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THE ZEROS OF CLASS OF ANALYTIC FUNCTIONS

P. RAMULU*[1], G. L. REDDY[2], C. GANGADHAR[3]

[1]Department of Mathematics,
Govt. Degree College, Wanaparthy, Mahabubnagar, Telangana, India-509103.

[2]School of Mathematics and Statistics,
University of Hyderabad, Telangana, India-500046.

[3]School of Mathematics and Statistics,
University of Hyderabad, Telangana, India-500046.

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ABSTRACT

In this paper we are finding the number of zeros of class of analytic functions, by considering more general coefficient conditions. As special cases the extended results yield much simpler expressions for the upper bounds of zeros of those of the existing results.

Mathematics Subject Classification: *30C10, 30C15.*

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1. INTRODUCTION

Let P(z)= $\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $0 < a_0 \le a_1 \le a_2 \le \cdots \le a_n$ then all the zeros of P(z) lie in |z|≤1. Finding approximate zeros of a polynomial related to analytic function is an important and well-studied problem. To find the number of zeros of a polynomial related to an analytic function has already been proved by Aziz and Mohamad [3], by extending Eneström-Kakeya theorem [1-2], is the following.

Theorem A: Let $F(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ *be an analytic function in* $|z| \le t$. *If* $a_i > 0$ *and* $a_i - t a_i \ge 0$, *for* $i = 1, 2, 3, \ldots$ *then F(z) does not vanish in* $|z| \le t$.

Here we establish the following results which are more interesting.

Theorem 1: Let $F(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ *be analytic function in* $|z| \le 1$ *such that*
$|\arg(a_i) - \beta| \le \alpha \le \frac{\pi}{2}, i = 0,1,2,\ldots,n,\ldots$ *for some real* $\beta, a_0 \ne 0, 0 < \tau \le 1, k_1 \ge 0, k_2 \ge 0$ *and*
$$|a_0| \le |a_1| \le \cdots \le |a_{m-1}| \le k_1|a_m| \ge |a_{m+1}| \ge \cdots \ge |a_{l-1}| \ge \tau|a_l| \le |a_{l+1}| \le |a_{l+2}| \le \cdots \le k_2 a_n \ge \cdots$$
for some m, l, 0 ≤ m ≤ l ≤ n then the number of zeros of F(z) in $|z| \le r$, $0 < r < 1$ does not exceed

$$\frac{1}{\log\frac{1}{r}} \log \frac{2[X + \sin\alpha \sum_{i=0}^{\infty}|a_i| - [|a_m| + |a_n|](1 + \sin\alpha) + |a_l|(1 - \sin\alpha)]}{|a_0|}.$$
where $X = (k_1|a_m| + k_2|a_n|)(1 + \cos\alpha + \sin\alpha) - \tau|a_l|(1 + \cos\alpha - \sin\alpha)$.

Corollary 1: Let $F(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ *be analytic function in* $|z| \le 1$ *such that*
$|\arg(a_i) - \beta| \le \alpha \le \frac{\pi}{2}, i = 0,1,2,\ldots,n,\ldots$ *for some real* $\beta, a_0 \ne 0$ *and*
$|a_0| \le |a_1| \ldots \le |a_{m-1}| \le |a_m| \ge |a_{m+1}| \ge \ldots \ge |a_{l-1}| \ge |a_l| \le |a_{l+1}| \le \cdots \le a_n \ldots$ for some m, l, 0 ≤ m ≤ l ≤ n then the number of zeros of F(z) in $|z| \le r$, $0 < r < 1$ does not exceed

Corresponding Author: P. Ramulu[1], [1]Department of Mathematics,*
Govt. Degree College, Wanaparthy, Mahabubnagar, Telangana, India-509103.

$$\frac{1}{\log\frac{1}{r}}\log\frac{2[(|a_m|-|a_l|+|a_n|)\cos\alpha+\sin\alpha\sum_{i=0}^{\infty}|a_i|]}{|a_0|}.$$

Remark 1: By taking $\tau=k_1=k_2=1$ in theorem 1, then it reduces to Corollary 1.

