International Journal of Mathematical Archive-6(6), 2015, 234-238
Available online through www.ijma.info ISSN 2229 – 5046

NOTES ON BIPOLAR-VALUED MULTI FUZZY SUBGROUPS OF A GROUP

V. K. SANTHI
Department of Mathematics,
Srimeenakshi Government Arts College, Madurai – 625 002 Tamilnadu, India.

G. SHYAMALA*
Department of Mathematics,
Arulmigu Palani Andavar College of Arts & Culture, Palani – 622001, Tamilnadu, India.

(Received On: 27-02-15; Revised & Accepted On: 06-05-15)

ABSTRACT

In this paper, we study some of the properties of bipolar-valued multi fuzzy subgroup and prove some results on these.

Key Words: *Bipolar-valued fuzzy subset, bipolar-valued multi fuzzy subset, bipolar-valued multi fuzzy subgroup.*

INTRODUCTION

In 1965, Zadeh [12] introduced the notion of a fuzzy subset of a set, fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of research in different domains, there have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, soft sets etc [5]. Lee [7] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [–1, 1]. In a bipolar-valued fuzzy subset, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree (0, 1] indicates that elements somewhat satisfy the property and the membership degree [–1, 0) indicates that elements somewhat satisfy the implicit counter property. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [7, 8]. We introduce the concept of bipolar-valued multi fuzzy subgroup and established some results.

1. PRELIMINARIES

1.1 Definition: A bipolar-valued fuzzy set (BVFS) A in X is defined as an object of the form $A = \{< x, A^+(x), A^-(x) >/ x\in X\}$, where $A^+ : X\rightarrow [0, 1]$ and $A^- : X\rightarrow [-1, 0]$. The positive membership degree $A^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set A and the negative membership degree $A^-(x)$ denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar-valued fuzzy set A. If $A^+(x) \neq 0$ and $A^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for A and if $A^+(x) = 0$ and $A^-(x) \neq 0$, it is the situation that x does not satisfy the property of A, but somewhat satisfies the counter property of A. It is possible for an element x to be such that $A^+(x) \neq 0$ and $A^-(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X.

1.2 Example: A = {< a, 0.5, –0.3 >, < b, 0.1, –0.7 >, < c, 0.5, –0.4 >} is a bipolar-valued fuzzy subset of X= {a, b, c}.

1.3 Definition: A bipolar-valued multi fuzzy set (BVMFS) A in X is defined as an object of the form $A = \{< x, A_i^+(x), A_i^-(x) >/ x\in X\}$, where $A_i^+ : X\rightarrow [0, 1]$ and $A_i^- : X\rightarrow [-1, 0]$. The positive membership degrees $A_i^+(x)$ denote the satisfaction degree of an element x to the property corresponding to a bipolar-valued multi fuzzy set A and the negative membership degrees $A_i^-(x)$ denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar-valued multi fuzzy set A. If $A_i^+(x) \neq 0$ and $A_i^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for A and if $A_i^+(x) = 0$ and $A_i^-(x) \neq 0$, it is the situation that x does not satisfy the property of A, but somewhat satisfies the counter property of A. It is possible for an element x to be such that $A_i^+(x) \neq 0$ and $A_i^-(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X, where i = 1 to n.

*Corresponding Author: G. Shyamala**

1.4 Example: A = {< a, 0.5, 0, 6, 0.3, –0.3, –0.6, –0.5 >, < b, 0.1, 0.4, 0.7, –0.7, –0.3, –0.6 >, < c, 0.5, 0.3, 0.8, –0.4, –0.5, –0.3 >} is a bipolar-valued multi fuzzy subset of X = {a, b, c}.

1.5 Definition: Let G be a group. A bipolar-valued multi fuzzy subset A of G is said to be a bipolar-valued multi fuzzy subgroup of G (BVMFSG) if the following conditions are satisfied

(i) $A_i^+(xy) \geq \min\{A_i^+(x), A_i^+(y)\}$
(ii) $A_i^+(x^{-1}) \geq A_i^+(x)$
(iii) $A_i^-(xy) \leq \max\{A_i^-(x), A_i^-(y)\}$
(iv) $A_i^-(x^{-1}) \leq A_i^-(x)$ for all x and y in G.

1.6 Example: Let G = {1, –1, i, –i} be a group with respect to the ordinary multiplication. Then A = {< 1, 0.5, 0.6, 0.4, –0.6, –0.5, –0.3 >, < –1, 0.4, 0.5, 0.3, –0.5, –0.4, –0.2 >, < i, 0.2, 0.3, 0.2, –0.4, –0.3, –0.1 >, < –i, 0.2, 0.3, 0.2, –0.4, –0.3, –0.1 >} is a bipolar-valued multi fuzzy subgroup of G.

