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# EXTENSIONS OF ORDERED SETS – CONSTRUCTIVE POINT OF VIEW[1]

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*ABSTRACT.*

*This investigation is in the Bishop's constructive mathematics. A theorem of the ideal extensions for ordered sets is given. If X and Y are ordered sets under a partial order and an anti-order, we construct the ordered sets* $V = X \cup Y$ *which has ideal A isomorphic to X, and anti-ideal B in V isomorphic to Y.*

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## 0 INTRODUCTIONS:

### 0.1 Setting and motivation:

The arguments in this paper conform to constructive mathematics in the sense of Bishop. Our setting is Bishop's constructive mathematics [2], [3], [5], [9], mathematics developed with Constructive logic (or Intuitionistic logic ([19]) – logic without the Law of Excluded Middle $P \vee \neg P$. We have to note that 'the crazy axiom' $\neg P \Rightarrow (P \Rightarrow Q)$ is included in the Constructive logic. Precisely, in Constructive logic the 'Double Negation Law' $P \Leftrightarrow \neg\neg P$ does not hold but the following implication $P \Rightarrow \neg\neg P$ holds even in the Minimal logic. In Constructive logic 'Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not hold also. It is interesting, in Constructive logic the following deduction principle $A \vee B, \neg A \vdash B$ holds, but this is impossible to prove without 'the crazy axiom'. As Intuitionistic logic is a fragment of Classical logic, our arguments should be valid from a classical point of view.

The extension problem for groups is as follows: given two groups H and K, construct all groups G which have a normal subgroup N such that N is isomorphic to H (in symbol, $N \cong H$) and $G/N \cong K$ (where G/N is the quotient of G by N). G is called the extension of H by K. Ideal extensions of semigroups have been considered by Clifford in [6] with exposition of the theory appearing in [7], [17]. Ideal extensions of totally ordered semigroups have been studied in [9], [10], and the ideal extensions of topological semigroups in [5], [8]. Ideal extensions of lattices have been considered in [11]. Ideal extensions of ordered semigroups have been studied in [12], [13] and [14].

In this paper we study ordered set under two compatible relations: partial order and anti-order relations.

### 0.2 Set with diversity:

Let $(X,=,\neq)$ be a set in the sense of books [2], [3], [5] and [9], where $\neq$ is a binary relation on X which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Rightarrow y \neq x,\ x \neq z \Rightarrow x \neq y \vee y \neq z$$

called *apartness* (A.Heyting). The relation $\neq$ must be extensional by the equality, in the following sense

$$x \neq y \wedge y = z \Rightarrow x \neq z.$$

Let Y be a subset of X and let $x \in X$. By $x \rhd\!\lhd Y$ we denote $(\forall y \in Y)(y \neq x)$ and by $Y^C$ we denote subset $\{x \in X : x \rhd\!\lhd Y\}$ – the strong complement of Y in X ([3]). The subset Y of X is *strongly extensional* ([19]) in X if and only if $y \in Y \Rightarrow y \neq x \vee x \in Y$.

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**Example I**: Let $\wp(X)$ be power-set of set X. If we for subsets A,B of X define $A \neq B$ if and only if $(\exists a\in A)\neg(a\in B)$ or $(\exists b\in B)\neg(b\in A)$, then the relation $\neq$ is diversity relation on $\wp(X)$ but it is not an apartness.
(2) ([9]) The relation $\neq$ defined on the set $\mathbf{Q}^{\mathbf{N}}$ by

$$f \neq g \Leftrightarrow (\exists k\in \mathbf{N})(\exists n\in \mathbf{N})(\forall \in \mathbf{N})(m \geq n \Rightarrow |f(m) - g(m)| > k^{-1})$$

is an apartness on $\mathbf{Q}^{\mathbf{N}}$. ♦

For subsets X and Y of A we say that set X is *set-set apartness* from Y, and it is denoted by $X \rhd\lhd Y$, if and only if $(\forall x\in X)(\forall y\in Y)(x \neq y)$. Sometime, we set $x \rhd\lhd Y$ instead $\{x\} \rhd\lhd Y$, and, of course, $x \neq y$ instead $\{x\} \rhd\lhd \{y\}$. With $S^{C} = \{x\in X: x \rhd\lhd S\}$ we denote *apartness complement* of S. So, $\rhd\lhd$ is relation between pairs of subsets of A. It is easy to see that the following holds:

