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GEODESIC 2-GRAPHOIDAL COVERING NUMBER OF A BICYCLIC GRAPHS

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ABSTRACT

A geodesic 2-graphoidal cover of a graph G is a collection ψ of shortest paths in G such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most two paths in ψ and every edge of G is in exactly one path in ψ. The minimum cardinality of a geodesic 2- graphoidal cover of G is called the geodesic 2-graphoidal covering number of G and is denoted by η_{2g}. In this paper we determine η_{2g} for bicyclic graphs.

Key words: *Graphoidal covers, Acyclic graphoidal cover, Geodesic Graphoidal cover, bicyclic graphs.*

1. INTRODUCTION

A graph is a pair $G=(V,E)$, where V is the set of vertices and E is the set of edges. Here we consider only nontrivial, finite, connected, undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Harary [4]. The concept of graphoidal cover was introduced by B.D Acharya and E. Sampathkumar [1] and the concept of acyclic graphoidal cover was introduced by Arumugam and Suresh Suseela [4].The reader may refer to [5] and [2] for the terms not defined here.

Let $P=(v_1,v_2,v_3,\ldots,v_r)$ be a path or a cycle in a graph $G=(V,E)$. Then vertices $(v_2,v_3,\ldots,v_{r-1})$ are called internal vertices of P and v_1 and v_r are called external vertices of P. Two paths P and Q of a graph G are said to be internally disjoint if no vertex of G is an internal vertex of both P and Q.

Definition: 1.1[1] A graphoidal cover of a graph G is called a collection ψ of (not necessarily open) paths in G satisfying the following conditions:
(i) Every path in ψ has at least two vertices.
(ii) Every vertex of G is an internal vertex of at most one path in ψ.
(iii) Every edge of G is in exactly one path in ψ.

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$.

Definition: 1.2 [3] A graphoidal cover ψ of a graph G is called an acyclic graphoidal cover if every member of ψ is an open path. The minimum cardinality of an acyclic graphoidal cover of G is called the acyclic graphoidal covering number of G and is denoted by $\eta_a(G)$ or η_a.

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Definition: 1.3 [4] A geodesic graphoidal cover of a graph G is a collection ψ of shortest paths in G such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most one path in ψ and every edge of G is an exactly one path in ψ. The minimum cardinality of a geodesic graphoidal cover of G is called the geodesic graphoidal covering number of G and is denoted by η_g.

Definition: 1.4 [1] Let ψ be a collection of internally disjoint paths in G. A vertex of G is said to be in the interior of ψ if it is an internal vertex of some path in ψ. Any vertex which is not in the interior of ψ is said to be an exterior vertex of ψ.

Theorem: 1.5 [7] For any graphoidal cover ψ of G, let t_ψ denote the number of exterior vertices of ψ. Let $t = \min t_\psi$ where the minimum is taken over all graphoidal covers of G. Then $\eta = q - p + t$

Corollary: 1.6[7] For any graph G, $\eta \geq q - p$. Morever the following are equivalent.

(i) $\eta = q - p$
(ii) There exists a graphoidal cover without exterior vertices.
(iii) There exists a set of internally disjoint and edge disjoint paths without exterior vertices.

In [4] it is given that $\eta \leq \eta_a \leq \eta_g$ and these inequalities can be strict and also for a tree $\eta = \eta_a = \eta_g = n-1$ and Theorem 1.5 and corollary 1.6 are true for geodesic graphoidal covers.

They observe that $\eta_g = q$ if and only if G is Complete. Further for a cycle $C_m, \eta_g = \begin{cases} 2 & \text{if m is even} \\ 3 & \text{if m is odd} \end{cases}$

Theorem: 1.7 [9] For any 2- graphoidal cover ψ of a (p,q) graph G, $\eta_2 = q - p - t_2 + t$

Corollary: 1.8 [9] For any graph G, $\eta_2 \geq q - p - t_2 + t$. Morever the following are equivalent.

(i) $\eta_2 = q - 2p$
(ii) There exists a 2-graphoidal cover in which every vertex is an internal vertex of exactly two paths.
(iii) There exists a set Q of edge disjoint 2-graphoidal cycle or path.(From such a set Q of paths, the required 2-graphoidal cover can be obtained by adding the edges which are not covered by the paths in Q).