1.7 Definition: Let $A = \langle A_i^+, A_i^- \rangle$ and $B = \langle B_i^+, B_i^- \rangle$ be any two bipolar-valued multi fuzzy subsets of sets G and H, respectively. The product of A and B, denoted by A×B, is defined as $A\times B = \{\langle (x, y), (A_i\times B_i)^+(x, y), (A_i\times B_i)^-(x, y)\rangle /$ for all x in G and y in H} where $(A_i\times B_i)^+(x, y) = \min\{A_i^+(x), B_i^+(y)\}$ and $(A_i\times B_i)^-(x, y) = \max\{A_i^-(x), B_i^-(y)\}$ for all x in G and y in H.

1.8 Definition: Let $A = \langle A_i^+, A_i^- \rangle$ be a bipolar-valued multi fuzzy subset in a set S, the strongest bipolar-valued multi fuzzy relation on S, that is a bipolar-valued multi fuzzy relation on A is $V = \{\langle (x, y), V_i^+(x, y), V_i^-(x, y)\rangle$ / x and y in S} given by $V_i^+(x, y) = \min\{A_i^+(x), A_i^+(y)\}$ and $V_i^-(x, y) = \max\{A_i^-(x), A_i^-(y)\}$ for all x and y in S.

2. PROPERTIES

2.1 Theorem: Let $A = \langle A_i^+, A_i^- \rangle$ be a bipolar-valued multi fuzzy subgroup of G. Then $A_i^+(x^{-1}) = A_i^+(x)$ and $A_i^-(x^{-1}) = A_i^-(x)$, $A_i^+(x) \leq A_i^+(e)$ and $A_i^-(x) \geq A_i^-(e)$ for all x in G and the identity element e in G.

Proof: Let x be in G. Now $A_i^+(x) = A_i^+((x^{-1})^{-1}) \geq A_i^+(x^{-1}) \geq A_i^+(x)$. Therefore $A_i^+(x) = A_i^+(x^{-1})$ for all x in G. And $A_i^-(x) = A_i^-((x^{-1})^{-1}) \leq A_i^-(x^{-1}) \leq A_i^-(x)$. Therefore $A_i^-(x^{-1}) = A_i^-(x)$ for all x in G.

Now $A_i^+(e) = A_i^+(xx^{-1}) \geq \min\{A_i^+(x), A_i^+(x^{-1})\} = A_i^+(x)$. Therefore $A_i^+(e) \geq A_i^+(x)$ for all x in G. And $A_i^-(e) = A_i^-(xx^{-1}) \leq \max\{A_i^-(x), A_i^-(x^{-1})\} = A_i^-(x)$. Therefore $A_i^-(e) \leq A_i^-(x)$ for all x in G.

2.2 Theorem: Let $A = \langle A_i^+, A_i^- \rangle$ be a bipolar-valued multi fuzzy subgroup of G. Then

(i) $A_i^+(xy^{-1}) = A_i^+(e)$ implies that $A_i^+(x) = A_i^+(y)$ for x and y in G.
(ii) $A_i^-(xy^{-1}) = A_i^-(e)$ implies that $A_i^-(x) = A_i^-(y)$ for x and y in G.

Proof: Now $A_i^+(x) = A_i^+(xy^{-1}y) \geq \min\{A_i^+(xy^{-1}), A_i^+(y)\} = \min\{A_i^+(e), A_i^+(y)\} = A_i^+(y) = A_i^+(yx^{-1}x) \geq \min\{A_i^+(yx^{-1}), A_i^+(x)\} = \min\{A_i^+(e), A_i^+(x)\} = A_i^+(x)$. Therefore $A_i^+(x) = A_i^+(y)$ for x and y in G. And $A_i^-(x) = A_i^-(xy^{-1}y) \leq \max\{A_i^-(xy^{-1}), A_i^-(y)\} = \max\{A_i^-(e), A_i^-(y)\} = A_i^-(y) = A_i^-(yx^{-1}x) \leq \max\{A_i^-(yx^{-1}), A_i^-(x)\} = \max\{A_i^-(e), A_i^-(x)\} = A_i^-(x)$. Therefore $A_i^-(x) = A_i^-(y)$ for x and y in G.

