International Journal of Mathematical Archive-2(6), June - 2011, Page: 964-968
Available online through www.ijma.info ISSN 2229 – 5046

A GENERALIZED FIXED POINT THEOREM FOR OCCASSIONALLY WEAKLY COMPATIBLE MAPPINGS

R. K. Verma*

Department of Mathematics, Govt. C. L. C. College, Patan, Dist- Durg, 491111 (C.G.), India

E-mail: rohitverma1967@rediffmail.com, rohitverma1967@sify.com

(Received on: 19-02-11; Accepted on: 27-02-11)

ABSTRACT.

Zhang (X. Zhang, Common fixed point theorems for some new generalized contractive type mappings, J. Math. Anal. Appl. ***333*** *(2) (2007) 780-786) introduced a new generalized contractive type conditions for a pair of mappings in a complete metric space, in which the integral operator was replaced by a monotone non-decreasing function* ***F****. In this paper, we prove a common fixed point theorem for a generalized contractive type condition for a quadruple of (owc) mappings under relaxed conditions.*

2000 Mathematics Subject Classification: *47H10, 54H25.*

Keywords and Phrases: *Common fixed points, compatible mappings, generalized contractive condition, weakly compatible mappings, occasionally weakly compatible (owc) mappings.*

---

1. INTRODUCTION:

Recently, some fixed point theorems ([1], [3], [11], [13] etc.) for mappings satisfying an integral type contractive condition are obtained. In this consequence, Zhang [14] introduced a generalized contractive type condition, in which the integral type operator was replaced by a monotone non-decreasing function ***F***, for mappings in a metric space to extend and unify the results of [3], [11] and [13]. In this paper, we extend the result of Zhang [14]. For this, we need following notions of Zhang [14]:

Let a € $(0, \infty]$, $R_a^+ = [0, a)$ and F: $R_a^+ \to R_+$ satisfying:

(i) F (0) =0 and F (t)>0 for each t € (0, a)

(ii) F is non-decreasing on R_a^+

(iii) F is continuous,

where R_+ denotes the set of non-negative real numbers. Define the family T [0, a) by:

$$T\,[0, a) := \{ F: F \text{ satisfies (i)-(iii)}\} \tag{1.1}$$

We refer to Ex. 1 of Zhang [14]. Further, we have the following Lemma:

Lemma 1.1 ([14]) *Let a€(0, ∞] and F€T [0, a). If* $\lim_{n\to\infty} F(\varepsilon_n) = 0$ *then* $\lim_{n\to\infty} \varepsilon_n = 0$, *for each* ε_n € R_a^+.

Definition 1.2 ([14]) Let a € $(0, +\infty]$, $R_a^+ = [0, a)$ and ψ: $R_a^+ \to R_+$ satisfying:

(i) ψ (t)<t for each t a € (0, a),

(ii) ψ is non-decreasing and right upper semi-continuous,

(iii) for each t € (0, a), $\lim_{n\to\infty} \psi^n(t) = 0$. Then define the family Ψ [0, a) by:

$$\Psi\,[0, a) := \{ \psi: \psi \text{ satisfies (i)-(iii)}\}. \tag{1.2}$$

Following Lemma is obtained for non-decreasing right u.s.c. function ψ € Ψ:

***Corresponding author:** *R. K. Verma*, ***E-mail:** *rohitverma1967@sify.com*

Lemma 1.3 ([14]) *If $\psi \in \Psi[0, a)$, then $\psi(0) = 0$.*

Zhang proved the following theorem:

Theorem 1.4 *Let X be a complete metric space and D= sup {d(x, y): x, y€ X}. Set a =D, if D=∞ and a>D, if D<∞. Suppose A, B: X→X, F €T [0, a) and $\psi \in \Psi[0, F(a-0))$ satisfy:*

$$F(d(Ax, By)) \leq \psi(F(m(x, y))), \quad \text{for each } x, y \in X, \tag{1.3}$$

where m(x, y)=max{d(x, y), d(Ax, x), d(By, y), ½[d(Ax, y)+d(By, x)]}. Then A and B has a unique common fixed point in X. Moreover, for each x_0 in X, the iterated sequence $\{x_n\}$ with $x_{2n+1}=Ax_{2n}$ and $x_{2n+2}=Bx_{2n+1}$ converges to the common fixed point of A and B.

