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COMMON FIXED POINT THEOREM FOR A PAIR OF COMPATIBLE SELFMAPS OF A G –METRIC SPACE WITH RATIONAL INEQUALITY

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ABSTRACT

In the present paper we prove a common fixed point theorem for a pair of compatible self maps of a G metric space which satisfy a rational inequality.

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***Key words**: G – Metric space, Compatible mappings, Fixed point, Associated sequence of a point relative to two self maps.*

1. INTRODUCTION

In an attempt to generalize fixed point theorems a metric space, Gahler [2, 3] introduced the notion of 2-metric spaces while Dhage [1] initiated the notion of D - metric spaces. Subsequently several researchers have proved that most of their claims made are not valid. As a probable modification to D - metric spaces Shaban Sedghi, Nabi Shobe and Haiyun Zhou [4] introduced D^* metric spaces. In 2006, Zead Mustafa and Brailey Sims [5] initiated G - metric spaces of these two generalizations, the G -metric space seen evinced interest in many researchers.

The purpose of this paper is to prove a common fixed point theorem for a pair of compatible self maps of a G -metric space. Now we recall some basic definitions and lemmas which will be useful in our later discussion.

2 . PRELIMINARIES

We begin with

Definition2.1: ([5], Definition 3) Let *X* be a non-empty set and $G : X^3 \to [0, \infty)$ be a function satisfying:

(G1) $G(x, y, z) = 0$ if $x = y = z$

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$

(G4) $G(x, y, z) = G(\sigma(x, y, z))$ for all $x, y, z \in X$, where $\sigma(x, y, z)$ is a permutation of the set $\{x, y, z\}$ and

(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$.

Then *G* is called a *G* - metric on *X* and the pair (X, G) is called a *G* - metric Space.

Definition 2.2: ([5], Definition 4) A *G*-metric Space (X, G) is said to be symmetric if

(G6) $G(x, y, y) = G(x, x, y)$ for all $x, y \in X$

The example given below is a non-symmetric *G*-metric space.

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Example 2.3: ([5], Example1**):** Let $X = \{a, b\}$. Define $G : X^3 \to [0, \infty)$ by $G(a,a,a) = G(b,b,b) = 0;$ $G(a,a,b) = 1, G(a,b,b) = 2$ and extend G to all of X^3 by using (G4). Then it is easy to verify that *(X, G)* is a *G* - metric space. Since $G(a,a,b) \neq G(a,b,b)$, the space (X, G) is non-symmetric, in view of (G6).

Remark 2.4: Suppose *(X, G)* is symmetric *G*-metric space. Then for any $x, y \in X$ define $d(x, y) = G(x, y, y)$ and note that *d* is a metric on *X*. In fact for any $x, y \in X$

(i) $d(x, y) = G(x, y, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow G(x, y, y) = 0 \Leftrightarrow x = y$

(ii) $d(x, y) = G(x, y, y) = G(y, x, x) = d(y, x)$

(iii) $d(x, y) = G(x, y, y) \leq G(x, z, z) + G(z, y, y) = d(x, z) + d(z, y)$

Thus every symmetric *G*-metric space *X* has a metric defined on it. From now onwards (*X, G)* is a *G*-metric space. We begin with some examples of *G*-metric spaces.

Example 2.5: Let *(X, d)* be a metric space. Define $G_s^d : X^3 \to [0, \infty)$ by

$G_s^d(x, y, z) = \frac{1}{3}\left[d(x, y) + d(y, z) + d(z, x)\right]$ for $x, y, z \in X$. Then (*X,* G_s^d) is a *G*-metric Space.

Lemma 2.6: ([5], p.292) If *(X, G)* is a *G*-metric space then $G(x, y, y) \leq 2G(y, x, x)$ for all $x, y \in X$

Example 2.7 ([5], p.291): Suppose *(X, G)* is a *G*-metric space. Define $d_G : X^2 \to [0,\infty)$ by $d_G(x, y) = G(x, y, y) + G(x, x, y)$ for $(x, y) \in X^2$. Then d_G is a metric on *X* giving a metric space *(X, d_G)*.

Remark 2.8: Using d_G, we can construct $G_s^{d_G} : X^3 \to [0,\infty)$ as given in Example 2.5. It has been proved in ([15], p.292]) that $G(x, y, z) \leq G_s^{d_G}(x, y, z) \leq 2G(x, y, z)$ for all $(x, y, z) \in X^3$

Definition 2.9: ([5], Definition 5) Let *(X, G)* be a *G*-metric space then for $x_0 \in X, r > 0$, the *G*-ball with centre x_0 and radius r is given by $B_G(x_0, r) = \{y \in X : G(x_0, y, y) < r\}$.