2.3 Theorem: Let $A = \langle A_i^+, A_i^- \rangle$ be a bipolar-valued multi fuzzy subgroup of a group G.

(i) If $A_i^+(xy^{-1}) = 1$, then $A_i^+(x) = A_i^+(y)$ for x and y in G.
(ii) If $A_i^-(xy^{-1}) = -1$, then $A_i^-(x) = A_i^-(y)$ for x and y in G.

Proof: Now $A_i^+(x) = A_i^+(xy^{-1}y) \geq \min\{A_i^+(xy^{-1}), A_i^+(y)\} = \min\{1, A_i^+(y)\} = A_i^+(y) = A_i^+(y^{-1}) = A_i^+(x^{-1}xy^{-1}) \geq \min\{A_i^+(x^{-1}), A_i^+(xy^{-1})\} = \min\{A_i^+(x^{-1}), 1\} = A_i^+(x^{-1}) = A_i^+(x)$. Therefore $A_i^+(x) = A_i^+(y)$ for x and y in G. Hence (i) is proved. Also $A_i^-(x) = A_i^-(xy^{-1}y) \leq \max\{A_i^-(xy^{-1}), A_i^-(y)\} = \max\{-1, A_i^-(y)\} = A_i^-(y) = A_i^-(y^{-1}) = A^-(x^{-1}xy^{-1}) \leq \max\{A_i^-(x^{-1}), A_i^-(xy^{-1})\} = \max\{A_i^-(x^{-1}), -1\} = A^-(x^{-1}) = A_i^-(x)$. Therefore $A_i^-(x) = A_i^-(y)$ for x and y in G. Hence (ii) is proved.

2.4 Theorem: Let $A = \langle A_i^+, A_i^- \rangle$ be a bipolar-valued multi fuzzy subgroup of a group G.

(i) If $A_i^+(xy^{-1}) = 0$, then either $A_i^+(x) = 0$ or $A_i^+(y) = 0$ for x and y in G.
(ii) If $A_i^-(xy^{-1}) = 0$, then either $A_i^-(x) = 0$ or $A_i^-(y) = 0$ for x and y in G.

Proof: Let x and y in G.

(i) By the definition $A_i^+(xy^{-1}) \geq \min\{A_i^+(x), A_i^+(y)\}$ which implies that $0 \geq \min\{A_i^+(x), A_i^+(y)\}$. Therefore either $A_i^+(x) = 0$ or $A_i^+(y) = 0$.

(ii) By the definition $A_i^-(xy^{-1}) \le \max\{A_i^-(x), A_i^-(y)\}$ which implies that $0 \le \max\{A_i^-(x), A_i^-(y)\}$. Therefore either $A_i^-(x) = 0$ or $A_i^-(y) = 0$.

2.5 Theorem: If $A = \langle A_i^+, A_i^-\rangle$ be a bipolar-valued multi fuzzy subgroup of G, then

(i) $A_i^+(xy) = A_i^+(yx)$ if and only if $A_i^+(x) = A_i^+(y^{-1}xy)$ for x and y in G.

(ii) $A_i^-(xy) = A_i^-(yx)$ if and only if $A_i^-(x) = A_i^-(y^{-1}xy)$ for x and y in G.

Proof: Let x and y be in G. Assume that $A_i^+(xy) = A_i^+(yx)$. So, $A_i^+(y^{-1}xy) = A_i^+(y^{-1}yx) = A_i^+(ex) = A_i^+(x)$. Therefore $A_i^+(x) = A_i^+(y^{-1}xy)$ for x and y in G. Conversely assume that $A_i^+(x) = A_i^+(y^{-1}xy)$. We get $A_i^+(xy) = A_i^+(xyxx^{-1}) = A_i^+(yx)$. Therefore $A_i^+(xy) = A_i^+(xy)$ for x and y in G. Hence $A_i^+(xy) = A_i^+(yx)$ if and only if $A_i^+(x) = A_i^+(y^{-1}xy)$ for x and y in G. Also assume that $A_i^-(xy) = A_i^-(yx)$. We get $A_i^-(y^{-1}xy) = A_i^-(y^{-1}yx) = A_i^-(ex) = A_i^-(x)$. Therefore $A_i^-(x) = A_i^-(y^{-1}xy)$ for x and y in G. Conversely assume that $A_i^-(x) = A_i^-(y^{-1}xy)$. So $A_i^-(xy) = A_i^-(xyxx^{-1}) = A_i^-(yx)$. Therefore $A_i^-(xy) = A_i^-(xy)$ for x and y in G. Hence $A_i^-(xy) = A_i^-(yx)$ if and only if $A_i^-(x) = A_i^-(y^{-1}xy)$ for x and y in G.