(0) $\neg(X \rhd\lhd X)$;
(1) $X \rhd\lhd Y \Rightarrow \neg(X = \emptyset \wedge Y = \emptyset)$;
(2) $X \rhd\lhd Y \Rightarrow X\cap Y = \emptyset$;
(3) $X \rhd\lhd Y \wedge Z \subseteq Y \Rightarrow X \rhd\lhd Z$
(3') $X \rhd\lhd (Y\cup Z) \Leftrightarrow X \rhd\lhd Y \wedge X \rhd\lhd Z$;
(4) $X \rhd\lhd Y \Rightarrow Y \rhd\lhd X$;

Let Y be a subset of $(S,=,\neq)$. We say that it is *detachable* if and only if

$$(\forall x)(x\in S \Rightarrow x\in Y \vee x \rhd\lhd Y).$$

For a function $f : (S,=,\neq) \to (T,=,\neq)$ we say that it is a *strongly extensional* if and only if $(\forall a,b\in S)(f(a) \neq_T f(b) \Rightarrow a \neq_S b)$.

## 0.3 Filled product:

Let X be a set with apartness and let $\alpha$, $\beta$ be relations on X. The *filed product* ([18], [20], [21]) of $\alpha$ and $\beta$ is the relation defined by

$$\beta*\alpha = \{(x,z)\in X\times X : (\forall y\in X)((x,y)\in \alpha \vee (y,z)\in \beta)\}.$$

For n ($\geq$ 2) let ${}^{n}\alpha = \alpha*\ldots*\alpha$ (n factors). Put ${}^{1}f = f$. By $c(\alpha)$ we denote the intersection $\cap_{n\in \mathbf{N}}\, {}^{n}\alpha$. The relation c(f) is a cotransitive relation on X, by [18], (or [20], [21]) called *cotransitive internal fulfillment* of the relation $\alpha$. Therefore, the relation $c(f \cap \neq)$ is the maximal consistent and cotransitive relation on X under $\alpha$.

## 0.4 Coequality relation:

Set with apartness was first defined and studied by Heyting. After that, several authors have worked on this important topic as for example: Bishop ([2]), Bridges and Richman ([4]), Mines, Richman and Ruitenburg ([16]), Troelstra and van Dalen ([24]), and this author ([18], [20], [21]). A relation q on X is a *coequality relation* on X if and only if

$$q \subseteq \neq,\ q^{-1} = q \text{ and } q \subseteq q*q.$$

If q is coequality on set S, then the strong complement $q^C$ of q is an equality on the set S compatible with q in the following sense:

$$(\forall x,y,z\in S)((x,y)\in q \wedge (y,z)\in q^C \Rightarrow (x,z)\in q).$$

In the case when we have a pair $(\rho,q)$ of compatible an equality and a coequality on set S, then we can construct the factor-set $S/(\rho,q)$ with

$$a\rho =_1 b\rho \Leftrightarrow (a,b)\in \rho,\ a\rho \neq_1 b\rho \Leftrightarrow (a.b)\in q,\ a\rho \cdot b\rho =_1 ab\rho.$$

Particularly, we can construct the factor-set $S/(q^C, q)$ in which equality and apartness defined by

$$a\, q^C =_1 b\, q^C \Leftrightarrow (a,b) \rhd\lhd q,\ aq \neq_1 bq \Leftrightarrow (a,b)\in q,\ a\, q^C \cdot b\, q^C =_1 ab\, q^C.$$

Except that, we can construct the factor-set $S/q = \{aq : a\in S\}$ where equality and apartness defined as above:

$$aq =_1 bq \Leftrightarrow (a,b) \rhd\lhd q,\ aq \neq_1 bq \Leftrightarrow (a,b)\in q,\ aq \cdot bq =_1 abq.$$

The mapping $\pi(\rho,q) : S \to S/(\rho,q)$ are strongly extensional and surjective function.