Lemma 2.10: ([15], Proposition 5) Let *(X, G)* be *G*-metric space, then for all $x_0 \in X$, and r > 0, we have

$$B_G(x_0, \frac{1}{3}r) \subseteq B_{d_G}(x_0, r) \subseteq B_G(x_0, r)$$

Consequently, the *G*-metric topology τ (*G*) coincides with the metric topology arising from d_G.

Definition 2.11: Let *(X, G)* be a *G*-metric Space. A sequence $\{x_n\}$ in *X* is said to be *G*-convergent if there is a $x_0 \in X$ such that to each $\varepsilon > 0$ there is a natural number *N* for which $G(x_n, x_n, x_0) < \varepsilon$ for all $n \geq N$.

Lemma 2.12: ([5], Proposition 6) : Let *(X, G)* be a *G*-metric Space, then for a sequence $\{x_n\} \subseteq X$ and point $x \in X$ the following are equivalent.

1. $\{x_n\}$ is *G*- convergent to x.
2. $d_G(x_n, x) \to 0 \quad as \quad n \to \infty$ (that is $\{x_n\}$ converges to x relative to the metric d_G)
3. $G\left(x_n, x_n x\right) \to 0 \quad as \quad n \to \infty$
4. $G(x_n, x, x) \to 0 \quad as \quad n \to \infty$
5. $G(x_m, x_n, x) \to 0 \quad as \quad m, n \to \infty$

Definition 2.13 :([5],Definition 8) Let *(X, G)* be a *G*-metric space, then a sequences $\{x_n\} \subseteq X$ is said to be *G*-Cauchy if for each ε > 0, there exists a natural number *N* such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$.

Note that every *G*-convergent sequence in a *G*-metric space *(X, G)* is *G*-Cauchy.

Definition 2.14: ([15], Definition 9) A *G*-metric space *(X, G)* is said to be *G*-complete if every G Cauchy sequence in *(X, G)* is *G*-convergent in *(X, G)*.

Definition 2.15: Suppose *f* and *g* are self maps of a *G*-metric space *(X, G)* such that $\lim_{n\to\infty} G\left(fgx_n, gfx_n, gfx_n\right) = 0$ for every sequence $\{x_n\} \subseteq X$ ith $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$ for some $t \in X$, then the pair *f* and *g* is said to be a compatible pair .

Definition 2.16: Let (X,G) be an G -metric space and g , f be two selfmaps of X such that $g(X) \subseteq f(X)$ .For any $x_0 \in X$,there is a sequence $\{x_n\}$ in X such that $fx_n = gx_{n-1}$ for $n \geq 1$. (In fact, $x_0 \in X$ then $gx_0 \in g(X) \subseteq f(X)$ so that there is a $x_1 \in X$ with $gx_0 = fx_1$; now $gx_1 \in g(X) \subseteq f(X)$ gives a $x_2 \in X$ with $gx_1 = fx_2$;and repeat this to obtain the sequence $\{x_n\}$) We shall call this sequence $\{x_n\}$ as an associated sequence of x_0 relative to g and f).

Example 2.14: If $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are defined by $fx = x^2, gx = \dfrac{x^2}{3}$ then $g(\mathbb{R}) \subseteq f(\mathbb{R})$ .For $x_0 \in \mathbb{R}$ we can find $x_1 \in \mathbb{R}$ with $gx_0 = fx_1$ is given by $x_1 = \pm\dfrac{x_0}{\sqrt{3}}$ .Again $x_2 \in \mathbb{R}$ with $gx_1 = fx_2$ is given by $x_2 = \pm\dfrac{x_0}{(\sqrt{3})^2}$ .More generally $x_n = \pm\dfrac{x_0}{(\sqrt{3})^n}$ for $n \geq 1$ .Therefore associated sequence $x_1, x_2, x_3, \ldots x_n, \ldots$ for a given $x_0 \in \mathbb{R}$ are infinitely many since each x_n has two choices $\dfrac{x_0}{(\sqrt{3})^n}, -\dfrac{x_0}{(\sqrt{3})^n}$ for $n \geq 1$ .Thus there may be more than one associated sequence of x_0 relative to g and f if $g(X) \subseteq f(X)$.

3. MAIN RESULTS

We now state our main theorem.

Theorem 3.1: Let *f* and *g* be selfmaps of a *G*- metric space *(X, G)* satisfying

(3.1.1) $g(X) \subset f(X)$

(3.1.2) $G(gx, gy, gy) \leq \dfrac{\alpha.G(fx, gy, gy)[1+G(fx, gx, gx)]}{[1+G(fx, fy, fy)]} + \beta G(fx, fy, fy)$ for all $x, y \in X$,

where $\alpha, \beta \geq 0$; $\alpha + \beta < 1$.