2.6 Theorem: If $A = \langle A_i^+, A_i^-\rangle$ is a bipolar-valued multi fuzzy subgroup of a group G, then $H = \{x\in G \mid A_i^+(x) = 1, A_i^-(x) = -1\}$ is either empty or a subgroup of G.

Proof: If no element satisfies this condition, then H is empty. If x and y in H, then $A_i^+(xy^{-1}) \ge \min\{A_i^+(x), A_i^+(y)\} = \min\{1, 1\} = 1$. Therefore $A_i^+(xy^{-1}) = 1$. And $A_i^-(xy^{-1}) \le \max\{A_i^-(x), A_i^-(y)\} = \max\{-1, -1\} = -1$. Therefore $A_i^-(xy^{-1}) = -1$. That is $xy^{-1} \in H$. Hence H is a subgroup of G. Hence H is either empty or a subgroup of G.

2.7 Theorem: If $A = \langle A_i^+, A_i^-\rangle$ is a bipolar-valued multi fuzzy subgroup of G, then $H = \{x\in G \mid A_i^+(x) = A_i^+(e)$ and $A_i^-(x) = A_i^-(e)\}$ is a subgroup of G.

Proof: Here $H = \{x\in G \mid A_i^+(x) = A_i^+(e)$ and $A_i^-(x) = A_i^-(e)\}$ by Theorem 2.1, $A_i^+(x^{-1}) = A_i^+(x) = A_i^+(e)$ and $A_i^-(x^{-1}) = A_i^-(x) = A_i^-(e)$. Therefore $x^{-1}\in H$. Now $A_i^+(xy^{-1}) \ge \min\{A_i^+(x), A_i^+(y)\} = \min\{A_i^+(e), A_i^+(e)\} = A_i^+(e)$ and $A_i^+(e) = A_i^+((xy^{-1})(xy^{-1})^{-1}) \ge \min\{A_i^+(xy^{-1}), A_i^+(xy^{-1})\} = A_i^+(xy^{-1})$. Hence $A_i^+(e) = A_i^+(xy^{-1})$. Also $A_i^-(xy^{-1}) \le \max\{A_i^-(x), A_i^-(y)\} = \max\{A_i^-(e), A_i^-(e)\} = A_i^-(e)$ and $A_i^-(e) = A_i^-((xy^{-1})(xy^{-1})^{-1}) \le \max\{A_i^-(xy^{-1}), A_i^-(xy^{-1})\} = A_i^-(xy^{-1})$. Therefore $A_i^-(e) = A_i^-(xy^{-1})$. Hence $A_i^+(e) = A_i^+(xy^{-1})$ and $A_i^-(e) = A_i^-(xy^{-1})$. Therefore $xy^{-1}\in H$. Hence H is a subgroup of G.

2.8 Theorem: Let G be a group. If $A = \langle A_i^+, A_i^-\rangle$ is a bipolar-valued multi fuzzy subgroup of G, then $A_i^+(xy) = \min\{A_i^+(x), A_i^+(y)\}$ and $A_i^+(xy) = \max\{A_i^-(x), A_i^-(y)\}$ for each x andy in G with $A_i^+(x) \ne A_i^+(y)$ and $A_i^-(x) \ne A_i^-(y)$.

Proof: Assume that $A_i^+(x) > A_i^+(y)$ and $A_i^-(x) < A_i^-(y)$. Then $A_i^+(y) = A_i^+(x^{-1}xy) \ge \min\{A_i^+(x^{-1}), A_i^+(xy)\} = \min\{A_i^+(x), A_i^+(xy)\} = A_i^+(xy) \ge \min\{A_i^+(x), A_i^+(y)\} = A_i^+(y)$. Therefore $A_i^+(xy) = A_i^+(y) = \min\{A_i^+(x), A_i^+(y)\}$. And $A_i^-(y) = A_i^-(x^{-1}xy) \le \max\{A_i^-(x^{-1}), A_i^-(xy)\} = \max\{A_i^-(x), A_i^-(xy)\} = A_i^-(xy) \le \max\{A_i^-(x), A_i^-(y)\} = A_i^-(y)$. Therefore $A_i^-(xy) = A_i^-(y) = \max\{A_i^-(x), A_i^-(y)\}$.

2.9 Theorem: If $A = \langle A_i^+, A_i^-\rangle$ and $B = \langle B_i^+, B_i^-\rangle$ are two bipolar-valued multi fuzzy subgroups of a group G, then their intersection $A\cap B$ is a bipolar-valued multi fuzzy subgroup of G.