Theorem 2: Let $F(z)=\sum_{i=0}^{\infty}a_iz^i\not\equiv 0$ *be analytic function in* $|z|\le 1$ *such that*

$$|\arg(a_i)-\beta|\le\alpha\le\frac{\pi}{2}, i=0,1,2,\dots,n,\dots \text{ for some real } \beta, a_0\ne 0, k\ge 0 \text{ and } k|a_0|\ge|a_1|\ge\cdots\ge|a_{m-1}|\ge|a_m| \le|a_{m+1}|\le\cdots\le|a_{n-1}|\le|a_n|\ge\cdots$$

for some m, $0\le m\le n$ then the number of zeros of F(z) in $|z|\le r$, $0<r<1$ does not exceed

$$\frac{1}{\log\frac{1}{r}}\log\frac{k|a_0|(\cos\alpha+\sin\alpha+1)+2(|a_n|-|a_m|)\cos\alpha+2\sin\alpha\sum_{i=1}^{\infty}|a_i|}{|a_0|}.$$

Corollary 2: Let $F(z)=\sum_{i=0}^{\infty}a_iz^i\not\equiv 0$ *be analytic function in* $|z|\le 1$ *such that*

$$|\arg(a_i)-\beta|\le\alpha\le\frac{\pi}{2}, i=0,1,2,\dots,n,\dots \text{ for some real } \beta, a_0\ne 0 \text{ and } |a_0|\ge|a_1|\ge\cdots\ge|a_{m-1}|\ge|a_m| \le|a_{m+1}|\le\cdots\le|a_{n-1}|\le|a_n|\ge\cdots$$

for some m, $0\le m\le n$, then the number of zeros of F(z) in $|z|\le\frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\log\frac{|a_0|(\cos\alpha+\sin\alpha+1)+2(|a_n|-|a_m|)\cos\alpha+2\sin\alpha\sum_{i=1}^{\infty}|a_i|}{|a_0|}.$$

Remark 2: By taking $k=1$ *and* $r=\frac{1}{2}$ in theorem 2, then it reduces to Corollary 2.

Theorem 3: Let $F(z)=\sum_{i=0}^{\infty}a_iz^i\not\equiv 0$ *be analytic function in* $|z|\le 1$ *such that*

$$Re(a_i)=\alpha_i, Im(a_i)=\beta_i, \text{for } i=0,1,\dots,n,$$
$$a_0\ne 0, 0<\vartheta\le 1, 0<\tau\le 1, k_1\ge 0, k_2\ge 0, k\ge 0 \text{ and}$$
$$\vartheta\alpha_0\le\alpha_1\le\cdots\le\alpha_{m-1}\le k_1\alpha_m\ge\alpha_{m+1}\ge\cdots\ge\alpha_{l-1}\ge\tau\alpha_l\le\alpha_{l+1}\le\cdots\le\alpha_{n-1}\le k_2\alpha_n\ge\alpha_{n+1}\ge\alpha_{n+2}\ge\cdots,$$
$$k\beta_0\ge\beta_1\ge\cdots\ge\beta_{n-1}\ge\beta_n\ge\cdots$$

for some m, l, $0\le m\le l\le n$, then the number of zeros of F(z) in $|z|\le r$, $0<r<1$ does not exceed

$$\frac{1}{\log\frac{1}{r}}\log\frac{2[|\alpha_0|-|\alpha_m|+|\alpha_l|-|\alpha_n|+Y]}{|a_0|},$$

where $Y=k_1(|\alpha_m|+\alpha_m)-\tau(\alpha_l+|\alpha_l|)+k_2(\alpha_n+|\alpha_n|)]+\frac{1}{2}[k(|\beta_0|+\beta_0)-\vartheta(|\alpha_0|+\alpha_0)].$

Corollary 3: Let $F(z)=\sum_{i=0}^{\infty}a_iz^i\not\equiv 0$ *be analytic function in* $|z|\le 1$ *such that*

$$Re(a_i)=\alpha_i, Im(a_i)=\beta_i, \text{for } i=0,1,\dots,n, a_0\ne 0, \text{and}$$
$$\alpha_0\le\alpha_1\le\cdots\le\alpha_{m-1}\le\alpha_m\ge\alpha_{m+1}\ge\cdots\ge\alpha_{l-1}\ge\alpha_l\le\alpha_{l+1}\le\cdots\le\alpha_{n-1}\le\alpha_n\ge\cdots,$$
$$\beta_0\ge\beta_1\ge\cdots\ge\beta_{n-1}\ge\beta_n\ge\cdots$$

for some m, l, $0\le m\le l\le n$ then the number of zeros of F(z) in $|z|\le\frac{1}{2}$, does not exceed

$$\frac{1}{\log 2}\log\frac{2[\alpha_m-\alpha_l+\alpha_n]-\alpha_0+\beta_0+|\beta_0|}{|a_0|},$$

