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SKOLEM MEAN LABELING, RELAXED SKOLEM MEAN LABELING AND SKOLEM DIFFERENCE MEAN LABELING OF BISTARS

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ABSTRACT

The concept of skolem mean labeling, relaxed skolem mean labeling and skolem difference mean labeling were already introduced. In this paper, the skolem mean labeling, relaxed skolem mean labeling and skolem difference mean labeling of bistars is studied.

Key words: *Skolem mean labeling, skolem mean graph, relaxed skolem mean labeling, relaxed skolem mean graph, skolem difference mean labeling and skolem difference mean graph*

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1. INTRODUCTION

Throughout this paper, by a graph, we mean a finite, undirected and simple one. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. Terms and notations not defined here are used in the sense of Harary[8]. For number theoretic terminology,[1] is followed.

A graph labeling is an assignment of integers to the vertices or edges or both vertices and edges subject to certain conditions. There are several types of graph labeling. A vertex labeling of a graph G is an assignment of labels to the vertices of G that induces for each edge xy a label depending on the vertex labels $f(x)$ and $f(y)$. Similarly, an edge labeling of a graph G is an assignment of labels to the edges of G that induces for each vertex v, a label depending on the edge labels incident on it.. Total labeling involves a function from the vertices and edges to some set of labels. A detailed survey of graph labeling is available in [9].

Labeled graphs are becoming an increasing useful family of mathematical models for a broad range of applications like designing good RADAR type codes with optimal auto correlation properties, determining ambiguities in X-ray crystallographic analysis, formulating a communication network addressing system, determining optimal circuit layouts, problems in additive number theory etc. A systematic presentation of diverse application of graph labeling is given in [6] and [7].

The concept of skolem mean labeling and relaxed skolem mean labeling were introduced by V.Balaji, D.S.T. Ramesh and A.Subramanian in [2,4] and further results were discussed in [3,5]. The concept of skolem difference mean labeling was due to K.Murugan and A.Subramanian [10]. Following definitions are necessary for the present study.

1.1 Definition: Let the graphs G_1 and G_2 have disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then their union $G = G_1 \cup G_2$ is a graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.Clearly $G_1 \cup G_2$ has $p_1 + p_2$ vertices and $q_1 + q_2$ edges.

1.2 Definition: The Bistar $B_{m,n}$ is the graph obtained from K_2 by joining m pendant edges to one end of K_2 and n pendant edges to the other end of K_2.

1.3 Definition: A graph $G = (V, E)$ with p vertices and q edges is said to be a skolem mean graph if there exists a function f from the vertex set of G to $\{1,2,3,\dots,p\}$ such that the induced map f^* from the set of G to $\{2,3,4,\dots p\}$ defined by $f^*(e = uv) = \frac{f(u)+f(v)}{2}$ if $f(u) + f(v)$ is even and $\frac{f(u)+f(v)+1}{2}$ if $f(u) + f(v)$ is odd, then the resulting edges get distinct labels from the set $\{2,3,4,\dots,p\}$.

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1.4 Definition: A graph G = (V, E) with p vertices and q edges is said to be a relaxed skolem mean graph if there exists a function f from the vertex set of G to $\{1,2,3, \dots p+1\}$ such that the induced map f^* from the set of G to $\{2,3,4, \dots, p+1\}$ defined by $f^*(e = uv) = \frac{f(u)+f(v)}{2}$ if $f(u) + f(v)$ is even and $\frac{f(u)+f(v)+1}{2}$ if $f(u) + f(v)$ is odd, then the resulting edges get distinct labels from the set $\{2,3,4, \dots, p+1\}$.

1.5 Definition: A graph *G*= (V, E) with p vertices and q edges is said to have skolem difference mean labeling if it is possible to label the vertices x ϵ V with distinct elements $f(x)$ from the set $\{1,2,3, \dots, p+q\}$ in such a way that the edge $e = uv$ is labeled with $\frac{|f(u)-f(v)|}{2}$ if $|f(u) - f(v)|$ is even and $\frac{|f(u)-f(v)|+1}{2}$ if $|f(u) - f(v)|$ is odd and the resulting labels of the edges are distinct and are from $\{1,2,3, \dots, q\}$. A graph that admits skolem difference mean labeling is called a skolem difference mean graph.

2. MAIN RESULTS

In this section, the skolem mean labeling, relaxed skolem mean labeling and skolem difference mean labeling of bistars is studied.