Proof: Let $A = \{< x, A_i^+(x), A_i^-(x) > / x\in G\}$, $B = \{< x, B_i^+(x), B_i^-(x) > / x\in G\}$. Let $C = A\cap B$ and $C = \{< x, C_i^+(x), C_i^-(x) > / x\in G\}$. Now $C_i^+(xy^{-1}) = \min\{A_i^+(xy^{-1}), B_i^+(xy^{-1})\} \ge \min\{\min\{A_i^+(x), A_i^+(y)\}, \min\{B_i^+(x), B_i^+(y)\}\} \ge \min\{\min\{A_i^+(x), B_i^+(x)\}, \min\{A_i^+(y), B_i^+(y)\}\} = \min\{C_i^+(x), C_i^+(y)\}$. Therefore $C_i^+(xy^{-1}) \ge \min\{C_i^+(x), C_i^+(y)\}$. Also $C_i^-(xy^{-1}) = \max\{A_i^-(xy^{-1}), B_i^-(xy^{-1})\} \le \max\{\max\{A_i^-(x), A_i^-(y)\}, \max\{B_i^-(x), B_i^-(y)\}\} \le \max\{\max\{A_i^-(x), B_i^-(x)\}, \max\{A_i^-(y), B_i^-(y)\}\} = \max\{C_i^-(x), C_i^-(y)\}$. Therefore $C_i^-(xy^{-1}) \le \max\{C_i^-(x), C_i^-(y)\}$. Hence $A\cap B$ is a bipolar-valued multi fuzzy subgroup of G.

2.10 Theorem: The intersection of a family of bipolar-valued multi fuzzy subgroups of a group G is a bipolar-valued multi fuzzy subgroup of G.

Proof: The Theorem is true by Theorem 2.9.

2.11 Theorem: If $A = \langle A_i^+, A_i^-\rangle$ and $B = \langle B_i^+, B_i^-\rangle$ are any two bipolar-valued multi fuzzy subgroups of the groups G_1 and G_2 respectively, then $A\times B = \langle (A_i\times B_i)^+, (A_i\times B_i)^-\rangle$ is a bipolar-valued multi fuzzy subgroup of $G_1\times G_2$.

Proof: Let A and B be two bipolar-valued multi fuzzy subgroups of the groups G_1 and G_2 respectively. Let x_1 and x_2 be in G_1, y_1 and y_2 be in G_2. Then (x_1, y_1) and (x_2, y_2) are in $G_1\times G_2$. Now, $(A_i\times B_i)^+[(x_1, y_1)(x_2, y_2)^{-1}] = (A_i\times B_i)^+(x_1x_2^{-1}, y_1y_2^{-1}) = \min\{A_i^+(x_1x_2^{-1}), B_i^+(y_1y_2^{-1})\} \ge \min\{\min\{A_i^+(x_1), A_i^+(x_2)\}, \min\{B_i^+(y_1), B_i^+(y_2)\}\} = \min\{\min\{A_i^+(x_1), B_i^+(y_1)\},$

min{ $A_i^+(x_2)$, $B_i^+(y_2)$}} = min {$(A_i\times B_i)^+(x_1, y_1)$, $(A_i\times B_i)^+(x_2, y_2)$}. Therefore $(A_i\times B_i)^+$ $[(x_1, y_1)(x_2, y_2)^{-1}]$ ≥ min {$(A_i\times B_i)^+(x_1, y_1)$, $(A_i\times B_i)^+(x_2, y_2)$}. Also $(A_i\times B_i)^-[(x_1, y_1)(x_2, y_2)^{-1}] = (A_i\times B_i)^-(x_1x_2^{-1}, y_1y_2^{-1})$ = max {$A_i^-(x_1x_2^{-1})$, $B_i^-(y_1y_2^{-1})$ } ≤ max {max {$A_i^-(x_1)$, $A_i^-(x_2)$}, max {$B_i^-(y_1)$, $B_i^-(y_2)$ }} = max {max {$A_i^-(x_1)$, $B_i^-(y_1)$}, max {$A_i^-(x_2)$, $B_i^-(y_2)$}} = max {$(A_i\times B_i)^-(x_1, y_1)$, $(A_i\times B_i)^-(x_2, y_2)$}. Therefore $(A_i\times B_i)^-[(x_1, y_1)(x_2, y_2)^{-1}]$ ≤ max {$(A_i\times B_i)^-(x_1, y_1)$, $(A_i\times B_i)^-(x_2, y_2)$}. Hence A×B is a bipolar-valued multi fuzzy subgroup of $G_1\times G_2$.