Remark 3: By taking $k_1=k_2=k=\vartheta=\tau=1$ *and* $r=\frac{1}{2}$ in theorem 3, then it reduces to Corollary 3.

Theorem 4: Let $F(z)=\sum_{i=0}^{\infty}a_iz^i\not\equiv 0$ *be analytic function in* $|z|\le 1$ *such that*

$$Re(a_i)=\alpha_i, Im(a_i)=\beta_i, \text{for } i=0,1,\dots,n,$$
$$a_0\ne 0, 0<\vartheta\le 1, 0<\tau\le 1, k_1\ge 0, k_2\ge 0, k\ge 0 \text{ and}$$
$$k_1\alpha_0\ge\alpha_1\ge\cdots\ge\alpha_{m-1}\ge\tau\alpha_m\le\alpha_{m+1}\le\cdots\le\alpha_{n-1}\le k_2\alpha_n\ge\alpha_{n+1}\ge\cdots,$$
$$\vartheta\beta_0\le\beta_1\le\cdots\le\beta_{n-1}\le k\beta_n\ge\cdots$$

for some m, l, $0\le m\le n$ then the number of zeros of F(z) in $|z|\le r$, $0<r<1$ does not exceed

$$\frac{1}{\log\frac{1}{r}}\log\frac{2[|\beta_0|+|\alpha_m|-|\alpha_n|-|\beta_n|+W]}{|a_0|},$$

where $W=k_2(\alpha_n+|\alpha_n|)+k(\beta_n+|\beta_n|)-\tau(\alpha_m+|\alpha_m|)+\frac{1}{2}[k_1(|\alpha_0|+\alpha_0)-\vartheta(|\beta_0|+\beta_0)].$

Corollary 4: Let $F(z)=\sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ *be analytic function in* $|z|\le 1$ *such that*

$$Re(a_i)=\alpha_i, Im(a_i)=\beta_i, for\ i=0,1,\dots,n, a_0\neq 0, and$$
$$\alpha_0 \ge \alpha_1 \ge \cdots \ge \alpha_{m-1} \ge \alpha_m \le \alpha_{m+1} \le \cdots \le \alpha_{n-1} \le \alpha_n \ge \alpha_{n+1} \ge \cdots,$$
$$\beta_0 \le \beta_1 \le \cdots \le \beta_{n-1} \le \beta_n \ge \cdots$$

for some m, l, $0 \le m \le n$ then the number of zeros of F(z) in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{log2} log \frac{[|\alpha_0| + |\beta_0| + \alpha_0 - \beta_0 + 2(\alpha_n + \beta_n - \alpha_m)]}{|a_0|},$$

Remark 4: By taking $k_1 = k_2 = k = \vartheta = \tau = 1\ and\ r = \frac{1}{2}$ in theorem 4, then it reduces to Corollary 4.

2. LEMMAS

Lemma 1 [4]: Let $P(z)=\sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ *be analytic function in* $|z|\le 1$ *such that*

$$|\arg(a_i)-\beta| \le \alpha \le \frac{\pi}{2}; |a_{i-1}| \le |a_i|\ for\ i=0,1,2,\dots,n,\dots$$
$$\text{then} |a_i - a_{i-1}| \le (\ |a_i| - |a_{i-1}|)cos\alpha + (\ |a_i| + |a_{i-1}|)sin\alpha.$$

Lemma 2.[5]: f f(z)is regular $f(0)\neq 0\ and\ f(z) \le M\ in\ |z| \le 1, then\ the\ nuber\ of\ zeros\ of\ f(z) in$ $|z| \le r, 0<r<1\ does\ not\ exceed\ \frac{1}{log\frac{1}{r}} log \frac{M}{|a_0|}.$