 

## 0.5 Goals of this article:

The aim of this paper is to construct the ideal extensions of ordered sets. "We are often interested in building more complex semigroups, lattices, ordered sets, and ordered or topological semigroups out of some of "simpler" structure and this can be sometimes achieved by constructing the ideal extensions" (Kehayopulu, [13]). If X and Y are two disjoint ordered sets, an ordered set V is called an ideal extension (or just an extension) of X by Y if there exists an ideal A of V which is isomorphic to X and the complement $X^C$ of A to V is isomorphic to Y. We give the main theorem of such extensions, which is the following: If $(X,=_X,\neq_X,\leq_X,\alpha_X)$ and $(Y,=_Y,\neq_Y,\leq_Y,\alpha_Y)$ are two disjoint ordered sets, $\theta$ an arbitrary subset of $X\times Y$, and

$$\Theta(\theta) = \{(a,b)\in X \times Y \mid (\exists(x,y)\in \theta \subseteq X\times Y)(a \leq_X x \wedge y \leq_Y b\},$$

and

$$\Omega(\theta) = c((\Theta(\theta)^C) \cap ((X \times Y)\cup(Y\times X))$$

then the set $V = X \cup Y$ endowed with the order "$\leq$", defined by $\leq = \leq_X \cup \leq_Y \cup \Theta$, and with the antiorder "$\Xi$", defined by $\Xi = \alpha_X \cup \alpha_Y \cup \Omega(\theta)$, is an ordered set and it is an extension of X by Y.

Conversely, if $(V,=_V,\neq_V,\leq_V,\Xi)$ is an extension of $(X,=_X,\neq_X,\leq_X,\alpha_X)$ by $(Y,=_Y,\neq_Y,\leq_Y,\alpha_Y)$, then the set $X \cup Y$, endowed with the relations "=", "$\neq$", "$\leq$" and "$\Sigma$" defined by

$$\neq = \neq_X \cup \neq_Y \cup (X\times Y)\cup(Y\times X), \quad \leq = \leq_X \cup \leq_Y \cup \Theta, \quad \Sigma = \alpha_X \cup \alpha_Y \cup \Omega,$$

is an ordered set and there exists strongly extensional, embedding, injective, order isotone and reverse isotone, anti-order isotone and reverse isotone function

$$f: (X \cup Y, =, \neq, \leq, \Sigma) \to (V,=_V, \neq_V, \leq_V, \Xi).$$

First, notion and elementary properties of anti-order relation of sets are introduced. The basic definitions and properties of ordered sets under order and anti-order are presented in section 2. Elementary properties of ordered anti-ideals of ordered set are given in the above mentioned section. In the section 3 we give some preliminaries results and, in section 4, we give main results (Theorem 4.1 and Theorem 4.2).

## 0.6 References:

For undefined notions and notations of classical ordered set we referred to books [1], [3], [7], [15], [17] and papers [5], [6], [8], [10]-[14], [23]. For constructive items we referred to well-known books [2], [4], [16], and [24], and to author's papers [18]-[22].

## 1 ORDERED SET:

This section we start with the following definitions:

Let $(S, =_S, \neq_S)$ be a set with apartness. For S we say that it is an *ordered set* if S equipped with relation $\leq$ (*partial order*) or $\Theta$ (*anti-order*) such that:

(1) The relation $\leq$ satisfies the following conditions:

$$=_S \subseteq \leq \text{ (reflexivity)}, \ \leq \circ \leq^{-1} \subseteq =_S \text{ (anti-symmetric)}, \ \leq \circ \leq \ \subseteq \leq \text{ (transitivity)};$$

The relation $\rho$ on S is a *quasi-order* if it is reflexive and transitive. If $\rho$ is a quasi-order on set S, then the relation $\rho\cap\rho^{-1}$ is an equality relation on S.

(2) As in [19] we define the notion of anti-order on set with apartness: The relation $\Theta$ satisfies the following conditions:

$$\Theta \subseteq \neq_S \text{ (consistency)}, \neq_S \subseteq \Theta \cup \Theta^{-1} \text{ (linearity) and } \Theta \subseteq \Theta * \Theta \text{ (cotransitivity)}.$$

As in [18], [20], [21] the relation $\omega$ is a *quasi-antiorder* on set S if it is consistent and cotransitive. If $\omega$ is a quasi-antiorder on set S, then the relation $q = \omega \cup\omega^{-1}$ is a coequality S.

(3) Relations $\leq$ and $\Theta$ are *compatible* in the following sense:

$$(\forall x,y,z\in S)(x \leq y \wedge z\Theta y \Rightarrow z\Theta x).$$

**NOTES:** Compatibility of order and anti-order relations in set is important notion for our work.

(i) The implication $x \leq y \wedge z\Theta y \Rightarrow z\Theta x$ is equivalent to condition $\neg(x \leq y \wedge x\Theta y)$. Indeed: Suppose that implication $x \leq y \wedge z\Theta y \Rightarrow z\Theta x$ holds and suppose that $x \leq y$ and $x\Theta y$. Then, by compatibility of relations, we have $x\Theta x$. It is impossible, because the relation $\Theta$ is consistent. So, should be $\neg(x \leq y \wedge x\Theta y)$. Opposite, let condition $\neg(x \leq y \wedge x\Theta y)$ holds. If $x \leq y \wedge z\Theta y$, then, by cotransitivity of $\Theta$, we have $z\Theta x \vee x\Theta y$. Thus we conclude $z\Theta x$, because $x \leq y$ and $x\Theta y$ is impossible. So, the implication $x \leq y \wedge z\Theta y \Rightarrow z\Theta x$ is consequent of the condition $\neg(x \leq y \wedge x\Theta y)$.