2.12 Theorem: Let A = ⟨A_i^+, A_i^-⟩ and B = ⟨B_i^+, B_i^-⟩ be any two bipolar-valued multi fuzzy subsets of the groups G and H respectively. Suppose that e and e^l are the identity elements of G and H respectively. If A×B is a bipolar-valued multi fuzzy subgroup of G×H, then at least one of the following two statements must hold.

(i) $B_i^+(e^l) \geq A_i^+(x)$ for all x in G and $B_i^-(e^l) \leq A_i^-(x)$ for all x in G

(ii) $A_i^+(e) \geq B_i^+(y)$ for all y in H and $A_i^-(e) \leq B_i^-(y)$ for all y in H.

Proof: Let A×B is a bipolar-valued multi fuzzy subgroup of G×H. By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find a in G and b in H such that $A_i^+(a) > B_i^+(e^l)$, $A_i^-(a) < B_i^-(e^l)$ and $B_i^+(b) > A_i^+(e)$, $B_i^-(b) < A_i^-(e)$. We have $(A_i\times B_i)^+(a, b)$ = min {$A_i^+(a)$, $B_i^+(b)$ } > min {$A_i^+(e)$, $B_i^+(e^l)$} = $(A_i\times B_i)^+(e, e^l)$. Also $(A_i\times B_i)^-(a, b)$ = max {$A_i^-(a)$, $B_i^-(b)$ }< max {$A_i^-(e)$, $B_i^-(e^l)$} = $(A_i\times B_i)^-(e, e^l)$. Thus A×B is not a bipolar-valued multi fuzzy subgroup of G×H. Hence either $B_i^+(e^l) \geq A_i^+(x)$ for all x in G and $B_i^-(e^l) \leq A_i^-(x)$ for all x in G or $A_i^+(e) \geq B_i^+(y)$ for all y in H and $A_i^-(e) \leq B_i^-(y)$ for all y in H.

2.13 Theorem: Let A= ⟨A_i^+, A_i^-⟩ and B = ⟨B_i^+, B_i^-⟩ be any two bipolar-valued multi fuzzy subsets of the groups G and H, respectively and A×B is a bipolar-valued multi fuzzy subgroup of G×H. Then the following are true:

(i) If $A_i^+(x) \leq B_i^+(e^l)$ for all x in G and $A_i^-(x) \geq B_i^-(e^l)$ for all x in G, then A is a bipolar-valued multi fuzzy subgroup of G where e^l is identity element of H.

(ii) If $B_i^+(x) \leq A_i^+(e)$ for all x in H and $B_i^-(x) \geq A_i^-(e)$ for all x in H, then B is a bipolar-valued multi fuzzy subgroup of H where e is identity element of G.

(iii) either A is a bipolar-valued multi fuzzy subgroup of G or B is a bipolar-valued multi fuzzy subgroup of H where e and e^l are the identity elements of G and H respectively.