3. PROOFS OF THE THEOREMS

Proof of the Theorem 1: Let $F(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m + \cdots + a_l z^l + \cdots + a_n z^n + \cdots$ be analytic function

Let us consider the polynomial $G(z) = (1-z)F(z)$ so that

$$G(z) = (1-z)(a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m + \cdots + a_l z^l + \cdots + a_n z^n + \cdots) = a_0 + \sum_{i=1}^{\infty}(a_i - a_{i-1})z^i$$

Now for $|z| \le 1$, we have

$$\begin{aligned}
|G(z)| &\le |a_0| + \sum_{i=1}^{m-1}|a_i - a_{i-1}| + |a_m - k_1 a_m + k_1 a_m - a_{m-1}| + |a_{m+1} - k_1 a_m + k_1 a_m - a_m| + \sum_{i=m+2}^{l-1}|a_i - a_{i-1}| \\
&\quad + |a_l - \tau a_l + \tau a_l - a_{l-1}| + |a_{l+1} - \tau a_l + \tau a_l - a_l| + \sum_{i=l+2}^{n-1}|a_i - a_{i-1}| \\
&\quad + |a_n - k_2 a_n + k_2 a_n - a_{n-1}| + |a_{n+1} - k_2 a_n + k_2 a_n - a_n| + \sum_{i=n+2}^{\infty}|a_i - a_{i-1}| \\
&\le |a_0| + \sum_{i=1}^{m-1}|a_i - a_{i-1}| + 2(k_1-1)|a_m| + |k_1 a_m - a_{m-1}| + |k_1 a_m - a_{m+1}| + \sum_{i=m+2}^{l-1}|a_i - a_{i-1}| \\
&\quad + 2(1-\tau)|a_l| + |a_{l-1} - \tau a_l| + |a_{l+1} - \tau a_l| + \sum_{i=l+2}^{n-1}|a_i - a_{i-1}| + 2(k_2-1)|a_n| + |k_2 a_n - a_{n-1}| \\
&\quad + |k_2 a_n - a_{n+1}| + \sum_{i=n+2}^{\infty}|a_i - a_{i-1}|
\end{aligned}$$

By using lemma 1 we get

$$\begin{aligned}
|G(z)| &\le |a_0| + 2[(k_1-1)|a_m| + (k_2-1)|a_n| + (1-\tau)|a_l|] + \sum_{i=1}^{m-1}(|a_i| - |a_{i-1}|)cos\alpha \\
&\quad + \sum_{i=1}^{m-1}(|a_i| + |a_{i-1}|)sin\alpha + (k_1|a_m| - |a_{m-1}|)cos\alpha + (k_1|a_m| + |a_{m-1}|)sin\alpha + (k_1|a_m| - |a_{m+1}|)cos\alpha \\
&\quad + (k_1|a_m| + |a_{m+1}|)sin\alpha + \sum_{i=m+2}^{l-1}(|a_{i-1}| - |a_i|)cos\alpha + \sum_{i=m+2}^{l-1}(|a_i| + |a_{i-1}|)sin\alpha + (\ |a_{l-1}| - \tau|a_l|)cos\alpha \\
&\quad + (\ |a_{l-1}| + \tau|a_l|)sin\alpha + (\ |a_{l+1}| - \tau|a_l|)cos\alpha + (\ |a_{l+1}| + \tau|a_l|)sin\alpha + \sum_{i=l+2}^{n-1}(|a_i| - |a_{i-1}|)cos\alpha
\end{aligned}$$

$$+\sum_{i=l+2}^{n-1}(\,|a_i|+|a_{i-1}|\,)\,sin\alpha+(k_2|a_n|-|a_{n-1}|)cos\alpha+(k_2|a_n|+|a_{n-1}|)sin\alpha+(k_2|a_n|-|a_{n+1}|)cos\alpha$$

$$+(k_2|a_n|+|a_{n+1}|)sin\alpha+\sum_{i=n+2}^{\infty}(\;|a_{i-1}|-|a_i|\,)\,cos\alpha+\sum_{i=n+2}^{\infty}(\,|a_i|+|a_{i-1}|\,)\,sin\alpha$$