2.1 Theorem: The graph $B(m, n)$ where $m, n \geq 1$ and $m \neq n$ is
(i) skolem mean if and only if $|m - n| = 1$
(ii) relaxed skolem mean if and only if $|m - n| \leq 3$ and
(iii) skolem difference mean for all $m, n \geq 1$

Proof: Let *G* be the graph $B(m, n)$ with the given conditions. Without loss of generality, let us assume that $m > n$.

$(\boldsymbol{i})$: Suppose $|m - n| = 1$. Then $m = n + 1$.Then the graph *G* is $B(n + 1, n)$.

Let $V(G) = \{u, v, u_i, v_j; 1 \leq i \leq n+1, 1 \leq j \leq \text{n}\}$ and $E(G) = \{uu_i, uv, vv_j; 1 \leq i \leq n+1, 1 \leq j \leq \text{n}\}$ Then the graph *G* has *2n*+3 vertices and *2n*+2 edges. Let *f:* $V(G) \rightarrow \{1,2,3, \dots, 2n+3\}$ be defined as follows.
$f(u) = 2$
$f(u_i) = 2i - 1; 1 \leq i \leq n+1$
$f(v) = 2n + 3$
$f(v_i) = 2 + 2j; 1 \leq j \leq n$

Let f^* be the induced edge labeling of *f*. Then
$f^*(uu_i) = 1 + i; 1 \leq i \leq n+1$
$f^*(uv) = n + 3$
$f^*(vv_i) = n + 3 + i; 1 \leq i \leq n$

The induced edge labels are distinct and are *2, 3, 4,…, 2n*+3. Hence the graph $B(m, n)$ is skolem difference mean.

Conversely suppose that $|m - n| > 1$. Then $m = n + r$ where $r \geq 2$. Then the graph *G* is $B(n + r, n)$where $r \geq 2$.

Let $V(G) = \{u, v, u_i, v_j; 1 \leq i \leq n+r, 1 \leq j \leq \text{n}\}$ and $E(G) = \{uu_i, uv, vv_j; 1 \leq i \leq n+r, 1 \leq j \leq \text{n}\}$

Then the graph *G* has *2n*+$r + 2$ vertices and *2n*+$r + 1$ edges.Let *f:* $V(G) \rightarrow \{1,2,3, \dots, 2n + r + 2\}$ be defined as follows.

Suppose that $f(u) = k$ where $1 \leq k \leq 2n + r + 2$.Then the $(n + r)u_i's$ and "v" should be labeled with $\frac{2n+r+2}{2}$ consecutive integers to get distinct edge labels where r ≥ 2.

ie, Atleast the $(n + 2)u_i's$ and "v" should be labeled with $n + 2$ labels which is impossible. This is a contradiction.

$(\boldsymbol{ii})$: Suppose that $|m - n| \leq 3$. Then there are three cases $m = n + 1, m = n + 2, m = n + 3$.

Case-$(\boldsymbol{a})$: Let $m = n + 1$. Then *G* is $B(n + 1, n)$. Let $V(G) = \{u, v, u_i, v_j; 1 \leq i \leq n+1, 1 \leq j \leq n\}$ and
$E(G) = \{uu_i, uv, vv_j; 1 \leq i \leq n+1, 1 \leq j \leq n\}$The graph *G* has *2n*+3 vertices and *2n*+2 edges. Let *f:* $V(G) \rightarrow \{1,2,3, \dots 2n+4\}$ be defined as follows:
$f(u) = 2n + 3$
$f(u_i) = 2 + 2i; 1 \leq i \leq n+1$
$f(v) = 2$
$f(v_j) = 1 + 2j; 1 \leq j \leq n$

Let f^* be the induced edge labeling of *f*. Then
$f^*(uu_i) = n + 3 + i; 1 \leq i \leq n + 1$
$f^*(uv) = n + 3$
$f^*(vv_j) = 2 + j; 1 \leq j \leq n$

The edge labels are 3, 4, 5,…, 2n+4 which are distinct.

Case($\boldsymbol{b}$): Let $m = n + 2$. Then G is $B(n + 2, n)$. Let $V(G) = \{u, v, u_i, v_j; 1 \leq i \leq n + 2, 1 \leq j \leq n\}$ and $E(G) = \{uu_i, uv, vv_j; 1 \leq i \leq n + 2, 1 \leq j \leq n\}$. The graph G has $2n$+4 vertices and $2n$+3 edges.Let *f:* $V(G) \rightarrow \{1,2,3, \ldots, 2n + 5\}$ be defined as follows:
$f(u) = 2n + 4$
$f(u_i) = 1 + 2i; 1 \leq i \leq n + 1$
$f(v) = 1$
$f(v_j) = 2 + 2j; 1 \leq j \leq n$

Let *f** be the induced edge labeling of *f*. Then
$f^*(uu_i) = n + 3 + i; 1 \leq i \leq n + 2$
$f^*(uv) = n + 3$
$f^*(vv_j) = 2 + j; 1 \leq j \leq n$

The edge labels are 3, 4, 5,…2n+5 which are distibnct.