Proof: Let A×B be a bipolar-valued multi fuzzy subgroup of G×H and x and y in G. Then (x, e^l) and (y, e^l) are in G×H. Now using the property if $A_i^+(x) \leq B_i^+(e^l)$ for all x in G and $A_i^-(x) \geq B_i^-(e^l)$ for all x in G where e^l is identity element of H we get, $A_i^+(xy^{-1})$ = min{$A_i^+(xy^{-1})$, $B_i^+(e^l e^l)$}= $(A_i\times B_i)^+$ $((xy^{-1}), (e^le^l))$ = $(A_i\times B_i)^+$ $[(x, e^l) (y^{-1}, e^l)]$ ≥ min{$(A_i\times B_i)^+(x, e^l)$, $(A_i\times B_i)^+(y^{-1}, e^l)$} = min{min{$A_i^+(x)$, $B_i^+(e^l)$}, min{$A_i^+(y^{-1})$, $B_i^+(e^l)$}}= min{$A_i^+(x)$, $A_i^+(y^{-1})$}≥min{$A_i^+(x)$, $A_i^+(y)$}. Therefore $A_i^+(xy^{-1})$ ≥ min {$A_i^+(x)$, $A_i^+(y)$} for all x and y in G. Also $A_i^-(xy^{-1})$ = max{$A_i^-(xy^{-1})$, $B_i^-(e^l e^l)$} = $(A_i\times B_i)^-((xy^{-1}), (e^le^l))$ = $(A_i\times B_i)^-[(x, e^l)(y^{-1}, e^l)]$ ≤ max{$(A_i\times B_i)^-(x, e^l)$, $(A_i\times B_i)^-(y^{-1}, e^l)$} = max {$A_i^-(x)$, $B_i^-(e^l)$}, max {$A_i^-(y^{-1})$, $B_i^-(e^l)$}}= max {$A_i^-(x)$, $A_i^-(y^{-1})$} ≤ max {$A_i^-(x)$, $A_i^-(y)$}. Therefore $A_i^-(xy^{-1})$ ≤ max {$A_i^-(x)$, $A_i^-(y)$} for all x and y in G. Hence A is a bipolar-valued multi fuzzy subgroup of G. Thus (i) is proved. Now using the property $B_i^+(x) \leq A_i^+(e)$ for all x in H and $B_i^-(x) \geq A_i^-(e)$ for all x in H we get, $B_i^+(xy^{-1})$ = min {$B_i^+(xy^{-1})$, $A_i^+(e.e)$}= $(A_i\times B_i)^+$ $((e.e), (xy^{-1}))$ = $(A_i\times B_i)^+[(e, x)(e, y^{-1})]$ ≥ min {$(A_i\times B_i)^+(e, x)$, $(A_i\times B_i)^+(e, y^{-1})$} = min {min {$A_i^+(e)$, $B_i^+(x)$}, min {$A_i^+(e)$, $B_i^+(y^{-1})$}}= min {$B_i^+(x)$, $B_i^+(y^{-1})$} ≥ min {$B_i^+(x)$, $B_i^+(y)$}. Therefore $B_i^+(xy^{-1})$ ≥ min {$B_i^+(x)$, $B_i^+(y)$} for all x and y in H. Also $B_i^-(xy^{-1})$ = max {$B_i^-(xy^{-1})$, $A_i^-(ee)$} = $(A_i\times B_i)^-((ee), (xy^{-1}))$ = $(A_i\times B_i)^-$ $[(e, x)(e, y^{-1})]$ ≤ max {$(A_i\times B_i)^-(e, x)$, $(A_i\times B_i)^-(e, y^{-1})$} = max {max{$A_i^-(e)$, $B_i^-(x)$}, max{$A_i^-(e)$, $B_i^-(y^{-1})$}} = max{$B_i^-(x)$, $B_i^-(y^{-1})$} ≤ max{$B_i^-(x)$, $B_i^-(y)$}. Therefore $B_i^-(xy^{-1})$ ≤ max {$B_i^-(x)$, $B_i^-(y)$ } for all x and y in H. Hence B is a bipolar-valued multi fuzzy subgroup of H. Thus (ii) is proved. Hence (iii) is clear.

2.14 Theorem: Let A = ⟨A_i^+, A_i^-⟩ be a bipolar-valued multi fuzzy subset of a group (G, .) and V = ⟨V_i^+, V_i^-⟩ be the strongest bipolar-valued multi fuzzy relation of G. Then A is a bipolar-valued multi fuzzy subgroup of G if and only if V is a bipolar-valued multi fuzzy subgroup of G×G.