$$=|a_0|+2[(k_1-1)|a_m|+(k_2-1)|a_n|+(1-\tau)|a_l|]+(|a_{m-1}|-|a_0|)cos\alpha+\sum_{i=1}^{m-1}(\,|a_i|+|a_{i-1}|\,)\,sin\alpha$$

$$+(k_1|a_m|-|a_{m-1}|)cos\alpha+(k_1|a_m|+|a_{m-1}|)sin\alpha+(k_1|a_m|-|a_{m+1}|)cos\alpha$$

$$+(k_1|a_m|+|a_{m+1}|)sin\alpha+(|a_{m+1}|-|a_{l-1}|)cos\alpha+\sum_{i=m+2}^{l-1}(\,|a_i|+|a_{i-1}|\,)\,sin\alpha$$

$$+(\;|a_{l-1}|-\tau|a_l|)cos\alpha+(\;|a_{l-1}|+\tau|a_l|)sin\alpha+(\;|a_{l+1}|-\tau|a_l|)cos\alpha+(\;|a_{l+1}|+\tau|a_l|)sin\alpha$$

$$+(|a_{n-1}|-|a_{l+1}|)cos\alpha+\sum_{i=l+2}^{n-1}(\,|a_i|+|a_{i-1}|\,)\,sin\alpha+(k_2|a_n|-|a_{n-1}|)cos\alpha$$

$$+(k_2|a_n|+|a_{n-1}|)sin\alpha+(k_2|a_n|-|a_{n+1}|)cos\alpha+(k_2|a_n|+|a_{n+1}|)sin\alpha+|a_{n+1}|cos\alpha-|a_0|sin\alpha$$

$$+\sum_{i=n+2}^{\infty}(\,|a_i|+|a_{i-1}|\,)\,sin\alpha$$

$$=2(k_1|a_m|+k_2|a_n|)(1+cos\alpha+sin\alpha)-2\tau|a_l|(1+cos\alpha-sin\alpha)-|a_0|(cos\alpha+sin\alpha-1)$$

$$+2sin\alpha\sum_{i=0}^{\infty}|a_i|-2[|a_m|+|a_n|](1+sin\alpha)+2\,|a_l|\,(1-sin\alpha)$$

$$\le 2\left[X+sin\alpha\sum_{i=0}^{\infty}|a_i|-[|a_m|+|a_n|](1+sin\alpha)+\;|a_l|\,(1-sin\alpha)\right].$$

where $X=(k_1|a_m|+k_2|a_n|)(1+cos\alpha+sin\alpha)-\tau|a_l|(1+cos\alpha-sin\alpha)$

Apply lemma 2 to G(z), we get then number of zeros of G(z) in $|z|\le r, 0<r<1$ does not exceed

$$\frac{1}{log\frac{1}{r}}log\frac{2[X+sin\alpha\sum_{i=0}^{\infty}|a_i|-[|a_m|+|a_n|](1+sin\alpha)+\;|a_l|\,(1-sin\alpha)]}{|a_0|}.$$

where $X=(k_1|a_m|+k_2|a_n|)(1+cos\alpha+sin\alpha)-\tau|a_l|(1+cos\alpha-sin\alpha)$

All the number of zeros ofF(z) in $|z|\le r, 0<r<1$ is also equal to the number of zeros of G(z) in $|z|\le r, 0<r<1$

This completes the proof of theorem 1.

Proof of the Theorem 2: Let $F(z)=a_0+a_1z+a_2z^2+\cdots+\;a_mz^m+\cdots+\;a_nz^n+\cdots$ be analytic function. Let us consider the polynomial $G(z)=(1-z)F(z)$ so that

$$G(z)=\;(1-z)(a_0+a_1z+a_2z^2+\cdots+\;a_mz^m+\cdots+\;a_nz^n+\cdots)$$

$$=a_0+\sum_{i=1}^{\infty}(\,a_i-a_{i-1}\,)\,z^i$$

Now for $|z|\le 1$, we have

$$|G(z)|\le|a_0|+|\,a_0-ka_0+ka_0-a_1\,|+\sum_{i=2}^{\infty}|\,a_i-a_{i-1}\,|$$

$$\le|a_0|+(k-1)|\,a_0|+|ka_0-a_1\,|+\sum_{i=2}^{\infty}|\,a_i-a_{i-1}\,|$$

By using lemma 1 we get

$$|G(z)|\le|a_0|+(k-1)|a_0|+(k\,|a_0|-|a_1|)cos\alpha+(k|\,a_0|+|a_1|\,)\,sin\alpha+\sum_{i=2}^{\infty}(\,|a_i|-|a_{i-1}|\,)\,cos\alpha$$

