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SPLIT GEODETIC NUMBER OF A LICT GRAPH

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ABSTRACT

A set $\boldsymbol{S} \subseteq \boldsymbol{V}[\boldsymbol{\eta}(\boldsymbol{G})]$is a split geodetic set of $(\boldsymbol{G})$, if S is a geodetic set and $\langle \boldsymbol{V}-\boldsymbol{S}\rangle$ is disconnected. The split geodetic number of a lict graph$\boldsymbol{\eta}(\boldsymbol{G})$, is denoted by $g_s[\boldsymbol{\eta}(\boldsymbol{G})]$, is the minimum cardinality of a split geodetic set of $\boldsymbol{\eta}(\boldsymbol{G})$. In this paper we obtain the split geodetic number of lict graph of any graph. Also obtain many bounds on split geodetic number in terms of elements of G and covering number of G. We investigate the relationship between split geodetic number and geodetic number.

Mathematics Subject Classification: *05C05, 05C12.*

Keywords: *Cartesian product, Distance, Edge covering number, Lict graph, Split geodetic number, Vertex covering number.*

I. INTRODUCTION

In this paper we follow the notations of [3]. As usual n = |V | and m = |E| denote the number of vertices and edges of a graph G respectively. The graphs considered here are undirected and non complete. For any graph G= (V, E), the lict graph η(G) whose vertices correspond to the edges of G and two vertices in η(G) are adjacent if and only if the corresponding edges in G are adjacent. The distance d(u, v) between two vertices u and v in a connected graph G is the length of a shortest u − v path in G. It is well known that this distance is a metric on the vertex set V (G). For a vertex v of G, the eccentricity e(v) is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is radius, rad G, and the maximum eccentricity is the diameter, diam G. A u−v path of length d(u, v) is called a u − v geodesic. We define I[u, v] to the set (interval) of all vertices lying on some u − v geodesic of G and for a nonempty subset S of V (G), $I[S] = \cup_{u,v\in S} I[u,v]$.

A set S of vertices of G is called a geodetic set in G if I[S] = V (G), and a geodetic set of minimum cardinality is a minimum geodetic set. The cardinality of a minimum geodetic set in G is called the geodetic number of G, and we denote it by g(G).

A vertex v is an extreme vertex in a graph G, if the subgraph induced by its neighbours is complete. A vertex cover in a graph G is a set of vertices that covers all edges of G. The minimum number of vertices in a vertex cover of G is the vertex covering number $\alpha_0(G)$ of G. An edge cover of a graph G without isolated vertices is a set of edges of G that covers all the vertices of G. The edge covering number $\alpha_1(G)$ of a graph G is the minimum cardinality of an edge cover of G.

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Split geodetic number of a graph was studied by in [5]. A geodetic set S of a graph G = (V, E) is a split geodetic set if the induced subgraph $\langle V - S \rangle$ is disconnected. The split geodetic number $g_s(G)$ of G is the minimum cardinality of a split geodetic set. Geodetic number of a lict graph was studied by in [4]. Geodetic number of a lict graph η(G) of G is a set S' of vertices of η(G)= H is called the geodetic set in H if $I[S']$= V (H) and a geodetic set of minimum cardinality is the geodetic number of η(G) and is denoted by$g[\eta(G)]$. Now we define split geodetic number of a lict graph. A set S' of vertices of η(G)= H is called the split geodetic set in H if the induced subgraph V (H) – S' is disconnected and a split geodetic set of minimum cardinality is the split geodetic number of η(G)and is denoted by $g_s[\eta(G)]$.