2. PRELIMINARIES:

In 1986, Jungck [5] introduced the notion of compatible mappings in metric space as a generalization of weakly commuting maps of Sessa [12]. This was further generalized to weakly compatible maps by Jungck [6] and to occassionally weakly compatible (owc) maps by Al-Thagafi and N. Shahzad [2].

Definition 2.1 Let A and S be two self-maps of a metric space (X, d). The pair (A, S) is said to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t$, for some t€ X.

Definition 2.2 Let A, S:X→X, then the pair (A, S) is said to weakly compatible if they commute at their coincidence points; i.e., ASu = SAu whenever Au = Su, for some u € X.

Definition 2.3 (Al-Thagafi and Shahzad [2]) Two self-mappings A and S are called occassionally weak compatible (owc) if ASu = SAu for some u in X whenever Au = Su. The point u 2 X is called the (owc) point of the pair. The set of (owc) points of (A, S) is denoted by *C* (A, S).

It is to be noted that compatible maps are weakly compatible but the converse need not true (see, [7], [9]).Further, weakly compatible maps implies (owc) but not conversely. Thus, (owc) is the weakest form of commuting type mappings.

In this paper, we will extend this theorem for a quadruple of mappings, by relax-ing the completeness of whole space X, and by replacing the iterated sequence on X by set-inclusion relation. We will take the (owc) mappings, as it is the weakest form of commuting type mappings.

Throughout this paper, let us denote the sets of natural and real numbers by **N** and **R** respectively. Let D be the diameter of a metric space (X, d). Put a = D if D = ∞, and a > D if D < ∞. Here is the main result:

3. MAIN RESULTS:

Theorem 3.1 *Let (X, d) be a metric space and A, B, S and T be four self-mappings on X. Let D = sup{d(x, y):x, y €X} . Set a = D if D = ∞, and a > D if D< ∞. Suppose F €ך [0, a) and $\psi \in \Psi[0, F(a-0))$ be functions satisfying:*

$$F(d(Ax, By)) \leq \psi(F(m(x, y))), \text{ for each } x, y \in X, \tag{1.3}$$

where m(x, y) = max{d(Sx,T y), d(Ax, Sx), d(By,T y), ½[d(Ax, Ty)+d(By, Sx)]}.

If the pairs (A, S) and (B, T) are occasionally weakly compatible (owc), then A, B, S and T have a unique common fixed point.

Proof Suppose that (A, S) and (B, T) are (owc). Let λ, μ € X be the (owc) point of the pairs (A, S) and (B, T), respectively. Then, by definition, ASλ = SAλ, whenever Aλ = Sλ. Next, the pair (B, T) is (owc), so by definition, BTμ = TBμ, whenever Bμ = Tμ.

We claim Aλ=Bμ. If not, i.e., d(Aλ, Bμ) > 0, then putting x=λ and y=μ in (3.1), we have

.

$$\begin{aligned} F(d(A\lambda, B\mu)) &\leq \psi(F(m(\lambda, \mu)) \\ &= \psi(F(\max\{d(S\lambda, T\mu), d(A\lambda, S\lambda), d(B\mu, T\mu), ½[d(B\mu, S\lambda)+d(A\lambda, T\mu)]\})) \\ &= \psi(F(\max\{d(A\lambda, B\mu), 0, 0, d(A\lambda, B\mu)\})) \\ &= \psi(F(d(A\lambda, B\mu))) \\ &< F(d(A\lambda, B\mu)), \end{aligned}$$

a contradiction. Thus $A\lambda = B\mu$ (= z say). Therefore $A\lambda = S\lambda = B\mu = T\mu = z$; also $Az = Sz$ and $Bz = Tz$. Now, let us show that z is a common fixed point of A, B, S and T. For, putting $x = z$ and $y = \mu$ in (3.1), we have

$$\begin{aligned} F(d(Az, z)) &= F(d(Az, B\mu) \leq \psi(F(m(z, \mu))) \\ &= \psi(F(\max\{(Sz, T\mu), d(Az, Sz), d(B\mu, T\mu), ½[d(B\mu, Sz)+d(Az, T\mu)]\})) \\ &= \psi(F(\max\{d(Az, B\mu), 0, 0, d(Az, B\mu)\})) \\ &= \psi(F(d(Az, B\mu))) \\ &< F(d(Az, B\mu)) = F(d(Az, z)), \end{aligned}$$

a contradiction. Thus $Az = z = Sz$. Further, we show that $Az = Bz$. For, putting $x = z$ and $y = z$, in (3.1), we have

$$\begin{aligned} F(d(Az, Bz)) &\leq \psi(F(m(z, z))) \\ &= \psi(F(\max\{d(Sz, Tz), d(Az, Sz), d(Bz, Tz), ½[d(Bz, Sz) + d(Az, Tz)]\})) \\ &= \psi(F(\max\{d(Az, Bz), 0, 0, d(Az, Bz)\})) \\ &= \psi(F(d(Az, Bz))) \\ &< F(d(Az, Bz) \end{aligned}$$

a contradiction. Thus $Az = Bz = Sz = Tz = z$. Hence z is a common fixed point of A, B, S and T. Uniqueness of common fixed point follows easily. This completes the proof.