$$+\sum_{i=2}^{\infty}(\,|a_i|+|a_{i-1}|\,)\,sin\alpha$$

$$|G(z)| \leq |a_0| + (k-1)|a_0| + (k\,|a_0| - |a_1|)cos\alpha + (k|\,a_0| + |a_1|\,)\,sin\alpha + \sum_{i=2}^{m}(\,|\,a_{i-1}| - |a_i|\,)\,cos\alpha$$
$$+ \sum_{i=m+1}^{n}(\,|\,a_i| \;- |a_{i-1}|\,)\,cos\alpha + \sum_{i=n+1}^{\infty}(\,|\,a_{i-1}| - |a_i|\,)\,cos\alpha + \sum_{i=2}^{\infty}(\,|\,a_i| + |a_{i-1}|\,)\,sin\alpha$$
$$\leq k|a_0| + (k\,|a_0| - |a_1|)cos\alpha + (k|\,a_0| + |a_1|\,)\,sin\alpha + (|a_1| - |a_m|)cos\alpha + (|a_n| - |a_m|)cos\alpha + |a_n|cos\alpha$$
$$+ |a_0|sin\alpha + 2sin\alpha\sum_{i=1}^{\infty}|\,a_i|$$
$$= k|a_0|(cos\alpha + sin\alpha + 1) + 2(|a_n| - |a_m|)cos\alpha + 2sin\alpha\sum_{i=1}^{\infty}|\,a_i|$$
$$\leq k|a_0|(cos\alpha + sin\alpha + 1) + 2(|a_n| - |a_m|)cos\alpha + 2sin\alpha\sum_{i=1}^{\infty}|a_i|.$$

Apply lemma 2 to G(z), we get then number of zeros of G(z) in $|z| \leq r, 0 < r < 1$ does not exceed

$$\frac{1}{log\frac{1}{r}}log\frac{k|a_0|(cos\alpha + sin\alpha + 1) + 2(|a_n| - |a_m|)cos\alpha + 2sin\alpha\sum_{i=1}^{\infty}|\,a_i|}{|a_0|}.$$

All the number of zeros ofF(z) in $|z| \leq r, 0 < r < 1$ is also equal to the number of zeros of G(z) in $|z| \leq r, 0 < r < 1$.

This completes the proof of theorem 2.

Proof of the Theorem 3:

Let $F(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m + \cdots + a_l z^l + \cdots + a_n z^n + \cdots$ be analytic function. Let us consider the polynomial $G(z) = (z-1)F(z)$ so that

$$G(z) = (z-1)(a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m + \cdots + a_l z^l + \cdots + a_n z^n + \cdots)$$
$$= -a_0 + \sum_{i=1}^{\infty}(\,a_{i-1} - a_i\,)\,z^i$$

Now for$|z| \leq 1$, we have

$$|G(z)| \leq |a_0| + \sum_{i=1}^{\infty}|a_{i-1} - a_i|$$
$$\leq |\alpha_0| + |\beta_0| + \sum_{i=1}^{\infty}|\alpha_{i-1} - \alpha_i| + \sum_{i=1}^{\infty}|\beta_{i-1} - \beta_i|$$
$$= |\alpha_0| + |\beta_0| + |\alpha_0 - \vartheta\alpha_0 + \vartheta\alpha_0 - \alpha_1| + |\beta_0 - k\beta_0 + k\beta_0 - \beta_1| + \sum_{i=2}^{m-1}|\alpha_{i-1} - \alpha_i|$$
$$+ |\alpha_{m-1} - k_1\alpha_m + k_1\alpha_m - \alpha_m| + |\alpha_m - k_1\alpha_m + k_1\alpha_m - \alpha_{m+1}| + \sum_{i=m+2}^{l-1}|\alpha_{i-1} - \alpha_i|$$
$$+ |\alpha_{l-1} - \tau\alpha_l + \tau\alpha_l - \alpha_l| + |\alpha_l - \tau\alpha_l + \tau\alpha_l - \alpha_{l+1}| + \sum_{i=l+2}^{n-1}|\alpha_{i-1} - \alpha_i|$$
$$+ |\alpha_{n-1} - k_2\alpha_n + k_2\alpha_n - \alpha_n| + |\alpha_n - k_2\alpha_n + k_2\alpha_n - \alpha_{n+1}| + \sum_{i=2}^{\infty}|\beta_{i-1} - \beta_i|$$

