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CERTAIN UNIFIED FRACTIONAL INTEGRALS AND DERIVATIVES PERTAINING TO A PRODUCT OF SPECIAL FUNCTIONS

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ABSTRACT

Saigo and Maeda (Transform Methods and Special Functions, Varna, Bulgaria, pp. 386-400, 1996) introduced and investigated certain generalized fractional integral and derivative operators involving the Appell function F_3. Here we aim at presenting four unified fractional integral and derivative formulas of Saigo and Maeda type, which are involved in a product of Aleph functions. On account of the most general nature of functions involved herein, a large number of (known or new) results on fractional integral and derivative involving simpler functions can be obtained.

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***Key Words**: Generalized fractional calculus operators, Aleph ($\aleph$)-function.*

1. INTRODUCTION

Fractional calculus deals with the investigations of integrals and derivatives of arbitrary orders. A remarkably large number of works on the subject of fractional calculus have given interesting account of the theory and applications of fractional calculus operators in many different areas of mathematical analysis (see, for very recent works [2, 3, 4, 13].

The fractional integral operators, especially, involving various special functions have found significant importance and applications in various fields of applied mathematics. Since last five decades, a number of researchers like Love [5], Srivastava and Saxena [16], Saxena *et al.* [11,12], Saigo [7] and Samko *et al.* [9] and so on have studied, in depth, certain properties, applications and different extensions of various hypergeometric operators of fractional integration.

Throughout this paper, let C, R, R_+, Z_0^- and N denote the sets of complex numbers, real numbers, positive real numbers, nonpositive integers and positive integers respectively, and $N_0 = N \cup \{0\}$.

Let $\alpha, \alpha', \beta, \beta', \gamma \in C$. Then the fractional integral operators $I_{0,x}^{\alpha,\alpha',\beta.\beta',\gamma}$ and $I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}$ of a function f(x) are defined, for $R(\gamma) > 0$, as follows (see Saigo and Maeda [8]):

$$(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, dt \tag{1.1}$$

and

$$(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t) dt, \tag{1.2}$$

where F_3 is one of the Appell series defined by (see, e.g., [15, p.23, Eq.(4)])

$$F_3(a, a', b, b'; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad (\max\{|x|, |y|\}) < 1) \tag{1.3}$$

and $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in C$) by (see [14]):

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$$(\lambda)_n = \begin{cases} 1 & (n=0), \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in N) \end{cases}$$
$$= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \qquad (\lambda \in C \setminus Z_0^-), \tag{1.4}$$

here Γ denotes the familiar gamma function. These operators reduce to the following simpler fractional integral operators (see [7]):

$$I_{0,x}^{\alpha,0,\beta,\beta',\gamma} f(x) = I_{0,x}^{\gamma,\alpha-\gamma,-\beta} f(x) \quad (\gamma \in C) \tag{1.5}$$

and

$$I_{x,\infty}^{\alpha,0,\beta,\beta',\gamma} f(x) = I_{x,\infty}^{\gamma,\alpha-\gamma,-\beta} f(x) \quad (\gamma \in C) \tag{1.6}$$

Let $\alpha, \alpha', \beta, \beta', \gamma \in C$ with $R(\gamma) > 0$ and $x \in R_+$. Then the generalized fractional differentiation operators involving the Appell function F_3 in the kernel are defined as follows (see [8]):

$$(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = (I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f)(x) \tag{1.7}$$

$$= \left(\frac{d}{dx}\right)^n (I_{0+}^{-\alpha',-\alpha,-\beta'+n,-\beta,-\gamma+n} f)(x) \tag{1.8}$$

Here $(R(\gamma) > 0 \ ; \ n = [R(\gamma)] + 1)$

$$= \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dx}\right)^n (x^{\alpha'}) \int_0^x (x-t)^{n-\gamma-1} t^{\alpha}$$
$$\times F_3\left(-\alpha', -\alpha, n-\beta', -\beta, n-\gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, dt \tag{1.9}$$

and

$$(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = (I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f)(x) \tag{1.10}$$

$$= \left(-\frac{d}{dx}\right)^n (I_{-}^{-\alpha',-\alpha,-\beta',-\beta+n,-\gamma+n} f)(x) \tag{1.11}$$