Corollary: 1.9 [9] Let G be any (p,q) –graph such that $\eta_2 = q - 2p$ .Then $\delta \geq 4$ and $\Delta \geq 5$

Remark: 1.10 [9] If $\Delta \leq 3$, then $t_2 = 0$ and hence $\eta_2(G) = \eta(G)$, where η is the minimum graphoidal covering number.

Hence $\eta_{2g}(G) = \eta_g(G)$

Remark: 1.11 [9] In [4] given that $\eta \leq \eta_a \leq \eta_g$ and these inequalities can be strict and also for a tree $\eta = \eta_a = \eta_g = n-1$ and Theorem 1.5 and corollary 1.6 are true for geodesic graphoidal covers.

They observe that $\eta_g = q$ if and only if G is Complete. Further for a cycle $C_m, \eta_g = \begin{cases} 2 & \text{if m is even} \\ 3 & \text{if m is odd} \end{cases}$

Remark: 1.12

(i) $\eta_{2g} \leq \eta_g$ and these inequalities can be strict

(ii) If G is Complete $\eta_{2g} = q$

Theorem: 1.13 [9] If G is a graph with $\delta \geq 3$, then there exists a graphoidal cover ψ of G such that every vertex of G is an internal vertex of some paths in ψ.

Corollary: 1.14 [9] If G is a graph with $\delta \geq 5$, then $\eta_2 = q - 2p.$

Definition: 1.15 [8] A connected (*p*, *p*+1) graph *G* is called a bicyclic graph.

Definition: 1.16 [8] A one – point union of two cycles is a simple graph obtained from two cycles, say C_l and C_m where $l,m \geq 3$, by identifying one and the same vertex from both cycles. Without loss of generality, we may assume the *l*-cycle to be $u_0 u_1 \ldots u_{l-1} u_0$ and the *m*-cycle to be $u_0 u_l u_{l+1} \ldots u_{m+l-2} u_0$. We denote this graph by *U(l; m)*

Definition: 1.17 [8] A long dumbbell graph is a simple graph obtained by joining two cycles C_l and C_m where $l,\ m \geq 3$, with a path of length *i*, $i \geq 1$. Without loss of generality, we may assume $C_l = u_0 u_1 \ldots u_{l-1} u_0$, $P_i = u_{l-1} u_l u_{l+1} \ldots u_{l+i-1}$ and $C_m = u_{l+i-1} u_{l+i} \ldots u_{l+m+i-2} u_{l+i-1}$.We denote this graph by *D(l, m, i)*

Definition: 1.18 [8] A cycle with a long chord is a simple graph obtained from an *m*-cycle, $m \geq 4$, by adding a chord of length *l* where $l \geq 1$. Let the *m*-cycle be $u_0 u_1 \ldots u_{m-1} u_0$. Without loss of generality, we may assume the chord joins u_0 with u_i, where $2 \leq i \leq m-2$. That is , $u_0 u_m u_{m+1} \ldots u_{l+m-2} u_i$ is the chord. We denote this graph by $C_m(i;\ l)$

In this paper we determine η_{2g} for bicyclic graphs containing a *U*(*l; m*), *D*(*l, m, i*), $C_m(i;\ l)$.

Remark: 1.19

(i) $\eta_{2g} \leq \eta_g$ and these inequalities can be strict

(ii) If *G* is Complete then $\eta_{2g} = q$

(iii) Every geodesic 2-graphoidal cover is a 2-graphoidal cover.

$(i.e)\, \eta_{2g} \geq \eta_2$

4.2 Geodesic 2-graphoidal covering number of bicyclic graphs

Theorem: 4.2.1 Let *G* be a bicyclic graph containing a $U(l,m)$ and both the cycles are of even length. Let *n* denote the number of pendant vertices of *G* and let *k* denote the number of vertices of degree greater than 4 on $U(l,m)$ other than u_0 .Then

$$\eta_{2g}(G) = \begin{cases} 2 & \text{if } k = 0 \\ n - t_2 + 3 & \text{if } k = 0 \,\&\, \deg u_0 = 5, \text{a tree attached with } u_0 \,\& \\ & \text{if } k \geq 1 \,\&\, (v,w) \text{ section is not a shortest path} \\ n - t_2 + 2 & \text{if } k = 1 \,\&\, (v,w) \text{ section is a shortest path} \,\&\, \text{if } k \geq 3 \\ n - t_2 + 1 & \text{if } k = 2 \,\&\, (v,w) \text{ section is a shortest path} \end{cases}$$