(ii) Except that, if relations $\leq$ and $\Theta$ are compatible, then implication $x\Theta y \wedge z \leq y \Rightarrow x\Theta z$ holds too. In fact, from $x\Theta y$ follows $x\Theta z$ or $z\Theta y$. Since $z \leq y$ and $z\Theta y$ is impossible, we deduce $x\Theta z$.

(iii) Let us note that the apartness on set S is an antiorder relation on S.

Essence of connection between partial order and anti-order relation in set with apartness is given in following lemma:

**Lemma 1.1** *Let* $\Theta$ *be an anti-order on set* $(S,=,\neq,\cdot)$. *Then* $\Theta^C$ *is an order on* $(S,\neg\neq, \neq,\cdot)$. *If the order relations* $\leq$ *on set* S and $\Theta$ *are compatible, then* $\leq \subseteq \Theta^C$.

**Proof**: (i) Let $a = b$ and let $(u,v)$ be an arbitrary element of $\Theta$. Then $(u,a)\in\Theta \vee (a,b)\in\Theta \vee (b,v)\in\Theta$. Thus $u \neq a \vee a \neq b \vee b \neq v$. Since $a \neq b$ is impossible, we have $(a,b) \neq (u,v)\in\Theta$. So, $= \subseteq \Theta^C$, i.e. the relation $\Theta^C$ is reflexive relation.

(ii) Let $(a,b)\in \Theta^C \wedge (b,a)\in \Theta^C$. From $a \neq b$ we conclude that $(a,b)\in\Theta$ or $(b,a)\in\Theta$. It is a contradiction. So, we have to $\neg(a \neq b)$.

(iii) Let $(a,b)\in \Theta^C \wedge (b,c)\in \Theta^C$ and let $(u,v)$ be an arbitrary element of $\Theta$. Thus $(u,a)\in\Theta \vee (a,b)\in\Theta \vee (b,c)\in\Theta \vee (c,v)\in\Theta$. Thus $u \neq a \vee c \neq v$ because $(a,b)\in \Theta^C \wedge (b,c)\in \Theta^C$ holds. So, $(a,c)\in \Theta^C$.

(iv) Let a,b,c be arbitrary element of S such that $(a,b)\in \Theta^C$ and let $(u,v)$ be an arbitrary element of $\Theta$. Then $(u,ac)\in\Theta \vee (ac,bc)\in\Theta \vee (bc,v)\in\Theta$. In the second case we should have $(a,b)\in \Theta$ which is impossible. So, have to $u \neq ac$ or $bc \neq v$. Therefore, $(ac,bc)\in \Theta^C$.

(v) Let $a \leq b$ and let $(u,v)$ be an arbitrary element of $\Theta$. Then $(u,a)\in\Theta$ or $(a,b)\in\Theta$ or $(b,v)\in\Theta$. Since $(a,b)\in\Theta$ is impossible, then $u \neq a$ or $b \neq v$. So, $(a,b)\in \Theta^C$.

**Corollary 1.1.1** *Let* $\Theta$ *be a quasi anti-order on set* $(S,=,\neq,\cdot)$. *Then* $\Theta^C$ *is an quasi-order on* $(S,\neg\neq, \neq,\cdot)$. *If the quasi-order relations* $\alpha$ *on set* S *exists and* $\alpha$ *and* $\Theta$ *are compatible, then* $\alpha \subseteq \Theta^C$.

**Example II**: Let $S = \{a,b,c,d,e\}$ with apartness. Relation $\alpha$, defined by

$\alpha = \{(a,c),(a,d),(a,e),(b,a),(b,c),(b,d),(b,e),(c,a),(c,b),(c,d),(c,e),(d,a),(d,e),(e,a),(e,b),(e,d)\}$,

is an antiorder relation on set S and the relation

$$\beta = \{(a,e),(b,e),(c,a),(c,b),(c,d),(c,e),(d,e),(e,a),(e,b),(e,d)\}$$

is a quasi-antiorder relation on set S.