Remark 3.1 Every contractive condition of integral type automatically includes a corresponding contractive condition, not involving integrals, by setting $\varphi(t) \equiv 1$. So, we have obtained so many results concerning the compatible mappings and dealing with a contractive condition, in the literature of fixed point theory.

Remark 3.2 Our Theorem 3.1 is a generalization of the main Theorem of Pathak and Verma [10] in the sense of weakening the condition of weakly compatibility of pairs of mappings to (owc) mappings.

Remark 3.3 If we put A=B=f and S=T=g in our Theorem 3.1, and take X a complete metric space, we get main Theorem 1 of Zhang [14].

Following is an example of our main result:

Example 3.2 Let X= [0, 2), so that D = diam(X) = 2. Let $F(t) = t^{1/t}$, for each t > 0 and F(0) = 0, so that F€J[0, a), where a = e > D = 2. Here R^+_a =[0, a). Let $\psi(t) = t/2$ so that $\lim_{n\to\infty}\psi^n(t) = 0$ and ψ€Ψ[0, F(a-0)). Define d on X by: d(x, y) =0, if x = y, and d(x, y)
= max{x, y}, if $x \neq y$. Note that (X, d) is a metric space (see Example p.928 of Jachymski [4]). Let A, B, S and T: X→ X be mappings defined by:

Ax=1/(n+4), Bx=1/(n+3), Sx=1/(n+2), Tx=1/(n+1) if x=1/n, n€ **N**

Ax = 0, otherwise, Bx = 0, otherwise Sx = 0 otherwise, Tx = 0 otherwise; but x€ X.

Then we observe that:

(i) (A, S) and (B, T) are (owc) mappings. The set of (owc) points of (A, S) is *C*(A, S) = X-{1/2, 1/3, …}, and that of (B, T) is *C*(B, T) = X-{1, 1/2, 1/3, ...}.

(ii) Though the observation of common fixed point of this example is easy (t = 0 € X), but it is important to satisfy the Inequality (3.1). The generalized contractive condition (3.1) satisfy, for all x, y € X, as follows:

(a) If x=0, y=1/n, then Ax = Sx = 0, and so that d(Ax, By)=1/(n+3) and
M(x, y)={1/(n+1), 0, 1/(n+1), ½[1/(n+3)+ 1/(n+1)] =1/(n+1).

Therefore $F(d(Ax, By)) = [1/(n+3)]^{n+3} \leq ½\,[1/(n+2)]^{n+2} \leq ½\,[1/(n+1)]^{n+1} = \psi(F(M(x, y)))$

(b) If y=0, x=1/n, then By = Ty = 0, and so that d(Ax, By)=1/(n+4) and
M(x, y)={1/(n+2), 1/(n+2), 0, ½[1/(n+2)+ 1/(n+4)] =1/(n+2).

Therefore $F(d(Ax, By)) = [1/(n+4)]^{n+4} \leq ½\,[1/(n+2)]^{n+2} = \psi(F(M(x, y)))$.

(c) If x=1/n and y is any rational number (not the form of 1/n , n € N), then By=Ty=0;
so that d(Ax, By)=1/(n+4) and M(x, y)=1/(n+2).

Therefore $F(d(Ax, By)) = [1/(n+4)]^{n+4} \le ½ [1/(n+2)]^{n+2} = \psi(F(M(x, y)))$.

(d) If x=1/n and y is any irrational number, then d(Ax, By)=1/(n+4) and M(x, y) =1/(n+2). Therefore $F(d(Ax, By)) = [1/(n+4)]^{n+4} \le ½ [1/(n+2)]^{n+2} = \psi(F(M(x, y)))$.