$$\leq |\alpha_0| + |\beta_0| + (1-\vartheta)|\alpha_0| + (\alpha_1 - \vartheta\alpha_0) + (k-1)|\beta_0| + (k\beta_0 - \beta_1) + \sum_{i=2}^{m-1}(\alpha_i - \alpha_{i-1}) + (k_1\alpha_m - \alpha_{m-1})$$
$$+ 2(k_1 - 1)|\alpha_m| + (k_1\alpha_m - \alpha_{m+1}) + \sum_{i=m+2}^{l-1}(\alpha_{i-1} - \alpha_i) + (\alpha_{l-1} - \tau\alpha_l) + 2(1-\tau)|\alpha_l|$$

$$+(\alpha_{l+1}-\tau\alpha_l)+\sum_{i=l+2}^{n-1}(\alpha_i-\alpha_{i-1})+(k_2\alpha_n-\alpha_{n-1})+2(k_2-1)|\alpha_n|+(k_2\alpha_n-\alpha_{n+1})$$
$$+\sum_{i=2}^{\infty}(\beta_{i-1}-\beta_i)$$

$$\leq 2[|\alpha_0|-|\alpha_m|+|\alpha_l|-|\alpha_n|+Y]$$

$$where\ \ Y=k_1(|\alpha_m|+\alpha_m)-\tau(\alpha_l+|\alpha_l|)+k_2(\alpha_n+|\alpha_n|)]+\frac{1}{2}[k(|\beta_0|+\beta_0)-\vartheta(|\alpha_0|+\alpha_0)].$$

Apply lemma 2 to G(z), we get then number of zeros of G(z) in $|z|\leq r, 0<r<1$ does not exceed

$$\frac{1}{log\frac{1}{r}}log\frac{2[|\alpha_0|-|\alpha_m|+|\alpha_l|-|\alpha_n|+Y]}{|a_0|},$$
$$where\ \ Y=k_1(|\alpha_m|+\alpha_m)-\tau(\alpha_l+|\alpha_l|)+k_2(\alpha_n+|\alpha_n|)]+\frac{1}{2}[k(|\beta_0|+\beta_0)-\vartheta(|\alpha_0|+\alpha_0)].$$

All the number of zeros ofF(z) in $|z|\leq r, 0<r<1$ is also equal to the number of zeros of G(z) in $|z|\leq r, 0<r<1$.

This completes the proof of theorem 3.

Proof of the Theorem 4:

Let F(z) $= a_0+a_1z+a_2z^2+\cdots+a_mz^m+\cdots+a_nz^n+\cdots$ be analytic function

Let us consider the polynomial G(z) $=(\mathrm{z}-1)F(z)$ so that

$$G(\mathrm{z})=(\mathrm{z}-1)(a_0+a_1z+a_2z^2+\cdots+a_mz^m+\cdots a_nz^n+\cdots)$$
$$=-a_0+\sum_{i=1}^{\infty}(a_{i-1}-a_i)z^i$$

