^{\infty} A_n$ and $B = \sum_{n=0}^{\infty} B_n$ then by the above Theorem, A is quasi-similar to B.*

Remark 2.2.9: *Recall that an operator $X \in B(H, K)$ intertwines $A \in B(H)$ to $B \in B(K)$ if $XA = BX$. If A is densely intertwined to B, then there exists an operator with dense range intertwining A to B.*

Potential Conflicts of Interest

The authors declare no conflict of interest.

REFERENCES

1. Hoover T.B. *Quasisimilarity of operators*, Illinois. Math. 16 (1972), pp 678-688.
2. Jibril A.A. *On almost similar Operators*, Arabian J.Sci.Engrg.21 (1996), 443-449.
3. Kryeyszig E. *Introductory Functional Analysis with Applications*, Wiley, New York (1978).
4. Kubrusly C.S. *An Introduction to Models and Decomposition in Operator Theory*, Birkhauser (1997).
5. Kubrusly C.S. *Hilbert Space Operators*, Birkhauser (2003).
6. Lee W.Y. *Lecture Notes on Operator Theory*, Seoul National University, Korea (2008).
7. Musundi S.W, Sitati I.N, Nzimbi B.M, Murwayi A.L. *On Almost Similarity Operator Equivalence Relations*, IJRRAS, Vol.15 Issue 3 (2013), pp 293-299.
8. Nzimbi B.M, Pokhariyal G.P and Khalagai J.M. *A note on Similarity, and Equivalence of Operators*, FJMS, Vol.28 No.2 (2008), pp 305-317.
9. Sitati I.N, Musundi S.W, Nzimbi B.M, Kirimi J. *On Similarity and Quasisimilarity Equivalence Relations*, BSOMASS, Vol.1No.2 (2012), pp 151-171.
10. Sz-Nagy B, Foias C, Bercovivi H and Kerchy L, *Harmonic Analysis of operators on Hilbert Space*, Springer New York Dordrecht Heidelberg London (2010).

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