Here $(R(\gamma) > 0 \ ; \ n = [R(\gamma)] + 1)$

$$= \frac{1}{\Gamma(n-\gamma)} \left(-\frac{d}{dx}\right)^n (x^{\alpha}) \int_x^\infty (t-x)^{n-\gamma-1} t^{\alpha'}$$
$$\times F_3\left(-\alpha', -\alpha, -\beta', n-\beta, n-\gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\, dt. \tag{1.12}$$

These operators reduce to the Saigo derivative operators as follows (see [8,7]):

$$(D_{0+}^{0,\alpha',\beta,\beta',\gamma} f)(x) = (D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f)(x) \qquad (R(\gamma) > 0) \tag{1.13}$$

and

$$(D_{-}^{0,\alpha',\beta,\beta',\gamma} f)(x) = (D_{-}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f)(x) \qquad (R(\gamma) > 0). \tag{1.14}$$

Furthermore, we also have (see [8, p.394, Eqs. (4.18) and (4.19)])

$$I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} = \Gamma\left[\begin{matrix} \rho, \rho+\gamma-\alpha-\alpha'-\beta, \rho+\beta'-\alpha' \\ \rho+\gamma-\alpha-\alpha', \rho+\gamma-\alpha'-\beta, \rho+\beta' \end{matrix}\right] x^{\rho-\alpha-\alpha'+\gamma-1}$$

Here $\quad (R(\gamma) > 0, R(\rho) > \max\{0, R(\alpha+\alpha'+\beta-\gamma), R(\alpha'-\beta')\})$ (1.15)

and $\quad I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} = \Gamma\left[\begin{matrix} 1+\alpha+\alpha'-\gamma-\rho, 1+\alpha+\beta'-\gamma-\rho, 1-\beta-\rho \\ 1-\rho, 1+\alpha+\alpha'+\beta'-\gamma-\rho, 1+\alpha-\beta-\rho \end{matrix}\right] x^{\rho-\alpha-\alpha'+\gamma-1}$

Here $$(R(\gamma)>0, R(\rho)<1+\min\{R(-\beta), R(\alpha+\alpha'-\gamma), R(\alpha+\beta'-\gamma)\}), \tag{1.16}$$

where the notation Γ[…] represents the fraction of gamma functions, for example,

$$\Gamma\begin{bmatrix}\alpha,\beta,\gamma\\ a,b,c\end{bmatrix}=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(a)\Gamma((b)\Gamma(c)}$$

A lot of research work has been recently come up on the study and development of a function that is more general than I-function and familiar H-function, known as the Aleph (ℵ)-function. The Aleph (ℵ)-function, introduced by Südland *et al.* [17], however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integrals [see also, 18]:

$$\aleph[z]=\aleph^{m,n}_{p_i,q_i,\tau_i;r}[z]=\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r}\\ (b_j,B_j)_{1,m},[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)z^{-\xi}\,d\xi, \tag{1.17}$$

for all z ≠ 0, where $i=\sqrt{-1}$ and

$$\Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j+B_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j-A_j\xi)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}\xi)}, \tag{1.18}$$

the integration path $L=L_{i\gamma\infty}, \gamma\in R,$ extends from γ–i∞ to γ+i∞, and is such that the poles, assumed to be simple of $\Gamma(1-a_j-A_j\xi), j=1,...,n$ do not coincide with the poles of $\Gamma(b_j+B_j\xi), j=1,...,m$ the parameter p_i, q_i are non-negative integers satisfying $0 \le n \le p_i$, $1 \le m \le q_i$, $\tau_i > 0$ for i = 1,…,r. The parameter $A_j, B_j, A_{ji}, B_{ji}>0$ and $a_j, b_j, a_{ji}, b_{ji}\in C.$ The empty product in (1.18) is interpreted as unity. The existence conditions for the defining integral (1.17) are given below:

$$\phi_i>0, |\arg(z)|<\frac{\pi}{2}\phi_i, i=1,...,r \tag{1.19}$$

$$\phi_i\ge 0, |\arg(z)|<\frac{\pi}{2}\phi_i \text{ and } R\{\zeta_i\}+1<0, \tag{1.20}$$

where

$$\phi_i=\sum_{j=1}^{n}A_j+\sum_{j=1}^{m}B_j-\tau_i\left(\sum_{j=n+1}^{p_i}A_{ji}+\sum_{j=m+1}^{q_i}B_{ji}\right), \tag{1.21}$$