(3.1.3) one of *f* and *g* is continuous

(3.1.4) *f* and *g* are compatible and

(3.1.5) an associated sequence $\{x_n\}$ of a point $x_0 \in X$ relative to the self maps f and g is such that $\{fx_n\}$ converges to t for some point $t \in X$, then t is the unique common fixed point of f and g .

To prove the theorem, we need the following lemma.

Lemma 3.2: Let f and g be compatible selfmaps of a *G*- metric space *(X, G)*. Suppose that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$ and some sequence $\{x_n\}$ in *X*. Then $\lim_{n\to\infty} gfx_n = fx$, if f is continuous.

Proof: Suppose f and g are compatible mappings and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$ .Then $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n)$ = 0, this implies

(3.2.1) $G(\lim_{n\to\infty} fgx_n, \lim_{n\to\infty} gfx_n, \lim_{n\to\infty} gfx_n) = 0$, Since f is continuous and $gx_n \to x$ as $n \to \infty$ we have

(3.2.2) $\lim_{n\to\infty} fgx_n = fx$.

From (3.2.1) and (3.2.2), we get $G(fx, \lim_{n\to\infty} gfx_n, \lim_{n\to\infty} gfx_n) = 0$, this implies $\lim_{n\to\infty} gfx_n = fx$, proving the lemma.

Proof: From (3.1.5), the sequence $\{x_n\}$ of x_0 relative to the selfmaps f and g such that $fx_n = gx_{n-1}$ for $n = 1, 2, 3, ...$ and $fx_n \to t$ as $n \to \infty,$ it follows that $gx_n \to t$ as $n \to \infty$.

Case-(i): Suppose that f is continuous. Then we have by Lemma 3.2 that

(3.2.3) $\lim_{n\to\infty} gfx_n = ft$ and also

(3.2.4) $\lim_{n\to\infty} f^2 x_n = ft,$

Now from (3.1.2) we get

$$G(gfx_n, gx_{n-1}, gx_{n-1}) \le \frac{\alpha.G(f^2x_n, gx_{n-1}, gx_{n-1})[1+G(f^2x_n, gfx_n, gfx_n)]}{[1+G(f^2x_n, fx_{n-1}, fx_{n-1})]} + \beta G(f^2x_n, fx_{n-1}, fx_{n-1})$$

where $\alpha, \beta \ge 0$; $\alpha + \beta < 1$,by letting $n \to \infty$ in the above inequality and using (3.2.3) and (3.2.4), we get

$$G(ft,t,t) \le \frac{\alpha.G(ft,t,t)[1+G(ft, ft, ft)]}{[1+G(ft,t,t)]} + \beta\, G(ft,t,t)$$

$$= \frac{\alpha G(ft,t,t)}{[1+G(ft,t,t)]} + \beta G(ft,t,t)$$

$$\le \alpha G(ft,t,t) + \beta G(ft,t,t)$$

$$= (\alpha + \beta) G(ft,t,t)$$

Which implies $G(ft,t,t) = 0$ and hence *ft = t*.

(Since $1 + G(ft,t,t) > 1 \Rightarrow \frac{1}{1+G(ft,t,t)} < 1$ and $\alpha + \beta < 1$)

Again from (3.1.2), we get

$$G(gt, gx_{n-1}, gx_{n-1}) \le \frac{\alpha.G(ft, gx_{n-1}, gx_{n-1})[1+G(ft, gt, gt)]}{[1+G(ft, fx_{n-1}, fx_{n-1})]} + \beta G(ft, fx_{n-1}, fx_{n-1})$$

where $\alpha, \beta \ge 0$; $\alpha + \beta < 1.$

Letting $n \to \infty$ in the above inequality, we obtain

$$G(g\ t,t,t) \le \frac{\alpha.G(ft,t,t)[1+G(ft,gt,gt)]}{[1+G(ft,t,t)]} + \beta G(ft,t,t).$$

Since $ft = t$, we get $G(g\ t,t,t) = 0$ which implies that $gt = t$, showing that t is a common fixed point of f and g

Case-(ii): Suppose that g is continuous. Then we have by Lemma 3.2, that

(3.2.5) $\lim_{n\to\infty} fgx_n = gt$ and also

(3.2.6) $\lim_{n\to\infty} g^2 x_n = gt.$

Now from (3.1.2) we get

$$G(g^2x_n, gx_{n-1}, gx_{n-1}) \le \frac{\alpha.G(fgx_n, gx_{n-1}, gx_{n-1})[1+G(fgx_n, ggx_n, ggx_n)]}{[1+G(fgx_n, fx_{n-1}, fx_{n-1})]} + \beta G(fgx_n, fx_{n-1}, fx_{n-1})$$

where $\alpha, \beta \ge 0$; $\alpha + \beta < 1$, by letting $n \to \infty$ in the above inequality and using (3.2.5) and (3.2.6), we get

$$\begin{aligned} G(gft,t) &\le \frac{\alpha.G(gt,t,t)[1+G(gt,gt,gt)]}{[1+G(gft,t)]} + \beta G(gft,t) \\ &= \frac{\alpha G(gft,t)}{[1+G(gft,t)]} + \beta G(gft,t) \\ &\le \alpha G(gft,t) + \beta G(gft,t) \\ &= (\alpha+\beta) G(gft,t) \end{aligned}$$

