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ON THE DEGREE OF APPROXIMATION OF FUNCTION BELONGING TO THE LIPSCHITZ CLASS BY (E, q) (C, δ) PRODUCT MEANS OF ITS CONJUGATE FOURIER SERIES

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ABSTRACT

In this paper a theorem on the degree of Approximation of function belonging to the Lipschitz Class by (E, q) (C, δ) Product Means of its Conjugate Fourier Series have been established.

Keywords: *Cesâro matrix, Euler matrix, Degree of Approximation.*

1. DEFINITION AND NOTATIONS

Let f be 2π - periodic and L-integrable over $[-\pi, \pi]$. The Fourier series of f at a point is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.1}$$

The conjugate series of the Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) \tag{1.2}$$

A function $f \in \mathrm{Lip}\,\alpha$ $(0 < \alpha \le 1)$ if

$$f(x+t) - f(x) = \left(|t|^{\alpha}\right). \tag{1.3}$$

The degree of Approximation of a function $f : R \to R$ by a trigonometric polynomial t_n of order n is defined by zygmund [1, p-114],

$$\|t_n - f\| = \sup\{|t_n(x) - f(x)| : x \in R\} \tag{1.4}$$

Let $\sum_{n=0}^{\infty} a_n$ be given infinite series with the sequence (s_n) of partial sums of its first (n+1) terms. The Euler means of the sequence (s_n) are defined by

$$(E,q) = E_n^q = (q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}S_k, (q \ge 0),$$

Where E_n^0 is defined to be s_n. If $t_n \to s$: as $n \to \infty$, we say that (S_n) or $\sum_{n=0}^{\infty} a_n$ is summable $(E,q)(q>0)$ to S or symbolically we write $(S_n) \in S(E,q)$,for q>0 see Hardy [2,p-180] and for real and complex values of $q \ne -1$,see Chandra [4].

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The sequence (S_n) is said to be summable (C, δ) $(\delta > -1)$ to limit S if, $\left(A_n^{\delta}\right)^{-1} \sum_{k=0}^{n} A_{n-k}^{\delta-1} S_k \to S \; as \, n \to \infty$

Where A_n^{δ} are the binomial coefficients. See Zygmund [1, p-76]

The (E, q) transform of the (C, δ) transform defines the (E, q) (C, δ) transform of the partial sums S_n of the series $\sum_{n=0}^{\infty} a_n$.

The Transform (E, q) (C, δ) reduces to (E, q) and (C, δ) respectively for δ = 0 and q = 0. Thus if

$$(E_q c_\delta)_n = (1+q)^{-n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} (A_v^{\delta})^{-1} \sum_{k=0}^{n} A_{v-k}^{\delta-1} S_k \to S \text{ as } n \to \infty .$$

Then the series $\sum_{n=0}^{\infty} a_n$ is said to be summable by (E, q) (C, δ) means or simply summable (E, d) (C, δ) to S.

Let S_n (f; x) be the n^{th} partial sum of the series (1.1).Then (E, q) (C, δ) means of (S_n (f; x)), Where q > 0 and δ >-1, is given by

$$(E_q c_\delta)_n (f; x) = (1+q)^{-n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} (A_v^{\delta})^{-1} \sum_{k=0}^{n} A_{v-k}^{\delta-1} S_k (f; x) \tag{1.5}$$

We shall use the following notations for each $x \in R$.

$$\psi(t) = f(x+t) - f(x-t) \tag{1.6}$$

$$\bar{K}_n(t) = \frac{1}{2\pi(1+q)^n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} (A_v^{\delta})^{-1} \sum_{k=0}^{n} A_{v-k}^{\delta-1} \frac{\text{Sin}(2k+1)(t/2)}{2\sin(t/2)} \tag{1.7}$$

$$\tilde{f}(x) = -\frac{1}{4\pi} \int_0^{\pi} \psi(t) \tag{1.8}$$

MAIN THEOREM

Theorem 2.1: If $f : R \to R$ is 2π periodic and Lebesgue integrable on $[-\pi, \pi]$ and $f \in Lip\,\alpha$ class then the degree of approximation of function f by (E, q) (C, δ) product means of its conjugate Fourier Series (1.2) of f satisfies, for n = 0, 1, 2......

$$\left\| (E_q C_\delta)_n (\tilde{f} : x) - \tilde{f}(x) \right\|_\infty = \begin{cases} o\left(\dfrac{1}{(n+1)\alpha}\right) & ;\,(0<\alpha<\delta\le 1) \\ & ;\,(0<\alpha\le 1, \delta>1) \\ o\left(\dfrac{\log(n+1)}{n+1, \alpha}\right) & ;(0 < \alpha \le \delta \le 1) \end{cases} \tag{2.1}$$

3. For the proof of our theorem, we need the following lemmas:

Lemma 1: [1, p-94]: For $(0 < \delta \le 1), n = 1, 2, 3....\;\; 0 < t \le \pi$

$$\left| \tilde{k}_v^{\delta}(t) \right| \le A_\delta y^{-\delta} t^{-(\delta+1)} \tag{3.1}$$

Where A_δ depending on δ only

Lemma 2: [5] For q > 0

$$\sum_{v=0}^{n} \binom{n}{v} q^{n-v} (v+1)^{-1} = o\{(1+q)^{n+1} / (n+1)\} \tag{3.2}$$

Lemma 3: For $\delta > 1$,

$$\left|\tilde{k}_v^{\delta}(t)\right| = o(1)\left(\frac{\delta}{(v+1)t^2}\right) \tag{3.3}$$