and

$$\zeta_i=\sum_{j=1}^{m}b_j-\sum_{j=1}^{n}a_j+\tau_i\left(\sum_{j=m+1}^{q_i}b_{ji}-\sum_{j=n+1}^{p_i}a_{ji}\right)+\frac{1}{2}(p_i-q_i), i=1,...,r \tag{1.22}$$

for detailed account of the Aleph (ℵ)-function see südland *et al.* [17,18]. Series representation of Aleph (ℵ)-function is given by [1]:

$$\aleph^{m',n'}_{p_i',q_i',\tau_i';r'}[u]=\aleph^{m',n'}_{p_i',q_i',\tau_i';r'}\left[u\left|\begin{matrix}(a_j',A_j')_{1,n'},[\tau_i'(a_{ji}',A_{ji}')]_{n'+1,p_i';r'}\\ (b_j',B_j')_{1,m'},[\tau_i'(b_{ji}',B_{ji}')]_{m'+1,q_i';r'}\end{matrix}\right.\right]$$

$$=\sum_{h=1}^{m'}\sum_{k=0}^{\infty}\frac{(-1)^k\phi(s)}{B_h'\,k!}u^{-s} \tag{1.23}$$

where

$$\phi(s)=\frac{\prod_{j=1}^{m'}\Gamma(b_j'+B_j's)\prod_{j=1}^{n'}\Gamma(1-a_j-A_j's)}{\sum_{i=1}^{r}\tau_i'\left\{\prod_{j=m'+1}^{q_i'}\Gamma(1-b_{ji}'-B_{ji}'s)\prod_{j=n'+1}^{p_i'}\Gamma(a_{ji}'+A_{ji}'s)\right.}, \tag{1.24}$$

and

$$s=\eta_{h,k}=\frac{b_h'+k}{B_h'},\ p_i'<q_i',\ |u|<1. \tag{1.25}$$

2. FRACTIONAL INTEGRAL FORMULAS

Here we establish two fractional integration formulas for the product of two $\aleph$-function defined by (1.17) and (1.23).

Theorem 1: Suppose that $\alpha,\alpha',\beta,\beta',\gamma,z,y,\rho\in C, R(\gamma)>0,\mu>0,\lambda\in R_+$ and

$$R(\rho)+\mu\min_{1\le j\le m}\frac{R(b_j)}{B_j}>\max\{0,R(\alpha+\alpha'+\beta-\gamma),R(\alpha'-\beta')\}.$$

If the conditions given in (1.19) – (1.22) are satisfied, then the following relation holds true

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}\aleph_{p_i',q_i',\tau_i';r'}^{m',n'}\left[yt^{\lambda}\left|\begin{matrix}(a_j',A_j')_{1,n'},[\tau_i'(a_{ji}',A_{ji}')_{n'+1,p_i';r'}\\(b_j',B_j')_{1,m'},[\tau_i'(b_{ji}',B_{ji}')]_{m'+1,q_i';r'}\end{matrix}\right.\right]\right.\right.$$

$$\left.\left.\times\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[z\,t^{\mu}\left|\begin{matrix}(a_j,A_j)_{1,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right]\right)\right\}(x)$$

$$=x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{h=1}^{m'}\sum_{k=0}^{\infty}\frac{(-1)^k\phi(s)}{B_h'\,k!}y^{-s}x^{-\lambda s}$$

$$\times\aleph_{p_i+3,q_i+3,\tau_i;r}^{m,n+3}\left[z\,x^{\mu}\left|\begin{matrix}(1-\rho+\lambda s,\mu),(1-\rho+\alpha+\alpha'+\beta-\gamma+\lambda s,\mu), & (1-\rho+\alpha'-\beta'+\lambda s,\mu),(a_j,A_j)_{1,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r}, & (1-\rho+\alpha+\alpha'-\gamma+\lambda s,\mu),(1-\rho+\alpha'+\beta-\gamma+\lambda s,\mu),(1-\rho-\beta'+\lambda s,\mu)\end{matrix}\right.\right]. \tag{2.1}$$