Case($\boldsymbol{c}$): Let $m = n + 3$. Then G is $B(n + 3, n)$.Let $V(G) = \{u, v, u_i, v_j; 1 \leq i \leq n + 3, 1 \leq j \leq n\}$ and $E(G) = \{uu_i, uv, vv_j; 1 \leq i \leq n + 3, 1 \leq j \leq n\}$.The graph G has $2n$+5 vertices and $2n$+4 edges.Let *f:* $V(G) \rightarrow \{1,2,3, \ldots, 2n + 6\}$ be defined as follows.

$f(u) = 2n + 5$
$f(u_i) = 2i; 1 \leq i \leq n + 3$
$f(v) = 1$
$f(v_j) = 3 + 2j; 1 \leq j \leq n$

Let *f** be the induced edge labeling of *f*. Then
$f^*(uu_i) = n + 3 + i; 1 \leq i \leq n + 3$
$f^*(uv) = n + 3$
$f^*(vv_j) = 2 + j; 1 \leq j \leq n$

The edge labels are 3, 4, 5,…, 2n+5 which are distinct.

Conversely suppose that $|m - n| > 3$. Let $m = n + r$ where $r \geq 4$. Consider the graph $G = B(m, n)$ for $m = n + r$ where $r \geq 4$. Let $V(G) = \{u, v, u_i, v_j; 1 \leq i \leq n + r, 1 \leq j \leq n\}$ and $E(G) = \{uu_i, uv, vv_j; 1 \leq i \leq n + r, 1 \leq j \leq n\}$

Then the graph G has $2n$+$r + 2$ vertices and $2n$+$r + 1$ edges. Let *f:* $V(G) \rightarrow \{1,2,3, \ldots, 2n + r + 3\}$ be defined as follows.

Suppose that $f(u) = k$ where $1 \leq k \leq 2n + r + 3$

The $(n + r)u_i's$ and a "v" should be labeled with $\frac{2n+r+3}{2}$ consecutive integers to get distinct edge labels where $r \geq 4$.
i.e., At least $(n + 4)u_i's$ and a "v" should be labeled with n+4 labels which is impossible.

($\boldsymbol{iii}$): Let *f:* $V(B_{m,n}) \rightarrow \{1,2,3, \ldots, 2m + 2n + 3\}$ be defined as follows:
$f(u) = 1$
$f(v) = 2m + 2n + 3$
$f(u_i) = 2i; 1 \leq i \leq m$
$f(v_j) = 2j + 1; 1 \leq j \leq n$

Let f^* be the induced edge labeling of *f*. Then
$f^*(uu_i) = i; 1 \leq i \leq m$
$f^*(uv) = m + n + 1$
$f^*(vv_j) = m + n + 1 - j; 1 \leq j \leq n$

Then the induced edge labels are distinct and are $1, 2, 3, \ldots, m+n+1$. Hence the theorem.

2.2 Theorem: The bistar $B(n,n)$ is skolem mean, relaxed skolem mean and skolem difference mean for all $n \geq 1$.

Proof: Let G be the graph $B(n,n)$.Let $V(G) = \{u, v, u_i, v_i; 1 \leq i \leq n\}$ and $E(G) = \{uu_i, uv, vv_i; 1 \leq i \leq n\}$

The graph G has $2n+2$ vertices and $2n+1$ edges.

$(\boldsymbol{i})$**:** Let the function $f: V(G) \to \{1,2,3, \ldots, 2n+2\}$ be defined as follows:

$f(u) = 1$
$f(u_i) = 2i; 1 \leq i \leq n$
$f(v) = 2n+2$
$f(v_i) = 2i+1; 1 \leq i \leq n$

Let f^* be the induced edge labeling of f. Then
$f^*(uu_i) = 1+i; 1 \leq i \leq n$
$f^*(uv) = n+2$
$f^*(vv_i) = n+2+i; 1 \leq i \leq n$

The induced edge labels are distinct and are $2, 3, 4, \ldots, 2n+2$.