Proof: Let $V(U(l,m)) = \{u_0, u_1, u_2, \ldots, u_{l-1}, u_l, u_{l+1}, \ldots, u_{l+m-2}\}$

$V(C_l) = \{u_0, u_1, u_2, \ldots, u_{l-1}, u_0\}$

$V(C_m) = \{u_0, u_l, u_{l+1}, \ldots, u_{l+m-2}, u_0\}$ where *l* and *m* are even.

Case-1: *k* = 0

Case-1(a): $G = U(l,m)$

The geodesic 2-graphoidal cover of *G* is as follows

$P_1=\{u_i,u_{i-1},\ldots,u_1,u_0,u_l,u_{l+1},\ldots,u_j\}$ $[i=\frac{l}{2}\ \&\ j=l+\frac{m}{2}-1]$

$P_2=\{u_i,u_{i+1},\ldots,u_{l-1},u_0,u_{l+m-2},,\ldots,u_j\}$ $[i=\frac{l}{2}\ \&\ j=l+\frac{m}{2}-1]$

$\psi=\{P_1,P_2\}$ is a geodesic 2-graphoidal cover of G

$\Rightarrow \eta_{2g}(G)\le 2$

Since at least two vertices on $U(l;m)$ are exterior vertices in any minimum geodesic 2-graphoidal cover so that $t\ge 2$ and $t_2=1$

Hence $\eta_{2g}(G)=q-p-t_2+t\Rightarrow \eta_g\ge 2$

Thus $\eta_{2g}(G)=2$

Case-1(b): $G=U(l,m)$ with $\deg u_0=5$ and there is a tree attached at u_0 with n pendant vertices.

Let $P_1=\{u_i,u_{i-1},\ldots,u_1,u_0,u_l,u_{l+1},\ldots,u_j\}$ $[i=\frac{l}{2}\ \&\ j=l+\frac{m}{2}-1]$

$P_2=\{u_i,u_{i+1},\ldots,u_{l-1},u_0,u_{l+m-2},,\ldots,u_j\}$ $[i=\frac{l}{2}\ \&\ j=l+\frac{m}{2}-1]$

Consider the graph $G_1=G-\{u_1,u_2\ldots,,u_{l-2},u_l,u_{l+1},\ldots,u_{l+m-2}\}$ is a tree with n +1 pendant vertices.

Let ψ_1 be a minimum geodesic 2- graphoidal cover of G_1 and let $t_2(G_1)=\max t_2(\psi_1)$

Using corollary 4.1.9, $\eta_{2g}(G_1)=n-t_2(G_1)$

Now $\psi=\psi_1\cup\{P_1,P_2\}$ is geodesic 2- graphoidal cover of G.

$$\Rightarrow \eta_{2g}(G)\le n-t_2(G_1)+2 \text{ where } t_2=t_2(G_1)+1$$
$$\le n-(t_2-1)+2$$
$$\eta_{2g}(G)\le n-t_2+3$$

Since in any minimum geodesic 2-graphoidal cover of G , all the n pendant vertices, u_i and u_j exterior vertices so that the number of exterior points $t\ge n+2$

$$\eta_{2g}(G)=q-p-t_2+t\ge 1-t_2+n+2$$

$\eta_{2g}\ge n-t_2+3$

$\therefore \eta_{2g}(G)=n-t_2+3$

Case 2: $k=1$

Let u_t be the unique vertex of degree greater than 4 on $U(l,m)$ other than u_0

Without loss of generality assume that u_t lies on C_l

Sub Case-2(a):

Let $t=\frac{l}{2}(u_t=u_i)$

Let $P_1=\{u_i,u_{i-1},\ldots,u_1,u_0,u_l,u_{l+1},\ldots,u_j\}$ $[i=\frac{l}{2}\ \&\ j=l+\frac{m}{2}-1]$

Consider the graph $G_1 = G - \{u_{i-1}, \dots, u_1, u_l, u_{l+1}, \dots, u_{j-1}\}$ is a tree with n +1 pendant vertices.