Proof: In order to prove (2.1), first expressing the Aleph ($\aleph$)-function occurring on its left-hand side as the series given by (1.23), replacing the $\aleph$-function in terms of Mellin-Barnes contour integral with the help of (1.17), interchanging the order of summation and integration (which is permissible under the conditions of validity stated above), we obtain the following form (say I):

$$I=\sum_{h=1}^{m'}\sum_{k=0}^{\infty}\frac{(-1)^k\phi(s)}{B_h'\,k!}y^{-s}\left\{\frac{1}{2\pi i}\int_L\Omega_{p_i,q_i,\tau_i;r}^{m,n}(\xi)z^{-\xi}\times(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-\lambda s-\mu\xi-1})(x)d\xi\right\}$$

$$=\sum_{h=1}^{m'}\sum_{k=0}^{\infty}\frac{(-1)^k\phi(s)}{B_h'\,k!}y^{-s}\frac{1}{2\pi i}\int_L x^{\rho-\alpha-\alpha'+\gamma-\lambda s-1}(zx^{\mu})^{-\xi}$$

$$\times\frac{\prod_{j=1}^{m}\Gamma(b_j+B_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j-A_j\xi)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}\xi)}$$

$$\times \frac{\Gamma(\rho-\lambda s-\mu\xi)\Gamma(\rho-\lambda s-\mu\xi+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho-\lambda s-\mu\xi+\beta'-\alpha')}{\Gamma(\rho-\lambda s-\mu\xi+\gamma-\alpha-\alpha')\Gamma(\rho-\lambda s-\mu\xi+\gamma-\alpha'-\beta)\Gamma(\rho-\lambda s-\mu\xi+\beta')} d\xi.$$

Finally, re-interpreting the Mellin-Barnes contour integral in terms of the $\aleph$-function, we are led to the right-hand side of (2.1). This completes proof of Theorem 1.

Theorem 2: Suppose that $\alpha, \alpha', \beta, \beta', \gamma, z, y, \rho \in C, R(\gamma) > 0, \mu > 0, \lambda \in R_+$ and

$$R(\rho) + \mu \max_{1\le i\le n}\left(\frac{R(a_i)-1}{A_i}\right) < 1 + \min\{R(-\beta), R(\alpha+\alpha'-\gamma), R(\alpha+\beta'-\gamma)\}.$$

If the conditions given in (1.19) – (1.22) are satisfied, then the following relation holds true

$$\left\{ I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\rho-1} \aleph^{m',n'}_{p_i',q_i',\tau_i';r'} \left[y t^{\lambda} \left| \begin{matrix} (a_j',A_j')_{1,n'}, [\tau_i'(a_{ji}',A_{ji}')]_{n'+1,p_i';r'} \\ (b_j',B_j')_{1,m'}, [\tau_i'(b_{ji}',B_{ji}')]_{m'+1,q_i';r'} \end{matrix} \right. \right] \right. \right.$$

$$\left. \left. \times \aleph^{m,n}_{p_i,q_i,\tau_i;r} \left[z t^{\mu} \left| \begin{matrix} (a_j,A_j)_{1,n}, [\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r} \\ (b_j,B_j)_{1,m}, [\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r} \end{matrix} \right. \right] \right) \right\}(x)$$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \sum_{h=1}^{m'} \sum_{k=0}^{\infty} \frac{(-1)^k \phi(s)}{B_h' \, k!} y^{-s} x^{-\lambda s}$$

$$\times \aleph^{m+3,n}_{p_i+3,q_i+3,\tau_i;r} \left[z x^{\mu} \left| \begin{matrix} (a_j,A_j)_{1,n}, [\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r} & (1-\rho+\lambda s,\mu),(1+\alpha-\beta-\rho+\lambda s,\mu),(1+\alpha+\alpha'+\beta'-\gamma-\rho+\lambda s,\mu \\ (1+\alpha+\alpha'-\gamma-\rho+\lambda s,\mu),(1+\alpha+\beta'-\gamma-\rho+\lambda s,\mu), & (1-\beta-\rho+\lambda s,\mu),(b_j,B_j)_{1,m},[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r} \end{matrix} \right. \right]. \tag{2.2}$$

Proof: A similar argument as in proving Theorem 1 will establish the result (2.2).

3. FRACTIONAL DERIVATIVE FORMULAS

Here we establish two fractional derivative formulas for the product of two $\aleph$-function defined by (1.17) and (1.23).