$(\boldsymbol{ii})$**:** Let $f: V(G) \to \{1,2,3, \ldots, 2n+3\}$ be defined as follows:

$f(u) = 2$
$f(u_i) = 1+2i; 1 \leq i \leq n$
$f(v) = 2n+3$
$f(v_i) = 2+2i; 1 \leq i \leq n$

Let f^* be the induced edge labeling of f. Then
$f^*(uu_i) = 2+i; 1 \leq i \leq n$
$f^*(uv) = n+3$
$f^*(vv_i) = n+3+i; 1 \leq i \leq n$

The edge labels are 3, 4, 5,…, 2n+3. which are distinct.

$(\boldsymbol{iii})$**:** The result follows immediately from (iii) of theorem 2.1. Hence the theorem.

2.3 Theorem: The graph $B(m,m) \cup B(n,n)$is skolem mean, relaxed skolem mean and skolem difference mean for all $m, n \geq 1$.

Proof: Let G be the graph $B(m,m) \cup B(n,n)$. Let $V(G) = \{u, v, u_i, v_i, s, t, s_j, t_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G) = \{uu_i, uv, vv_i, ss_j, st, tt_j; 1 \leq i \leq m, 1 \leq j \leq n\}$The graph G has $2m+2n+4$ vertices and $2m+2n+2$ edges.

$(\boldsymbol{i})$: Let $f: V(G) \to \{1,2,3, \ldots, 2m+2n+4\}$be defined as follows:
$f(u) = 2m+2n+4$
$f(u_i) = 2m+2n+5-2i; 1 \leq i \leq m$
$f(v) = 2n+3$
$f(v_i) = 2m+2n+4-2i; 1 \leq i \leq m$
$f(s) = 2n+2$
$f(s_j) = 2n+3-2j; 1 \leq j \leq n$
$f(t) = 1$
$f(t_j) = 2n+2-2j; 1 \leq j \leq n$

Let f^* be the induced edge labeling of f. Then
$f^*(uu_i) = 2m+2n+5-i; 1 \leq i \leq m$
$f^*(uv) = m+2n+4$
$f^*(vv_i) = m+2n+4-i; 1 \leq i \leq m$
$f^*(ss_j) = 2n+3-j; 1 \leq j \leq n$
$f^*(st) = n+2$
$f^*(tt_j) = n+2-j; 1 \leq j \leq n$

The induced edge labels are distinct and are 2, 3, 4,…, $2n+2, 2n+4, 2n+5, \ldots, 2n+2m+4$.

$(\boldsymbol{ii})$: Let f: $V(G) \to \{1,2,3,\ldots,2m+2n+5\}$ be defined as follows:
$f(u) = 2m+2n+5$

$f(u_i) = 2m+2n+6-2i; 1 \le i \le m$
$f(v) = 2n+4$
$f(v_j) = 2m+2n+5-2i; 1 \le i \le m$
$f(s) = 2n+3$
$f(s_j) = 2n+4-2j; 1 \le j \le n$
$f(t) = 2$
$f(t_j) = 2n+3-2j; 1 \le j \le n$

Let f^* be the induced edge labeling of f. Then
$f^*(uu_i) = 2m+2n+6-i; 1 \le i \le m$
$f^*(uv) = m+2n+5$
$f^*(vv_i) = m+2n+5-i; 1 \le i \le m$
$f^*(ss_j) = 2n+4-j; 1 \le j \le n$
$f^*(st) = n+3$
$f^*(tt_j) = n+3-j; 1 \le j \le n$

The induced edge labels are 3, 4, 5,…, $2n+3, 2n+5, 2n+6, \ldots, 2m+2n+5$ which are distinct.

$(\boldsymbol{iii})$: Let f: $V(G) \to \{1,2,3,\ldots,4m+4n+6\}$ be defined as follows:
$f(u) = 4m+4n+5$
$f(u_i) = 2i-1; 1 \le i \le m$
$f(v) = 2m+1$
$f(v_i) = 4m+4n+5-2i; 1 \le i \le m$
$f(s) = 4m+4n+4$
$f(s_j) = 4m+2j; 1 \le j \le n$
$f(t) = 4m+2n+2$
$f(t_j) = 4m+4n+4-2j; 1 \le j \le n$

Let f^* be the induced edge labeling of f. Then
$f^*(uu_i) = 2m+2n+3-i; 1 \le i \le m$
$f^*(uv) = m+2n+2$
$f^*(vv_i) = m+2n+2-i; 1 \le i \le m$
$f^*(ss_j) = 2n+2-j; 1 \le j \le n$
$f^*(st) = n+1$
$f^*(tt_j) = n+1-j; 1 \le j \le n$

The induced edge labels are distinct and are 1, 2, 3,…,$2m+2n+2$. Hence the theorem.