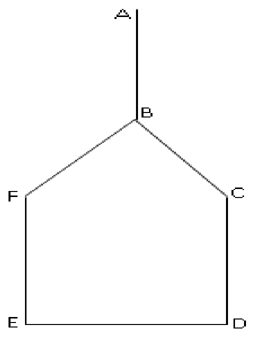

Figure 1.0(a)

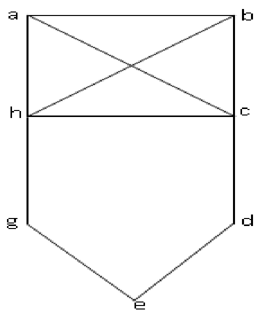

Figure 1.0(b)

For the graph G given in Figure 4.2.1(a), S = {A, D, E} is a minimum geodetic set so that g(G) = 3 and S_1 = {A,D,B,E} is a minimum split geodetic set so that $g_s(G)$ = 4. For the graph η (G) given in Figure 1.0(b), S′ = {a, b, e} is a minimum geodetic set so that g[η (G)] = 3 and S_2 = {a, b, e, h} is a minimum split geodetic set so that g_s[η (G)] = 4.

II. PRELIMINARY NOTES

We need the following results to prove further results.

Theorem 2.1: [1] Every geodetic set of a graph contains its extreme vertices.

Theorem 2.2: [2] For any path P_n of order n, the edge covering number

$$\alpha_1(P_n) = \begin{cases} \frac{n}{2} & if\ \ n\ \ is\ \ even \\ \frac{n+1}{2} & if\ \ n\ \ is\ \ odd \end{cases}$$

Theorem 2.3: [1] Let G be a connected graph of order at least 3. If G contains a minimum geodetic set S with a vertex x such that every vertex of G lies on some x − w geodesic in G for some $w \epsilon S$, then g(G) = g(G $\times K_2$).

Theorem 2.4: [5] For any graph G, $g(G) \leq g_s(G)$.

III. MAIN RESULTS

Theorem 3.1: For any tree T of order n, $g_s[\eta(T)] = n + 1$.

Proof: Let S be the set of all extreme vertices of a lict graph $\eta(T)$ of a tree T. Let v_i be a cut vertex in V− S and $S' = S \cup \{v_i\}$, by Theorem 2.1 $g_s[\eta(T)] \geq |S'|$. On the other hand, for an internal vertex v of $\eta(T)$, there exists x, y of $\eta(T)$ such that v lies on the unique x−y geodesic in $\eta(T)$. By the definition of lict graph pendant vertices and cut vertices of T are the extreme vertices of $\eta(T)$ and the induced subgraph V $[\eta(T)] - S'$ is a split geodetic set of $\eta(T)$. Thus $g_s[\eta(T)] \leq |S'|$. Also, every split geodetic set S_1 of $\eta(T)$ must contain S' which is the unique minimum split geodetic set. Therefore | S' | = $|S_1|$ = |S| + $|v_i|$ = n + 1. Hence $g_s[\eta(T)] = n + 1$.

Corollary 3.2: For any path P_n, $n \geq 6$, $g_s[\eta(P_n)] = n + 1$.

Proof: Clearly the set of two pendant vertices of a path P_n is its unique geodetic set. From Theorem 3.1 the results follow.

Theorem 3.3: For any graph G of order n, $g_s[\eta(G)] \leq n$.

Proof: Let S be the geodetic set of $\eta(G)$ such that $\langle V - S \rangle$ is connected. So S is not a split geodetic set of $\eta(G)$.
Now, we consider a set $S' = S \cup \{v_i\}$ where v_i be the vertex in $\langle V - S \rangle$ and is adjacent to at least one vertex in S. Thus $\langle V - S' \rangle$ is a split geodetic set of $\eta(G)$. Therefore $g_s[\eta(G)] \leq n$.

Theorem 3.4: For any path P_n, $n \geq 6$, $g_s[\eta(P_n)] = d + \triangle$ where $\triangle$ be the maximum degree and d be the diameter.

Proof: Since the set of two pendant vertices of a path is its unique geodetic set, the distance between those vertices is the diameter i.e d(u, v) =1n= d and the maximum degree of vertices in path is $\triangle$ = 2. Clearly, it follows that $g_s[\eta(P_n)] = n - 1 + 2 = d + \triangle$.

Theorem 3.5: For any graph G of order n, $g_s[\eta(G)] > m - \alpha_1(G) + 1$. Where α_1 is the edge covering number.