Theorem 3: Suppose that $\alpha, \alpha', \beta, \beta', \gamma, z, y, \rho \in C, R(\gamma) > 0, \mu > 0, \lambda \in R_+$ and

$$R(\rho) + \mu \min_{1\le j\le m}\left(\frac{R(b_j)}{B_j}\right) + \max\{0, R(\alpha-\beta), R(\alpha'+\beta'+\alpha-\gamma)\} > 0.$$

If the conditions given in (1.19) – (1.22) are satisfied, then the following relation holds true:

$$\left\{ D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\rho-1} \aleph^{m',n'}_{p_i',q_i',\tau_i';r'} \left[y t^{\lambda} \left| \begin{matrix} (a_j',A_j')_{1,n'}, [\tau_i'(a_{ji}',A_{ji}')]_{n'+1,p_i';r'} \\ (b_j',B_j')_{1,m'}, [\tau_i'(b_{ji}',B_{ji}')]_{m'+1,q_i';r'} \end{matrix} \right. \right] \right. \right.$$

$$= x^{\rho+\alpha+\alpha'-\gamma-1} \sum_{h=1}^{m'} \sum_{k=0}^{\infty} \frac{(-1)^k \phi(s)}{B_h' \, k!} y^{-s} x^{-\lambda s}$$

$$\times \aleph^{m,n+3}_{p_i+3,q_i+3,\tau_i;r} \left[z x^{\mu} \left| \begin{matrix} (1-\rho+\lambda s,,\mu),(1-\rho-\alpha+\beta+\lambda s,\mu), & (1-\rho-\alpha-\alpha'-\beta'+\gamma+\lambda s,\mu),(a_j,A_j)_{1,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r} \\ (b_j,B_j)_{1,m},[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r}, & (1-\rho-\alpha-\alpha'+\gamma+\lambda s,\mu),(1-\rho-\alpha-\beta'+\gamma+\lambda s,\mu),(1-\rho+\beta+\lambda s,\mu) \end{matrix} \right. \right]. \tag{3.1}$$

Proof: In order to prove (3.1), first expressing the Aleph (ℵ)-function occurring on its left-hand side as the series given by (1.23), replacing the ℵ-function in terms of Mellin-Barnes contour integral with the help of (1.17), interchanging the order of summation and integration, we obtain the following form (say I):

$$I=\sum_{h=1}^{m'}\sum_{k=0}^{\infty}\frac{(-1)^k\phi(s)}{B_h' \, k!}y^{-s}\left\{\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)z^{-\xi}\times(D^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}t^{\rho-\lambda s-\mu\xi-1})(x)d\xi\right\}$$

$$=\left(\frac{d}{dx}\right)^n\sum_{h=1}^{m'}\sum_{k=0}^{\infty}\frac{(-1)^k\phi(s)}{B_h' \, k!}y^{-s}\frac{1}{2\pi i}\int_L x^{\rho+\alpha+\alpha'-\gamma-\lambda s+n-1}(zx^{\mu})^{-\xi}$$

$$\times\frac{\prod_{j=1}^{m}\Gamma(b_j+B_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j-A_j\xi)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}\xi)}$$

$$\times\frac{\Gamma(\rho-\lambda s-\mu\xi)\Gamma(\rho-\lambda s-\mu\xi-\gamma+\alpha'+\alpha+\beta')\Gamma(\rho-\lambda s-\mu\xi-\beta+\alpha)}{\Gamma(\rho-\lambda s-\mu\xi-\gamma+\alpha'+\alpha)\Gamma(\rho-\lambda s-\mu\xi-\gamma+\alpha+\beta')\Gamma(\rho-\lambda s-\mu\xi-\beta)}d\xi.$$

Here $n=[-R(\gamma)]+1,$ and by using

$$\frac{d^{\ell}}{dx^{\ell}}x^m=\frac{\Gamma(m+1)}{\Gamma(m-\ell+1)}x^{m-\ell}\quad(m,\ell\in N_0;m\ge\ell), \tag{3.2}$$

and re-interpreting the Mellin-Barnes contour integral in terms of the ℵ-function, we are led to the right-hand side of (3.1). This complets the proof of Theorem 3.

Theorem 4: Suppose that $\alpha,\alpha',\beta,\beta',\gamma,z,y,\rho\in C, R(\gamma)>0,\mu>0,\lambda\in R_+$ and

$$R(\rho)+\mu\max_{1\le i\le m}\left(\frac{R(a_i)-1}{A_i}\right)<1+\min\{R(-\beta),R(\gamma-\alpha-\alpha'-\ell),R(-\alpha'-\beta+\gamma)\},$$