Let ψ_1 be a minimum geodesic 2- graphoidal cover of G_1 and let $t_2(G_1) = \max t_2(\psi_1)$

Using Corollary 4.1.9, $\eta_{2g}(G_1) = n - t_2(G_1)$

Now $\psi = \psi_1 \cup P_1$ is geodesic 2- graphoidal cover of G.

$$\Rightarrow \eta_{2g}(G) \le n - t_2(G_1) + 1 \text{ where } t_2 = t_2(G_1) + 1$$
$$\le n - (t_2 - 1) + 1$$

$$\eta_{2g}(G) \le n - t_2 + 2$$

Since in any minimum geodesic 2-graphoidal cover of G , all the n pendant vertices and u_j are exterior vertices so that the number of exterior points $t \ge n+1$

$$\eta_{2g}(G) = q - p - t_2 + t \ge 1 - t_2 + n + 1$$
$$\eta_{2g} \ge n - t_2 + 2$$
$$\therefore \eta_{2g}(G) = n - t_2 + 2$$

Sub Case-2(b):

If $u_t \ne u_i$

Without loss of generality assume that $t > \frac{l}{2}$

Let $P_1 = \{u_i, u_{i-1}, \dots, u_1, u_0, u_l, u_{l+1}, \dots, u_j\}$ $[i = \frac{l}{2}$ & $j = l + \frac{m}{2} - 1]$

Consider the graph $G_1 = G - \{u_{i-1}, \dots, u_1, u_l, u_{l+1}, \dots, u_{j-1}\}$ is a tree with n +2 pendant vertices.

Let ψ_1 be a minimum geodesic 2- graphoidal cover of G_1 and let $t_2(G_1) = \max t_2(\psi_1)$

Using Corollary 4.1.9, $\eta_{2g}(G_1) = n + 1 - t_2(G_1)$

Now $\psi = \psi_1 \cup P_1$ is geodesic 2- graphoidal cover of G.

$$\Rightarrow \eta_{2g}(G) \le n + 1 - t_2(G_1) + 1 \text{ where } t_2 = t_2(G_1) + 1$$
$$\le n - (t_2 - 1) + 2$$

$$\eta_{2g}(G) \le n - t_2 + 3$$

Since in any minimum geodesic 2-graphoidal cover of G , all the n pendant vertices, u_i and u_j are exterior vertices so that the number of exterior points $t \ge n+2$

$$\eta_{2g}(G) = q - p - t_2 + t \ge 1 - t_2 + n + 2$$
$$\eta_{2g} \ge n - t_2 + 3$$
$$\therefore \eta_{2g}(G) = n - t_2 + 3$$

Case-3*:* *k* = 2

Case-3(a): $k=2$ and every (*v*,*w*) section of each of the cycles on $U(l,m)$ in which all the vertices except *v* and *w* have degree 2 $(\deg_v = \deg_w \geq 4)[u_i = v = \frac{l}{2}, u_j = w = l + \frac{m}{2} - 1]$ and this (*v*,*w*) section is a shortest path.

Let $P_1 = \{u_i, u_{i-1}, \ldots, u_1, u_0, u_l, u_{l+1}, \ldots, u_j\}$ $[i = \frac{l}{2}$ & $j = l + \frac{m}{2} - 1]$

Consider the graph $G_1 = G - \{u_{i-1}, \ldots, u_1, u_l, u_{l+1}, \ldots, u_{j-1}\}$ is a tree with *n* pendant vertices.

Let ψ_1 be a minimum geodesic *2*- graphoidal cover of G_1 and let $t_2(G_1) = \max t_2(\psi_1)$

Using Corollary 4.1.9, $\eta_{2g}(G_1) = n - 1 - t_2(G_1)$

Now $\psi = \psi_1 \cup P_1$ is geodesic 2- graphoidal cover of *G*.

$$\Rightarrow \eta_{2g}(G) \leq n - 1 - t_2(G_1) + 1 \text{ where } t_2 = t_2(G_1) + 1$$
$$\leq n - (t_2 - 1)$$

$$\eta_{2g}(G) \leq n - t_2 + 1$$

Since in any minimum geodesic 2-graphoidal cover of *G* , all the *n* pendant vertices so that the number of exterior points $t \geq n$

$$\eta_{2g}(G) = q - p - t_2 + t \geq 1 - t_2 + n$$
$$\eta_{2g}(G) \geq n - t_2 + 1$$
$$\therefore \eta_{2g}(G) = n - t_2 + 1$$