4. PROOF OF THE THEOREM:

The n^{th} partial sum of the conjugate Fourier series [1, p-50] is $\overline{S}_n(f;x)$. Then

$$(E_q c_\delta)_n(\tilde{f};x)-\tilde{f}(x)=\frac{1}{\pi}\int_0^{\pi}\phi_x(t)(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}(A_v^{\delta})^{-1}\sum_{k=0}^{n}A_{v-k}^{\delta-1}D_k(t)d$$

$$\left|(E_q c_\delta)_n(\tilde{f};x)-\tilde{f}(x)\right|\le\frac{1}{\pi}\int_0^{\pi}\left|\phi_x(t)\right|\left|(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}(A_v^{\delta})^{-1}\sum_{k=0}^{n}A_{v-k}^{\delta-1}D_k(t)\right|d$$

$$\le \frac{1}{\pi}\left\{\int_0^{\frac{1}{(n+1)}}+\int_{\frac{1}{(n+1)}}^{\pi}\right\} \le |I_1|+|I_2|, say$$

Now, for $0 \le t \le 1/(n+1)$, $\sin nt \le n\sin t$, see [1, p-91]

$$|I_1|\le\frac{1}{\pi}\int_0^{\frac{1}{(n+1)}}\left|\phi_x(t)\right|\left|(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}(A_v^{\delta})^{-1}\sum_{k=0}^{n}A_{v-k}^{\delta-1}D_k(t)\right|dt$$

$$\le \frac{1}{2\pi}\int_0^{\frac{1}{(n+1)}}\left|\phi_x(t)\right|\left|(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}(A_v^{\delta})^{-1}\sum_{k=0}^{n}A_{v-k}^{\delta-1}(2k+1)\right|dt$$

We have by Boos [3, p-104]

$$|I_1|\le\frac{1}{2\pi}\int_0^{\frac{1}{(n+1)}}\left|\phi_x(t)\right|(1+d)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}(2v+1)dt,$$

$$\le \frac{1}{2\pi}\int_0^{\frac{1}{(n+1)}}\left|\phi_x(t)\right|(2n+1)dt,$$

By (1.3)

$$0(n+1)\int_0^{\frac{1}{(n+1)}}t^{\alpha}dt = o(n+1)^{-\alpha} \tag{4.1}$$

By (1.8), we have

$$|I_2|\le\frac{1}{\pi}\int_{\frac{1}{(n+1)}}^{\pi}\left|\phi_x(t)\right|(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}\left|\tilde{k}_v^{\delta}(t)\right|dt$$

Condition –I

For $\delta \le 1$, by lemma 1, we have

$$|I_2|\le\frac{1}{\pi}\int_{\frac{1}{(n+1)}}^{\pi}\left|\phi_x(t)\right|(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}A_\delta v^{-\delta}t^{-(\delta+1)}dt$$

$$\le \frac{A_\delta}{\pi}\int_{\frac{1}{(n+1)}}^{\pi}\left|\phi_x(t)\right|t^{-(\delta+1)}(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}(v+1)^{-\delta}dt$$

By Lemma 2 and (1.2), we get

$$|I_2| = o((n+1)^{-\delta}) \int_{\frac{1}{(n+1)}}^{\pi} t^{\alpha-(\delta+1)} dt$$

Case-I: When $\alpha = \delta$, then

$$|I_2| = o((n+1)^{-\alpha}) \int_{\frac{1}{(n+1)}}^{\pi} t^{-1} dt$$

$$= o((n+1)^{-\alpha}) \log(n+1) \qquad (4.2)$$

Case-II: When $\alpha < \delta$, then

$$|I_2| = o((n+1)^{-\delta})(t^{\alpha-\delta})^{\pi} \frac{1}{(n+1)}$$

$$= o((n+1)^{-\alpha}) \qquad (4.3)$$

Combining (4.2) and (4.3) we have,

$$|I_2| = \begin{cases} o\{(n+1)^{-\alpha}\}, & (0 < \alpha < \delta \le 1) \\ o\{(n+1)^{-\alpha} \log(n+1)\}, & (0 < \alpha \le \delta \le 1) \end{cases}$$

Condition-II:

For $\delta > 1$, by lemma 3, we have

$$|I_2| \le \frac{1}{\pi} \int_{\frac{1}{(n+1)}}^{\pi} |\phi_x(t)| (1+q)^{-n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} \frac{\delta}{(v+1)t^2} dt \qquad (4.4)$$

By lemma 3 and (1.2), we have

$$o((n+1)^{-1}) \int_{\frac{1}{(n+1)}}^{\pi} t^{\alpha-2} dt = o((n+1)^{-\alpha}) \qquad (4.5)$$

Now combining the estimate (4.1), (4.4) and (4.5) we get required result.

REFERENCES

1. A. Zygmund, Trigonometric series, Vol. I (Second Edition), Cambridge University Press, London /New York, (1968).
2. G.H. Hardy Divergent Series, Oxford (1949).
3. J.Boss, Classical and Modern Methods in Summability, Oxford Univ. Press, (2000).
4. P. Chandra, On Some Summability Methods, Bollettino della unione Mathematica Italiana, 4(3) (12) (1975), 211-224.
5. P. Chandra, On the absolute Euler Summability of conjugate series of a Fourier series, Periodica math. Hungaricu, 3(3-4) (1973)189-194.

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