Proof: Suppose that A is a bipolar-valued multi fuzzy subgroup of G. Then for any x = (x_1, x_2) and y = (y_1, y_2) are in G×G. We have V_i^+ (xy^{-1}) = V_i^+ $[(x_1, x_2)(y_1, y_2)^{-1}]$ = V_i^+ $(x_1y_1^{-1}, x_2y_2^{-1})$ = min{A_i^+ $(x_1y_1^{-1})$, A_i^+ $(x_2y_2^{-1})$} ≥ min {min{A_i^+ (x_1), A_i^+ (y_1)}, min{A_i^+ (x_2), $A_i^+(y_2)$}} = min{min{A_i^+ (x_1), A_i^+ (x_2)}, min{A_i^+ (y_1), A_i^+ (y_2)}}= min{$V_i^+(x_1, x_2)$, $V_i^+(y_1, y_2)$}= min {$V_i^+(x)$, $V_i^+(y)$}. Therefore $V_i^+(xy^{-1})$ ≥ min {$V_i^+(x)$, $V_i^+(y)$} for all x and y in G×G. Also we have $V_i^-(xy^{-1})$= $V_i^-[(x_1, x_2)(y_1, y_2)^{-1}]$ = $V_i^-(x_1y_1^{-1}, x_2y_2^{-1})$ = max {$A_i^-(x_1y_1^{-1})$, $A_i^-(x_2y_2^{-1})$ }≤ max {max {$A_i^-(x_1)$, $A_i^-(y_1)$}, max {$A_i^-(x_2)$, $A_i^-(y_2)$ }} = max{max {$A_i^-(x_1)$, $A_i^-(x_2)$ }, max {$A_i^-(y_1)$, $A_i^-(y_2)$ }} = max {$V_i^-(x_1, x_2)$, $V_i^-(y_1, y_2)$} = max{$V_i^-(x)$, $V_i^-(y)$}. Therefore $V_i^-(xy^{-1})$ ≤ max{$V_i^-(x)$, $V_i^-(y)$} for all x and y in G×G. This proves that V is a bipolar-valued multi fuzzy subgroup of G×G. Conversely, assume that V is a bipolar-valued multi fuzzy subgroup of G×G, then for any x = (x_1, x_2) and y = (y_1, y_2) are in G×G, we have min {$A_i^+(x_1y_1^{-1})$, $A_i^+(x_2y_2^{-1})$} = $V_i^+(x_1y_1^{-1}, x_2y_2^{-1})$ = $V_i^+[(x_1, x_2)(y_1, y_2)^{-1}]$ = $V_i^+(xy^{-1})$ ≥ min{$V_i^+(x)$, $V_i^+(y)$}= min{$V_i^+(x_1, x_2)$, $V_i^+(y_1, y_2)$}= min{min{$A_i^+(x_1)$, $A_i^+(x_2)$}, min{$A_i^+(y_1)$, $A_i^+(y_2)$}}. If we put $x_2 = y_2 = e$, we get, $A_i^+(x_1y_1^{-1})$ ≥ min {$A_i^+(x_1)$, $A_i^+(y_1)$} for all x_1 and y_1 in G. Also we have max {$A_i^-(x_1y_1^{-1})$, $A_i^-(x_2y_2^{-1})$} = V_i^- $(x_1y_1^{-1}, x_2y_2^{-1})$ = $V_i^-[(x_1, x_2)(y_1, y_2)^{-1}]$ = $V_i^-(xy^{-1})$ ≤ max {$V_i^-(x)$, $V_i^-(y)$}

$= \max \{V_i^-(x_1, x_2), V_i^-(y_1, y_2)\} = \max \{ \max \{A_i^-(x_1), A_i^-(x_2) \}, \max \{ A_i^-(y_1), A_i^-(y_2) \}\}$. If we put $x_2 = y_2 = e$, we get $A_i^-(x_1y_1^{-1}) \le \max \{A_i^-(x_1), A_i^-(y_1)\}$ for all x_1 and y_1 in G. Hence A is a bipolar-valued multi fuzzy subgroup of G.

REFERENCES

1. Anthony.J.M. and Sherwood.H (1979), Fuzzy groups Redefined, Journal of mathematical analysis and applications, 69,124 -130.
2. Arsham Borumand Saeid (2009), Bipolar-valued fuzzy BCK/BCI-algebras, World Applied Sciences Journal 7 (11): 1404-1411.
3. Azriel Rosenfeld (1971), Fuzzy groups, Journal of mathematical analysis and applications 35, 512-517.
4. Choudhury.F.P. and Chakraborty.A.B. and Khare.S.S., (1988) A note on fuzzy subgroups and fuzzy homomorphism, Journal of mathematical analysis and applications, 131, 537 -553.
5. Gau, W.L. and D.J. Buehrer(1993), Vague sets, IEEE Transactons on Systems, Man and Cybernetics, 23: 610-614.
6. Kyoung Ja Lee (2009), Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras, Bull. Malays.Math. Sci. Soc. (2) 32(3), 361–373.
7. Lee, K.M. (2000), Bipolar-valued fuzzy sets and their operations. Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand, pp: 307-312.
8. Lee, K.M. (2004), Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets and bipolar valued fuzzy sets. J. Multi fuzzy Logic Intelligent Systems, 14 (2): 125-129.
9. Mustafa Akgul (1988), some properties of fuzzy groups, Journal of mathematical analysis and applications, 133, 93 -100.
10. Samit Kumar Majumder (2012), Bipolar Valued fuzzy Sets in Γ-Semigroups, Mathematica Aeterna, Vol. 2, no. 3, 203 – 213.
11. Young Bae Jun and Seok Zun Song (2008), Subalgebras and closed ideals of BCH-algebras based on bipolar-valued fuzzy sets, Scientiae Mathematicae Japonicae Online, 427-437.
12. Zadeh, L.A. (1965), Fuzzy sets, Inform. And Control, 8: 338-353.

Source of support: Nil, Conflict of interest: None Declared