Case-3 (b): $k=2$ and the (*v*,*w*) section of each of the cycles on $U(l,m)$ in which all the vertices except *v* and *w* have degree 2 $(\deg_v = \deg_w \geq 4)$ and this (*v*,*w*) section is not a shortest path. Without loss of generality assume that $u_r = v, u_s = w$ with $r < s$

Let $P_1 = \{u_r, u_{r+1}, \ldots, u_i\}$
$P_2 = \{u_s, u_{s-1}, \ldots, u_i\}$

Consider the graph $G_1 = G - \{u_{r+1}, u_{r+2}, \ldots, u_{s-1}\}$ is a unicyclic graph with *n* pendant vertices.

Let ψ_1 be a minimum geodesic *2*- graphoidal cover of G_1 and let $t_2(G_1) = \max t_2(\psi_1)$

Using theorem 4.1.10, $\eta_{2g}(G_1) = n - t_2(G_1)$

Now $\psi = \psi_1 \cup \{P_1, P_2\}$ is geodesic 2- graphoidal cover of *G*.

$$\Rightarrow \eta_{2g}(G) \leq n - t_2(G_1) + 2 \text{ where } t_2 = t_2(G_1) + 1$$
$$\leq n - (t_2 - 1) + 2$$
$$\eta_{2g}(G) \leq n - t_2 + 3$$

Since in any minimum geodesic 2-graphoidal cover of G, all the n pendant vertices, u_i and u_j are exterior vertices so that the number of exterior points $t \geq n+2$

$$\eta_{2g}(G) = q - p - t_2 + t \geq 1 - t_2 + n + 2$$

$$\eta_{2g}(G) \geq n - t_2 + 3$$

$$\therefore \eta_{2g}(G) = n - t_2 + 3$$

Case-4:
If $k \geq 3$

The proof is similar to case 3.

Theorem: 4.2.2 Let G be a bicyclic graph containing a $U(l,m)$ and any one of the cycles is of odd length. Let n denote the number of pendant vertices of G and let k denote the number of vertices of degree greater than 4 on $U(l,m)$ other than u_0. Then

$$\eta_{2g}(G) = \begin{cases} 3 & \text{if } k = 0 \\ n - t_2 + 4 & \text{if } k = 0 \,\&\, \deg u_0 = 5, \text{a tree attached with } u_0 \\ n - t_2 + 3 & \text{if } k \geq 1 \,\&\, (v,w) \text{ section is not a shortest path} \\ n - t_2 + 2 & \text{if } k = 1 \,\&\, (v,w) \text{ section is a shortest path} \\ n - t_2 + 1 & \text{if } k \geq 2 \,\&\, (v,w) \text{ section is a shortest path} \end{cases}$$

Proof: Let $V(U(l,m)) = \{u_0, u_1, u_2, \ldots, u_{l-1}, u_l, u_{l+1}, \ldots, u_{l+m-2}\}$

$V(C_l) = \{u_0, u_1, u_2, \ldots, u_{l-1}, u_0\}$

$V(C_m) = \{u_0, u_l, u_{l+1}, \ldots, u_{l+m-2}, u_0\}$

Without loss of generality assume that l is odd and m is even.

Case-1: $k = 0$

Then $G = U(l,m)$

The geodesic 2-graphoidal cover is as follows

$P_1 = \{u_i, u_{i-1}, \ldots, u_1, u_0, u_l, u_{l+1}, \ldots, u_j\}$

$P_2 = \{u_i, u_{i+1}\}$

$P_3 = \{u_{i+1}, u_{i+2}, \ldots, u_{l-1}, u_0, u_{l+m-2}, ,\ldots, u_j\}$ where $[i = \frac{l-1}{2} \,\&\, j = l + \frac{m}{2} - 1]$

$\psi = \{P_1, P_2, P_3\}$ be The geodesic 2-graphoidal cover of G

$\therefore \eta_{2g} \leq 3$

Since atleast three vertices on $U(l;m)$ are exterior vertices in any minimum geodesic 2-graphoidal cover so that $t \geq 3, t_2 = 1$

Hence $\eta_{2g} = q - p - t_2 + t \Rightarrow \eta_{2g} \geq 3$

Thus $\eta_{2g} = 3$

Case1 (b): $G = U(l,m)$ with $\deg u_0 = 5$ and there is a tree attached at u_0 with *n* pendant vertices.