Proof: Suppose S = {e_1, e_2, ..., e_k} be the set of all pendant edges in G. Then $S \cup J$ where $J \subseteq E(G) - S$, be the minimal set of edges which covers all the vertices of G such that $|S \cup J| = \alpha_1(G)$. Now without loss of generality in $\eta(G)$, let I = {u_1, u_2, ..., u_p} $\subseteq V[\eta(G)]$ be the set of vertices in $\eta(G)$ formed by the pendant vertices and cut vertices in G. Suppose H = {u_1, u_2, ..., u_j} $\subseteq V[\eta(G)] - I$. Then $I \cup \{u_j\}$, where $u_j \in H$, forms a minimum split geodetic set of $\eta(G)$. Clearly it follows that $|I \cup \{u_j\}| > |E(G)| - |S \cup J| + 1$. Therefore $g_s[\eta(G)] > m - \alpha_1(G) + 1$.

Theorem 3.6: For any graph G of order n, $g_s[\eta(G)] \leq 3\alpha_0(G) + 1$.

Proof: Let S be a minimum set of vertices in G. Then S has at least two vertices and every vertex in S adjacent to some vertex in $\langle V - S \rangle$. Thus $\langle V - S \rangle$ is disconnected. Hence S is a split geodetic set of $\eta(G)$. Hence $g_s[\eta(G)] \leq 3\alpha_0(G) + 1$.

Theorem 3.7: For any graph G of order n, $g_s[\eta(G)] < n_1 - k[\eta(G)]$, where n_1 be the number of vertices in $\eta(G)$ and $k[\eta(G)]$ is a vertex connectivity.

Proof: Let $k(G)$ = k. Since $\eta(G)$is connected and each block is complete, $l \leq k(G) \leq n_1 - 2$. Let U = {u_1, u_2, ..., u_k} be a minimum cutset of G, G_1, G_2, ...,G_r $(r \geq 2)$ be the components of G− U and let W = V [$\eta(G)$] – U then every vertex u_i $(l \leq i \leq k)$ is adjacent to at least one vertex of G_j for every $(i \leq j \leq r)$. Therefore, every vertex u_i belongs to a W geodesic path. Thus $g_s[\eta(G)] < n_1 - k[\eta(G)]$.

Theorem 3.8: For any tree G of order n and diameter d then$g_s[\eta(G)] \leq n + d - 2$.

Proof: Let S = {v_1, v_2, ..., v_i, u_1, u_2, ..., u_j} be the geodetic set of $\eta(G)$ such that v_1, v_2, ..., v_i be the set of vertices in $\eta(G)$ corresponding to the pendant edges of G and also u_1, u_2, ..., $u_j \in \eta(G)$ corresponds to the cut vertices of G. Now, consider a set $S' = S \cup w_k$, where $w_k \in \langle v[\eta(G)] - S \rangle$. Hence $\langle v[\eta(G)] - S' \rangle$ be a split geodetic set of $\eta(G)$. It follows that $|S'| = |S| + 1$. Hence $g_s[\eta(G)] \leq n + d - 2$.

Theorem 3.9: For any graph G of order n, $g_s[\eta(G)] \leq \alpha_1[\eta(G)] + 4$.

Proof: Let S be the minimum set of edges covers all the vertices in $\eta(G)$. Then S has at least two vertices and every vertex in S is adjacent to some vertex in $\langle V[\eta(G)] - S \rangle$. Thus $\langle V[\eta(G)] - S \rangle$ is disconnected. Hence S is a split geodetic set of $\eta(G)$. Hence $g_s[\eta(G)] \leq \alpha_1[\eta(G)] + 4$.

Theorem 3.10: For any graph G of order n, $g[\eta(G)] + g_s[\eta(G)] > n$.
.
Proof: Let S = {v_1, v_2, ..., v_n} $\subseteq V[\eta(G)]$ be the minimum split geodetic set of $\eta(G)$. Now without loss of generality in $\eta(G)$, if F = {u_1, u_2, ..., u_p} be the set of all extreme vertices in $\eta(G)$. Then $F \cup I$ *where* $I \subseteq V[\eta(G)] - F$ forms a minimum split geodetic set of $\eta(G)$. Clearly, $|S| \cup |F \cup H| > n$. Therefore, $g[\eta(G)] + g_s[\eta(G)] > n$.