Now for $|z|\leq 1$, we have

$$|\mathrm{G(z)}|\leq|a_0|+\sum_{i=1}^{\infty}|a_{i-1}-a_i|$$
$$\leq|\alpha_0|+|\beta_0|+\sum_{i=1}^{\infty}|\alpha_{i-1}-\alpha_i|+\sum_{i=1}^{\infty}|\beta_{i-1}-\beta_i|$$
$$=|\alpha_0|+|\beta_0|+|\alpha_0-k_1\alpha_0+k_1\alpha_0-\alpha_1|+|\beta_0-\vartheta\beta_0+\vartheta\beta_0-\beta_1|+\sum_{i=2}^{m-1}|\alpha_{i-1}-\alpha_i|$$
$$+|\alpha_{m-1}-\tau\alpha_m+\tau\alpha_m-\alpha_m|+|\alpha_m-\tau\alpha_m+\tau\alpha_m-\alpha_{m+1}|+\sum_{i=m+2}^{n-1}|\alpha_{i-1}-\alpha_i|$$
$$+|\alpha_{n-1}-k_2\alpha_n+k_2\alpha_n-\alpha_n|+|\alpha_n-k_2\alpha_n+k_2\alpha_n-\alpha_{n+1}|+\sum_{i=2}^{n-1}|\beta_{i-1}-\beta_i|$$
$$+|\beta_{n-1}-k\beta_n+k\beta_n-\beta_n|+|\beta_n-k\beta_n+k\beta_n-\beta_{n+1}|+\sum_{i=n+2}^{\infty}|\beta_{i-1}-\beta_i|$$

$$\leq|\alpha_0|+|\beta_0|+(k_1-1)|\alpha_0|+(k_1\alpha_0-\alpha_1)+(1-\vartheta)|\beta_0|+(\beta_1-\vartheta\beta_0)+\sum_{i=2}^{m-1}(\alpha_{i-1}-\alpha_i)+(\alpha_{m-1}-\tau\alpha_m)$$
$$+2(1-\tau)|\alpha_m|+(\alpha_{m+1}-\tau\alpha_m)+\sum_{i=m+2}^{n-1}(\alpha_i-\alpha_{i-1})+\sum_{i=2}^{n-1}(\beta_i-\beta_{i-1})+(k_2\alpha_n-\alpha_{n-1})$$
$$+2(k_2-1)|\alpha_n|+(k_2\alpha_n-\alpha_{n+1})+(k\beta_n-\beta_{n-1})+2(k-1)|\beta_n|+(k\beta_n-\beta_{n+1})$$
$$+\sum_{i=n+2}^{\infty}(\beta_{i-1}-\beta_i)$$

$\leq 2[|\beta_0| + |\alpha_m| - |\alpha_n| - |\beta_n| + W]$

$where\ \ W = k_2(\alpha_n + |\alpha_n|) + k(\beta_n + |\beta_n|) - \tau(\alpha_m + |\alpha_m|) + \frac{1}{2}[k_1(|\alpha_0| + \alpha_0) - \vartheta(|\beta_0| + \beta_0)].$

Apply lemma 2 to G(z), we get then number of zeros of G(z) in $|z| \leq r, 0 < r < 1$ does not exceed

$$\frac{1}{log\frac{1}{r}} log \frac{2[|\beta_0| + |\alpha_m| - |\alpha_n| - |\beta_n| + W]}{|a_0|},$$

$where\ \ W = k_2(\alpha_n + |\alpha_n|) + k(\beta_n + |\beta_n|) - \tau(\alpha_m + |\alpha_m|) + \frac{1}{2}[k_1(|\alpha_0| + \alpha_0) - \vartheta(|\beta_0| + \beta_0)].$

All the number of zeros ofF(z) in $|z| \leq r, 0 < r < 1$is also equal to the number of zeros of G(z) in $|z| \leq r, 0 < r < 1$

This completes the proof of theorem 4.

REFERENCES

1. G.Eneström, Remarqueesur un théorèmerelatif aux racines de l'equationa_n+...+a_0=0 oü tous les coefficient sont et positifs, Tôhoku Math.J 18 (1920), 34-36.
2. S.KAKEYA, On the limits of the roots of an alegebraic equation with positive coefficient, TôhokuMath.J 2 (1912-1913), 140-142.
3. A. Aziz and Q.G. Mahammad, on the certain class of analytic class of polynomial and related analytic functions, J. Anal and Appl. 75,[1980], 495-502.
4. N.K.Govil and Q.I.Rehman, On the Enstrom Kakeya Theorem Tohoku Math .J. 20, (1968), 126-136.
5. K. K. Dewan, Extremal Properties and Coefficient Estimates for Polynomials with Restricted Zeros and on Location ofZeros of Polynomials, Ph.D Thesis, Indian Institutes of Technology, Delhi, 1980.

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