$P_1 = \{u_i, u_{i-1}, \ldots, u_1, u_0, u_l, u_{l+1}, \ldots, u_j\}$

$P_2 = \{u_i, u_{i+1}\}$

$P_3 = \{u_{i+1}, u_{i+2}, \ldots, u_{l-1}, u_0, u_{l+m-2}, , \ldots, u_j\}$ where $[i = \frac{l-1}{2}$ & $j = l + \frac{m}{2} - 1]$

Consider the graph $G_1 = G - \{u_1, u_2 \ldots, , u_{l-1}, u_l, u_{l+1}, \ldots, u_{l+m-2}\}$ is a tree with *n*+1 pendant vertices.

Let ψ_1 be a minimum geodesic 2- graphoidal cover of G_1 and let $t_2(G_1) = \max t_2(\psi_1)$

Using Corollary 4.1.9, $\eta_{2g}(G_1) = n - t_2(G_1)$

Now $\psi = \psi_1 \cup \{P_1, P_2, P_3\}$ is geodesic 2- graphoidal cover of *G*.

$$\Rightarrow \eta_{2g}(G) \le n - t_2(G_1) + 3 \text{ where } t_2 = t_2(G_1) + 1$$
$$\le n - (t_2 - 1) + 3$$

$$\eta_{2g}(G) \le n - t_2 + 4$$

Since in any minimum geodesic 2-graphoidal cover of *G* , the number of exterior points $t \ge n + 3$

$$\eta_{2g}(G) = q - p - t_2 + t \ge 1 - t_2 + n + 3$$
$$\eta_{2g} \ge n - t_2 + 4$$
$$\therefore \eta_{2g}(G) = n - t_2 + 4$$

The proof of the remaining cases is similar to that of theorem 4.2.1

Theorem: 4.2.3 Let *G* be a bicyclic graph containing a $U(l,m)$ and both the cycles is of odd length.

Let *n denote* the number of pendant vertices of *G* and let *k* denote the number of vertices of degree greater than 4 on $U(l,m)$ other than u_0. Then

$$\eta_{2g}(G) = \begin{cases} 4 & \text{if } k = 0 \\ n - t_2 + 5 & \text{if } k = 0 \,\&\, \deg u_0 = 5, \text{a tree attached with } u_0 \\ n - t_3 + 3 & \text{if } k \ge 1 \,\&\, (v,w) \text{ section is not a shortest path} \\ n - t_2 + 2 & \text{if } k = 2 \,\&\, (v,w) \text{ section is a shortest path} \\ n - t_2 + 1 & k \ge 3 \end{cases}$$

Proof: Let $V(U(l,m)) = \{u_0, u_1, u_2, \ldots, u_{l-1}, u_l, u_{l+1}, \ldots, u_{l+m-2}\}$

$V(C_l) = \{u_0, u_1, u_2, \ldots, u_{l-1}, u_0\}$

$V(C_m) = \{u_0, u_l, u_{l+1}, \ldots, u_{l+m-2}, u_0\}$ where *l* and *m* are odd.

Case-1: $k = 0$

Case-1(a):

Then $G = U(l,m)$

The geodesic 2- graphoidal cover is as follows

$P_1 = \{u_i, u_{i-1}, \ldots, u_1, u_0, u_l, u_{l+1}, \ldots, u_j\}$

$P_2 = \{u_{i+1}, u_{i+2}, u_0, u_{l+m-2}, \ldots u_{j+1}\}$ where $[i = \frac{l-1}{2}$ & $j = l + \frac{(m-1)}{2} - 1]$

$P_3 = \{u_i, u_{i+1}\}$

$P_4 = \{u_{j+1}, u_j\}$

$\psi = \{P_1, P_2, P_3, P_4\}$ be The geodesic 2-graphoidal cover of G

$\therefore \eta_{2g} \leq 4$

Since atleast four vertices on $U(l;m)$ are exterior vertices in any minimum geodesic 2-graphoidal cover so that $t \geq 4$

Hence $\eta_{2g} = q - p - t_2 + t \Rightarrow \eta_{2g} \geq 4$

Thus $\eta_{2g} = 4$

Case-1(b): $G = U(l,m)$ with $\deg u_0 = 5$ and there is a tree attached at u_0 with n pendant vertices.