IV. ADDING AN PENDANT EDGE

Definition: For an edge e = {u, v} of a graph G with deg(u) = 1 and deg(v) > 1, we call e an pendant edge and u an pendant-vertex. Let G' be the graph obtained by adding an pendant-edge {u, v} to a cycle C_n = G of order $n \geq 5$, with u $\in$ G and v $\notin$ G.

Let G' be the graph obtained by adding pendant edge (u_i, v_j), i = 1, 2... n, j = 1, 2, ..., k to each vertex of G of order $n \geq 5$ such that $u_i \in G$, $v_j \notin G$.

Theorem 4.1: G' be the graph obtained by adding k pendant edges {(u, v_1), (u, v_2), ..., (u, v_k)} to a cycle C_n = G of order $n \geq 5$, with u $\in$ G and {v_1, v_2, ..., v_k} $\notin$ G. Then

$$g_s[\eta(G')] = \begin{cases} K + 4 \; if \; n \; is \; even \\ k + 3 \; \; if \; n \; is \; odd \end{cases}$$

Proof: Let {e_1, e_2, ..., e_n, e_1} be a cycle with n vertices and let G' be the graph obtained from G = C_n by adding pendant edges (u, v_i), i = 1, 2, ..., k. Such that u $\in$ G and $v_i \notin$ G.

Case-1: Consider n is even.

By the definition of lict graph, $\eta(G')$as an $\langle K_{k+3}\rangle$ as an induced subgraph. Also the edges $(u, v_i) = e_i$, i = 1, 2, ..., k and the cut vertices u_i becomes the vertices of $\eta(G')$ and these belongs to some geodetic set of $\eta(G')$. Hence $\{e_1, e_2, ..., e_k, e_l, e_m, u_i\}$ are the vertices of $\eta(G')$ where e_l, e_m are the edges incident on the antipodal vertex of u in G' and these vertices belongs to some geodetic set of $\eta(G')$, also $\eta(G')= C_n \cup K_{k+3}$. Let $S = \{e_1, e_2, ..., e_k, e_l, e_m, u_i\}$ be the geodetic set of $\eta(G')$. Since $V[\eta(G^{'})]$ −S is connected, we consider a set $S^{'} = S \cup e_j$ where e_j be the edge incident on the cut vertex of G'. Such that $V[\eta(G^{'})] - S'$ is disconnected. So S' is the minimum split geodetic set of $\eta(G')$. Therefore $g_s[\eta(G')] = k + 4$.

Case-2: Consider n is odd.

By the definition of lict graph, $\eta(G')$ has $\langle K_{k+3}\rangle$ as an induced subgraph, also the edges $(u, v_i) = \{e_1, e_2, ..., e_k\}$ becomes vertices of $\eta\,(G')$. Let $e_l = (a, b) \epsilon$ G such that d(u, a) = d(u, b) in the graph $\eta(G')$. Let $S = \{e_1, e_2, ..., e_k, e_l\}$ be the geodetic set of $\eta(G')$.

Now, consider a set $S^{'} = S \cup \{e_j\}$ is a split geodetic set of $\eta(G')$ where e_j is the vertex from V $[\eta(G')] - S$ having deg= 2. It is clear that S' is the minimum split geodetic set of $\eta(G')$. Therefore $g_s\,[\eta(G')]$ = k+3.

Theorem 4.2: Let G' be the graph obtained by adding pendant edge (u_i, v_j), i = 1, 2, ..., n, j = 1, 2, ..., k to each vertex of G = C_n of order $n \geq 5$ such that $u_i \in G, v_j \notin G$. Then, $g_s[\eta(G')] = n_1 + 2$, where n_1 be the number of vertices in G'.