Which implies $G(gft,t) = 0$ and hence $gt = t.$

(Since $1+G(gft,t) > 1 \Rightarrow \dfrac{1}{1+G(gft,t)} < 1$ and $\alpha + \beta < 1$)

From (3.1.1), we can find a $w \in X$ such that $gt = fw$. Now from (3.1.2) we have

$$G(g^2x_n, gw, gw) \le \frac{\alpha.G(fgx_n, gw, gw)[1+G(fgx_n, g^2x_n, g^2x_n)]}{[1+G(fgx_n, fw, fw)]} + \beta G(fgx_n, fw, fw)$$

where $\alpha, \beta \ge 0$; $\alpha + \beta < 1$.Letting $n \to \infty$ in the above inequality and using (3.2.5) and (3.2.6), we obtain

$$G(gt, gw, gw) \le \frac{\alpha.G(gt, gw, gw)[1+G(gt,gt,gt)]}{[1+G(gt, fw, fw)]} + \beta G(gt, fw, fw),$$

Since
$t = gt = fw$, we obtain $G(gt, gw, gw) \le \alpha.G(fw, gw, gw),$ that is $G(gt, gw, gw) \le \alpha.G(gt, gw, gw)$, which implies that $G(gt, gw, gw) = 0$

Since $\alpha \in (0,1),$ hence $gt = gw$, thus $t = gt = fw = gw$.

Now put $y_n = w$ for $n = 1,2,3, \ldots$ then $fy_n \to fw$ and $gy_n \to gw$ as $n \to \infty$. Since $fw = gw$, f and g are compatible, $\lim\limits_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0.$

Since $y_n = w$ for $n = 1,2,3, \ldots$ we have $\lim\limits_{n\to\infty} G(fgw, gfw, gfw) = 0$, that is $G(fgw, gfw, gfw) = 0,$ which implies that $fgw = gfw$, since $fw = gw = t$, we get $ft = gt$. Since $gt = t$, it follows that $ft = gt = t,$ Showing that t is a common fixed point of f and g.

Finally to prove the uniqueness of common fixed point of f and g, suppose $u = fu = gu$ and $v = fv = gv$ for some $u, v \in X$. From (3.1.2), we get

$$G(u,v,v) = G(gu, gv, gv) \le \frac{\alpha.G(fu, gv, gv)[1+G(fu, gu, gu)]}{[1+G(fu, fv, fv)]} + \beta G(fu, fv, fv)$$

where $\alpha, \beta \ge 0; \alpha + \beta < 1;$

$$\begin{aligned} G(u,v,v) &\le \frac{\alpha.G(u,v,v)[1+G(u,u,u)]}{[1+G(u,v,v)]} + \beta G(u,v,v) \\ &= \frac{\alpha\, G(u,v,v)}{[1+G(u,v,v)]} + \beta G(u,v,v) \\ &\le \alpha\, G(u,v,v) + \beta G(u,v,v) \\ &= (\alpha+\beta) G(u,v,v) \end{aligned}$$

which implies that $G(u,v,v) = 0$,

Since $\frac{1}{1+G(u,v,v)}<1$ and $(\alpha+\beta)<1$, hence $u=v$, proving the theorem.

Example 3.3: Let *X*= [0, 1) and $G(x,y,z)=\max\{|x-y|,|y-z|,|z-x|\}$ for $x,y,z\in X$. Then *(X, G)* is a *G*- metric space.

Define $f:X\to X$ and $g:X\to X$ by $fx=x$ and $gx=\frac{x}{2}$ for all $x\in X$. Then

$g(X)=[0,\frac{1}{2})\subset[0,1)=f(X)$

Clearly $fg=gf$, so that f and g are compatible. Also an associated sequence of $x_0=0$ relative to the selfmaps f and g is given by $x_n=0$ for $n=0,1,2,...$ and since $\{fx_n\}$ is a constant sequence converging to 0, which is a point in *X*. Take $\alpha=0$, $\beta=\frac{1}{2}$, then f and g satisfies the inequality (3.1.2). Thus the conditions (3.1.3) to (3.1.5) of Theorem 3.1 are satisfied. Hence by Theorem 3.1, '0' is the unique common fixed point of f and g .

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