$P_1 = \{u_i, u_{i-1}, \ldots, u_1, u_0, u_l, u_{l+1}, \ldots, u_j\}$

$P_2 = \{u_{i+1}, u_{i+2}, u_0, u_{l+m-2}, \ldots u_{j+1}\}$ where $[i = \frac{l-1}{2}$ & $j = l + \frac{(m-1)}{2} - 1]$

$P_3 = \{u_i, u_{i+1}\}$

$P_4 = \{u_{j+1}, u_j\}$

Consider the graph $G_1 = G - \{u_1, u_2 \ldots,, u_{l-2}, u_l, u_{l+1}, \ldots, u_{l+m-2}\}$ is a tree with n+1 pendant vertices.

Let ψ_1 be a minimum geodesic 2- graphoidal cover of G_1 and let $t_2(G_1) = \max t_2(\psi_1)$

Using Corollary 4.1.9, $\eta_{2g}(G_1) = n - t_2(G_1)$

Now $\psi = \psi_1 \cup \{P_1, P_2, P_3, P_4\}$ is geodesic 2- graphoidal cover of G.

$\Rightarrow \eta_{2g}(G) \leq n - t_2(G_1) + 4$ where $t_2 = t_2(G_1) + 1$

$\leq n - (t_2 - 1) + 4$

$\eta_{2g}(G) \leq n - t_2 + 5$

Since in any minimum geodesic 2-graphoidal cover of G , all the n pendant vertices and at least four vertices are exterior so that the number of exterior points $t \geq n + 4$

$\eta_{2g}(G) = q - p - t_2 + t \geq 1 - t_2 + n + 4$

$\eta_{2g} \geq n - t_2 + 5$

$\therefore \eta_{2g}(G) = n - t_2 + 5$

The proof of the remaining cases is similar to that of theorem 4.2.1

Similar to the Theorem 4.2.4 to Theorem 4.2.7 we have the following results for the bicyclic graphs *D(l,m,i)* and $C_m(i;l)$

Theorem: 4.2.4 Let *G* be a bicyclic graph containing a long dumbbell graph *D(l,m,i)* if both cycles are of even length. Let *n* denote the number of pendant vertices of *G* and let *k* denote the number of vertices of degree greater than 4 on *D(l,m,i)* other than u_{l-1} & u_{l+i-1} .Then

$$\eta_{2g}\left(G\right)=\begin{cases}3 \quad \text{if } k=0\\ n-t_2+2 \text{ if } k=1 \text{and every } (v,w) \text{section is a shortest path}\\ n-t_2+3 \text{ if } k\geq 1 \text{and every } (v,w) \text{section is not a shortest path}\\ n-t_2+1 \text{otherwise}\end{cases}$$

Theorem: 4.2.5 Let G be a bicyclic graph containing a long dumbbell graph *D(l,m,i)* if both cycles are of odd length. Let *n* denote the number of pendant vertices of *G* and let *k* denote the number of vertices of degree greater than 4 on *D(l,m,i)* other than u_{l-1} & u_{l+i-1}.Then

$$\eta_{2g}\left(G\right)=\begin{cases}5 \quad \text{if } k=0\\ n-t_2+4 \text{if } k=1\,\&\,(v,w) \text{section is not a shortest path}\\ n-t_2+2 \text{ otherwise}\end{cases}$$

Theorem: 4.2.6 Let G be a bicyclic graph containing a $C_m(i;l)$ if both cycles are of even length. Let *n* denote the number of pendant vertices of *G* and let *k* denote the number of vertices of degree greater than 4 on $C_m\left(i,l\right)$ other than u_0 and u_i . Then

$$\eta_{2g}\left(G\right)=\begin{cases}3 \quad \text{if } k=0\\ n-t_2+3 \text{ if } k=0\,\&\,\deg u_0\geq 4, \text{a tree attached with } u_0\\ \qquad \&\,\text{if } k=1\\ n-t_2+4 \text{ if } k=2 \text{the } (v,w) \text{ section is not a shortest path}\\ n-t_2+2 \text{ otherwise}\end{cases}$$

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