Proof: Let $\{e_1, e_2, ..., e_n, e_1\}$ be a cycle with n vertices and G = C_n. Let G' be the graph obtained by adding pendant vertex (u_i, v_j), i = 1, 2, ..., n, j = 1, 2, ..., k to each vertex of G, such that $u_i \in G, v_j \notin G$. Clearly k be the number of end vertices of $'$. By the definition of lict graph $\eta(G')$ have n copies of K_4 as an induced subgraph. The edges $(u_i, v_j) = e_j$ for all j and u_i becomes vertices of $\eta(G')$ and those lies on geodetic set of $\eta(G')$. Since they forms the extreme vertices of $\eta(G')$. By Theorem 2.1 $S = \{e_1, e_2, ..., e_k, u_1, u_2, ..., u_i\}$ forms geodetic set . Now consider any two vertices $\{e_l, e_m\} \in V - S$ which are not adjacent. $S' = \{e_1, e_2, ..., e_k, u_1, u_2, ..., u_i, e_l, e_m\}$ forms a split geodetic set of S'. Suppose $P = \{e_1, e_2, ..., e_k, u_1, u_2, ..., u_i, e_l\}$ be the set of vertices of $\eta(G')$, such that $|P| < |\,S'\,|$, then V−P is connected. Hence it is clear that S' is the minimum split geodetic set of $S^{'}$. Therefore $g_s[\eta(G')] = n_1 + 2$.

Theorem 4.3: Let $|G'$ be the graph obtained by adding k end edges $\{(u, v_1), (u, v_2), ..., (u, v_k)\}$ to a cycle C_n = G of order $n \geq 5$, with u ∈ G and $\{v_1, v_2, ..., v_k\} \notin G$. Then$g_s[\eta(G')] \leq \alpha_1[\eta(G')]$, where α_1 be the edge covering number of G'.

Proof: Let S be the minimum set of edges which covers all the vertices in $\eta(G')$. Then S has at least two vertices and every vertex in S is adjacent to some vertex in $\langle V[\eta(G')] - S\rangle$. Thus $\langle V[\eta(G')] - S\rangle$ is disconnected. Hence S is a split geodetic set of $\eta(G')$. Hence $g_s[\eta(G')] \leq \alpha_1[\eta(G')]$.

Theorem 4.4: Let G' be the graph obtained by adding pendant edge (u_i, v_j), i = 1, 2, ..., n, j = 1, 2, ..., k to each vertex of G = C_n of order $n \geq 5$,such that $u_i \in$ G, $v_j \notin$ G. Then $g_s[\eta(G')] = \alpha_1[\eta(G')] + 4$, where α_1 be the edge covering number of $\eta(G')$.

Proof: Let S be the minimum set of edges which covers all the vertices in $\eta(G')$. Then S has at least two vertices and every vertex in S is adjacent to some vertex in $\langle V[\eta(G') - S]\rangle$. Thus $\langle V[\eta(G') - S\rangle]$ is disconnected. Hence S is a split geodetic set of $\eta(G')$. Hence $g_s[\eta(G')] = \alpha_1[\eta(G')] + 4$.

Theorem 4.5: Let G' be the graph obtained by adding k pendant edges $\{(u, v_1), (u, v_2), ..., (u, v_k)\}$ to a cycle C_n = G of order $n \geq 5$, with $u \in G$and $\{v_1, v_2, ..., v_k\} \notin G$. Then $g_s[\eta(G')] < n_1 - \kappa(G')$, where $\kappa(G')$ be the vertex connectivity of G'.

Proof: Let $\kappa(G')$= k. Since G' is connected but not complete. Let u_k be a cutset of G' and $G_1, G_2, ...,G_r$, $r \geq 2$ be the components of G− u $_k$ and let $P = V(G') - u_k$ then every vertex is adjacent to at least one vertex of G_j $(1 \leq j \leq r)$. Therefore every vertex u_i belongs to some geodetic set. Hence $g_s[\eta(G')] < n_1 - \kappa(G')$.