(e) If x=1/n, y=1/m, m<n, then $1/(m+3) \ge 1/(n+4)$. Thus

d (Ax, By)=max{1/(n+4), 1/(m+3)}=1/(m+3);
d (Sx, Ty) = max{1/(n+4),1/(n+2)}=1/(n+2);
d (Ax, Sx) = max{1/(m+3),1/(m+1)}=1/(m+1);
d (By, Sx) = max{1/(m+3),1/(n+2)}=1/(m+3);
d (Ax, Ty) = max{1/(n+4),1/(m+1)}=1/(m+1);
½[d (By, Sx) +d (Ax, Ty)] = ½[1/ (m+3), 1/(m+1)] = 1/(m+1).

So that

$$F (d (Ax, By)) = [1/(m+3)]^{m+3} \le ½ [1/(m+2)]^{m+2}$$
$$\le (½)^2.[1/(m+1)]^{m+1} \le ½.[1/(m+1)]^{m+1} = \psi(F(M(x, y))).$$

If x=1/n, y=1/m, m>n, then d (Ax, By) = 1/ (n+4) or 0, according as m+3 > n+4 or m+3 = n+4. So that F(d(Ax, By)) = $[1/(n+4)]^{n+4}$ or 0 according the case is chosen.

Now, for M(x, y), there arises following three cases:

(*) if m+3 > m+1 > n+4 is chosen, then M(x, y) = 1/(n+2).
(**) if m+3 > n+4 > m+1 is chosen, M(x, y) = 1/(n+2), as m+1>n+2.
(***) m+3 = n+4, then M(x, y) = 1/(n+2), as m+1=n+2.

Thus in each sub-cases M(x, y) = 1/ (n+2). We thus have

$$F(d(Ax, By)) \le [1/(n+2)]^{n+2} = \psi(F(M(x, y))).$$

For the rest of the cases, we have d (Ax, By) = 0 and M(x, y) = 0, so that $F (d (Ax, By)) = 0 = \psi(F(M(x, y)))$.

Therefore in each case the inequality (3.1) satisfies. Hence all the conditions of our Theorem 3.1 satisfy. So that A, B, S and T have its only common fixed point x = 0 € X.

REFERENCES:

[1] A. Aliouche, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type, *J. Math. Anal. Appl*. **322**(2006), 796–802.

[2] M. A. Al-Thagafi and N. Shahzad, Generalized I-nonexpansive self-maps and its invariant approximations, *Acta Math. Sinica* **24**(5)(2008), 867-876.

[3] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, *Internat. J. Math. Math. Sci.* **29**(9)(2002), 531–536; MR1900344 (2003c:54075).

[4] J. Jachymski, Common fixed point theorems for some families of maps, *Indian J. Pure Appl. Math.* **25**(9)(1994), 925–937; MR1294061 (95g:54038).

[5] G. Jungck, Compatible mappings and common fixed points, *Internat. J. Math. Math. Sci*. **9**(4)(1986), 771–779; MR0870534 (87m:54122).

[6] G. Jungck, Common fixed points for non-continuous non-self mappings on a non-numeric spaces, *Far East J. Math. Sci*. **4**(2)(1996), 199–212.

[7] G. Jungck, B. E. Rhoades, Fixed points for set-valued functions without continuity, *Indian J. Pure Appl. Math*., **29**(3)(1998), 227-238.

[8] G. Jungck and B. E. Rhoades, Fixed point theorems for occasionally weakly compatible mappings, *Fixed Point Theory* **7**(2)(2006), 287–296.

[9] H. K. Pathak, R. K. Verma, A common fixed point theorem for weak compatible mappings on normed vector space and its applications *J. Nonlinear Funct. Anal. & Diff. Eqns*. **2**(1)(2008), 11-27.

[10] H. K. Pathak, R. K. Verma, Common fixed point theorem for generalized contractive condition, *Bull. Math. Anal. Appl*.(Accepted).

[11] B. E. Rhoades, A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.* **226**(1977), 257-290; MR0433430 (55#6406).

[12] S. Sessa, On a weak commutative condition in fixed point consideration, *Publ. Inst. Math. (Beograd)* **32**(46)(1982), 146–153; MR0710984(85f:54107).

[13] P. Vijayaraju, B. E. Rhoades and R. Mohanraj, A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, *Intnat. J. Math. Math. Sci*. **15**(2005), 2359–2364; MR2184475 (2006g: 54050).

[14] X. Zhang, Common fixed point theorems for some new generalized contractive type mappings*, J. Math. Anal. Appl*. **333**(2)(2007), 780–786; MR2331693 (2008e:47150)

****