Here $\ell=[R(\gamma)]+1.$

If the conditions given in (1.19) – (1.22) are satisfied, then the following relation holds true:

$$\left\{D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}\aleph^{m',n'}_{p_i',q_i',\tau_i';r'}\left[yt^{\lambda}\left|\begin{matrix}(a_j',A_j')_{1,n'},[\tau_i'(a_{ji}',A_{ji}')]_{n'+1,p_i';r'}\\(b_j',B_j')_{1,m'},[\tau_i'(b_{ji}',B_{ji}')]_{m'+1,q_i';r'}\end{matrix}\right.\right]\right.\right.$$

$$\left.\left.\times\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z\,t^{\mu}\left|\begin{matrix}(a_j,A_j)_{1,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right]\right)\right\}(x)$$

$$=(-1)^{[R(\gamma)]+1}x^{\rho+\alpha+\alpha'-\gamma-1}\sum_{h=1}^{m'}\sum_{k=0}^{\infty}\frac{(-1)^k\phi(s)}{B_h' \, k!}y^{-s}x^{-\lambda s}$$

$$\times\aleph^{m+3}_{p_i+3,q_i+3,\tau_i;r}\left[z\,x^{\mu}\left|\begin{matrix}(a_j,A_j)_{1,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r}, & (1-\rho+\lambda s,\mu),(1-\alpha'+\beta'-\rho+\lambda s,\mu),(1-\alpha-\alpha'-\beta+\gamma-\rho+\lambda s,\mu\\(1-\alpha-\alpha'+\gamma-\rho+\lambda s,\mu),(1+\beta'-\rho+\lambda s,\mu), & (1-\alpha'-\beta+\gamma-\rho+\lambda s,\mu),(b_j,B_j)_{1,m},[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right]. \tag{3.3}$$

Proof: In order to prove (3.3), first expressing the Aleph (ℵ)-function occurring on its left-hand side as the series given by (1.23), replacing the ℵ-function in terms of Mellin-Barnes contour integral with the help of (1.17), interchanging the order of summation and integration, we obtain the following form (say I):

$$I=\sum_{h=1}^{m'}\sum_{k=0}^{\infty}\frac{(-1)^k\,\phi(s)}{B_h'\,k!}y^{-s}\left\{\frac{1}{2\pi i}\int_L \Omega_{p_i,q_i,\tau_i;r}^{m,n}(\xi)z^{-\xi}\times(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-\lambda s-\mu\xi-1})(x)\,d\xi\right\}$$

$$=\left(-\frac{d}{dx}\right)^{\ell}\sum_{h=1}^{m'}\sum_{k=0}^{\infty}\frac{(-1)^k\,\phi(s)}{B_h'\,k!}y^{-s}\frac{1}{2\pi i}\int_L x^{\rho+\alpha+\alpha'-\gamma+\ell-\lambda s-1}(zx^{\mu})^{-\xi}$$

$$\times\frac{\prod_{j=1}^{m}\Gamma(b_j+B_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j-A_j\xi)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}\xi)}$$

$$\times\frac{\Gamma(1-\alpha-\alpha'+\gamma-\ell-\rho+\lambda s+\mu\xi)\Gamma(1+\beta'-\rho+\lambda s+\mu\xi)\Gamma(1-\alpha'-\beta+\gamma-\rho+\lambda s+\mu\xi)}{\Gamma(1-\rho+\lambda s+\mu\xi)\Gamma(1-\alpha'+\beta'-\rho+\lambda s+\mu\xi)\Gamma(1-\alpha-\alpha'-\beta+\gamma+\lambda s+\mu\xi)}d\xi.$$

Here $\ell=[R(\gamma)]+1$, and by using (3.2) in the above expression, and re-interpreting the Mellin-Barnes contour integral in terms of the ℵ-function, we are led to the right-hand side of (3.3). This completes the proof of Theorem 4.

4. CONCLUSION

In the present paper, we have given the four theorems of generalized fractional integral and derivative operators given by Saigo-Maeda. The theorems have been developed in terms of the product of two ℵ-function. Most of the given results have been put in a compact form, avoiding the occurrence of infinite series and thus making them useful in applications.

In view of the generality of the ℵ-function, on specializing the various parameters, we can obtain from our results, several results involving a remarkably wide variety of useful functions, which are expressible in terms of the I-function [10], the H-function [6], the Mittag-Leffler function [6, p.25, Eq.(1.137)], the generalized Wright hypergeometric function [6, p.25, Eq. (1.140)], the generalized Bessel-Maitland function [6, p.25, Eq. (1.139)] and their various special cases. Thus, the results presented in this paper would at once yield a very large number of results involving a large variety of special functions occurring in the problems of science, engineering and mathematical physics etc.

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