V. CARTESIAN PRODUCT

The Cartesian product of the graphs H_1 and H_2, written as $H_1 \times H_2$, is the graph with vertex set $V(H_1) \times V(H_2)$, two vertices u_1, u_2 and v_1, v_2 being adjacent in $H_1 \times H_2$ if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E(H_2)$, or $u_2 = v_2$ and $(u_1, v_1) \in E(H_1)$.

Theorem 5.1: For any path P_n of order n, $g_s[K_3 \times \eta(P_n)] = n + 2$.

Proof: By the definition of lict graph $\eta(P_n)$ = K_3, K_3, ..., (n − 2) factors $(n \geq 3)$. Consider G = $\eta(P_n)$, let K_3 ×G be the graph formed from three copies G_1, G_2 and G_3 of G and S be a minimum geodetic set of K_3×G. Now, we define S' be the union of those vertices of G belonging to S. Let $v \in V(G_1)$lies on some x − y geodesic for any x, y ∈ S. Since S is a geodetic set by the Theorem 2.1 we have g[$\eta(P_n)$] = n at least one of x and y belongs to V_1. If both x, y ∈ V_1, then x, y ∈ S'. Hence we may assume that x ∈ V_1 and y ∈V_2. If corresponds to x then v = x ∈ S' where y ≠ x. Since d(x, y) = d(x, y′) + 1 and the vertex v lies on an x–y geodesic in K_3×G. Let S contains a vertex x with the property that every vertex of G_1 lies on an x–w geodesic in G_1 for some w ∈ S. Let S′ consists of x together with those vertices of G_3 and G_2 corresponding to those vertices in S− x. Thus $|S'| = |S| \cup \{a, b\}$ where $\{a, b\}$ be the vertices in G_3 and G_2. Hence, S' is a split geodetic set of K_3 × G.

Therefore | S'| = g[η (P_n)] + $\{a, b\}$
$\Rightarrow g_s[K_3 \times G] = n + 2$
$\Rightarrow g_s\big[K_3 \times [\eta(P_n)]\big] = n + 2.$

Theorem 5.2: For any path P_n of order n, then

$$g_s\big[K_3 \times [\eta(P_n)]\big] = \begin{cases} 2\alpha_1(P_n) + 1 & if\ n\ is\ even \\ 2\alpha_1(P_n) + 2 & if\ n\ \ is\ odd \end{cases},$$ where α_1 be the edge covering number.

Proof: Let α_1 (P_n) be a edge covering of a graph is a minimum cardinality of an edge cover of a path P_n. We have the following cases.

Case-1: Suppose n is odd, we have α_1 (P_n) = $\frac{n+1}{2}$

$\Rightarrow 2\alpha_1$ (P_n) = n+1

Since $g_s[K_3 \times \eta(G')] = n + 2$
$\Rightarrow g_s[K_3 \times \eta(G')]$ = n+1 + 1

$\Rightarrow g_s[K_3 \times \eta(G')] = 2\alpha_1(P_n) + 1$.

Case-2: Suppose n is even, we have α_1 (P_n) = $\frac{n}{2}$

$\Rightarrow 2\alpha_1$ (P_n) = n

Since $g_s[K_3 \times \eta(G')] = n + 2$
$\Rightarrow g_s[K_3 \times \eta(G')] = 2\alpha_1(P_n) + 2$.

Theorem 5.3: For any path P_n of order n, $g_s\big[K_3 \times [\eta(P_n)]\big] \leq 3\alpha_0(P_n) + 1$, where α_0 is a vertex covering number.

Proof: Let S be a minimum set of vertices in P_n. Then S has at least two vertices and every vertex in S adjacent to some vertex in $\langle V - S\rangle$. Thus $\langle V - S\rangle$ is disconnected. Hence S is a split geodetic set of K_3 × η (P_n).

Hence $g_s\big[K_3 \times [\eta(P_n)]\big] \leq 3\alpha_0(P_n) + 1$.

Corollary 5.4: For any path P_n of order n, $g_s[K_3 \times \eta(P_n)] > g_{ns}[K_3 \times \eta(P_n)]$.

Proof: Proof follows from the above Theorem.

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