2.4 Theorem: The graph $B(m,n) \cup B(n,m)$is skolem mean, relaxed skolem mean and skolem difference mean for all $m, n \ge 1$.

Proof: Let G be the graph $B(m,n) \cup B(n,m)$. Let $V(G) = \{u, v, u_i, v_j, s, t, s_j, t_i; 1 \le i \le m, 1 \le j \le n\}$ and $E(G) = \{uu_i, uv, vv_j, ss_j, st, tt_i; 1 \le i \le m, 1 \le j \le n\}$.The graph G has $2m+2n+4$ vertices and $2m+2n+2$ edges.
$(\boldsymbol{i})$: Let the function f: $V(G) \to \{1,2,3,\ldots,2\mathrm{m}+2n+4\}$be defined as follows.

$f(u) = 2\mathrm{m}+2n+4$
$f(u_i) = 2m+2n+5-2i; 1 \le i \le m$
$f(v) = 2n+3$
$f(v_i) = 2m+2n+4-2i; 1 \le i \le n$
$f(s) = 2m+2$
$f(s_j) = 2n+3-2j; 1 \le j \le n$
$f(t) = 1$
$f(t_j) = 2m+2-2j; 1 \le j \le m$

Let f^* be the induced edge labeling of *f*. Then
$f^*(uu_i) = 2m + 2n + 5 - i; 1 \leq i \leq m$
$f^*(uv) = m + 2n + 4$
$f^*(vv_i) = m + 2n + 4 - j; 1 \leq j \leq n$
$f^*(ss_j) = m + n + 3 - j; 1 \leq j \leq n$
$f^*(st) = m + 2$
$f^*(tt_j) = m + 2 - j; 1 \leq i \leq m$

The induced edge labels are 2, *3, 4,…, m+n+2*, $m + n + 4, m + n + 5, \ldots, 2n + 2m + 4$ which are distinct.
.
$(\boldsymbol{ii})$: Let *f:* $V(G) \rightarrow \{1,2,3, \ldots, 2m + 2n + 5\}$ be defined as follows:

$f(u) = 2m + 2n + 5$
$f(u_i) = 2m + 2n + 6 - 2i; 1 \leq i \leq m$
$f(v) = 2n + 4$
$f(v_j) = 2m + 2n + 5 - 2j; 1 \leq j \leq m$
$f(s) = 2m + 3$
$f(u_i) = 2n + 4 - 2j; 1 \leq j \leq n$
$f(t) = 2$
$f(t_j) = 2m + 3 - 2j; 1 \leq j \leq m$

Let *f ** be the induced edge labeling of *f*. Then
$f^*(uu_i) = 2m + 2n + 6 - i; 1 \leq i \leq m$
$f^*(uv) = m + 2n + 5$
$f^*(vv_j) = m + 2n + 5 - j; 1 \leq j \leq n$
$f^*(ss_j) = m + n + 4 - j; 1 \leq j \leq n$
$f^*(st) = m + 3$
$f^*(tt_j) = m + 3 - j; 1 \leq j \leq m$

The induced edge labels are distinct and are 3, 4, 5,…, m+n+3, $m + n + 5, m + n + 6, \ldots, 2m + 2n + 5$.

$(\boldsymbol{iii})$: Let *f:* $V(G) \rightarrow \{1,2,3, \ldots, 4m + 4n + 6\}$ be defined as follows:

$f(u) = 4m + 4n + 5$
$f(u_i) = 2i - 1; 1 \leq i \leq m$
$f(v) = 2m + 1$
$f(v_j) = 4m + 4n + 5 - 2j; 1 \leq j \leq n$
$f(s) = 4m + 4n + 4$
$f(s_j) = 2m + 2n + 2j; 1 \leq j \leq n$
$f(t) = 2m + 4n + 2$
$f(t_i) = 4m + 4n + 4 - 2i; 1 \leq i \leq m$

Let *f ** be the induced edge labeling of *f*. Then
$f^*(uu_i) = 2m + 2n + 3 - i; 1 \leq i \leq m$
$f^*(uv) = m + 2n + 2$
$f^*(vv_j) = m + 2n + 2 - j; 1 \leq j \leq n$
$f^*(ss_j) = m + n + 2 - j; 1 \leq j \leq n$
$f^*(st) = m + 1$
$f^*(tt_i) = m + 1 - i; 1 \leq i \leq n$

The induced edge labels are distinct and are 1, 2, 3,…, $2m + 2n + 2$. Hence the theorem.

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