**Example III**: Let A be a strongly extensional consistent subset of a semigroup $(S,=,\neq,\cdot)$. Then relation $\Theta_A \subseteq S\times S$, defined by $(a,b)\in \Theta_A \Leftrightarrow a \neq b \wedge a\in A$, is an quasi-antiorder relation on S but it is not antiorder relation on S.
Indeed: It is clear that $\Theta_A \subseteq \neq$. Let a,b,c be arbitrary elements of S such that $a\, \Theta_A\, c$, i.e. let $a \neq c$ and $a\in A$. From $a \neq c$ follows $a \neq b$ or $b \neq c$. If $a \neq b$, then $a\, \Theta_A\, b$. Suppose that $b \neq c$ and $a\in A$. Then $a \neq b$ or $b\in A$. If $a\in A$ and $a \neq b$, we have $a\, \Theta_A\, b$ again. If $b\in A$ and $b \neq c$, then $b\, \Theta_A\, c$. Let $xa\, \Theta_A\, xb$ $(x\in S)$, i.e. let $xa \neq xb$ and $xa\in A$. Thus $a \neq b$ and $a\in A$. So, $a\Theta_A b$. Similarly, we get $ax\, \Theta_A\, bx \Rightarrow a\, \Theta_A\, b$.
Suppose that $\neg(a\in A)$ and $\neg(b\in A)$ and $a \neq b$. We can not conclude that the implication $a \neq b \Rightarrow a\Theta_A b \vee b\Theta_A a$ holds. So, the relation $\Theta_A$ is not an antiorder relation on S.

**Example III** ([19]): Let $(R,=,\neq,+,\cdot)$ be a commutative ring with apartness.

**(1)** The subset D of R is a cosubring of R if and only if satisfies the following conditions:

$$0 \rhd\!\lhd D,\ 1 \rhd\!\lhd D,\ -a\in D \Rightarrow a\in D,\ a+b\in D \Rightarrow a\in D \vee b\in D, \text{ and } ab\in D \Rightarrow a\in D \vee b\in D.$$

It is clear that cosubring D of R is strongly extensional in R.

(0) Every anti-ideal of a ring is a cosubring of R.
(1) Suppose that M is an A-module. Let S = A×M, and for (a,x), (b,y) let define:

$$(a,x) = (b,y) \Leftrightarrow a =_A b \wedge x =_M y,\ (a,x) \neq (b,y) \Leftrightarrow a \neq_A b \vee x \neq_M y\ ;$$

$$(a,x)+(b,y) = (a+b,x+y),\ \ (a,x)\cdot(b,y) = (ab,bx + ay).$$

That S is a ring under these definitions. S has the identity if and only if a contains the identity and M is an unitary A-module. The set A' = $\{(a,0)\in S : a\in A\}$ is a subring of S isomorphic to A. The set A" = $\{(a,x)\in S : x \neq_M 0\}$ is a cosubring of s. The set M" = $\{(a,x)\in S : a \neq_A 0\}$ is a coideal of S.

(3) Let m be an integer. Then the set $C(m) = \cup_{i=1,\dots,m-1}(m\mathbf{Z}+i)$ is a cosubring of the ring **Z.**

(4) Let K be a Richman's field and x be an unknown variable under K. Then the sets $C = \{\Sigma a_i x^i \in K[x] : a \neq 0\}$ and $D = \{\Sigma a_i x^i \in K[x] : \Sigma a_i \neq 0\}$ are cosubrings of K[x].

Let R be a commutative ring with an apartness and D be a cosubring of R. Then the set $D^C$ is a subring of R compatible with D in sense that $a\in D \wedge b\in D^C \Rightarrow a+b\in D$.

**(2)** If (K,=,≠,+) be an additive Abelian group, for relation Θ we say that it is *compatible* with the group operation if

$$(\forall x,a,b\in K)((a+x,b+x)\in \Theta \Rightarrow (a,b)\in \Theta).$$

2.1 ([19], Proposition 4.1) If a subset P of an Abelian group (K,=,≠,+) satisfies the following conditions:

$$0 \rhd\!\lhd P,\ P\cup(-P) = K^*,\ P\cap(-P) = \{0\},\ (\forall a,b\in K)(a+b\in P \Rightarrow a\in P \vee b\in P),$$

then the relation Θ on K, defined by $(a,b)\in \Theta \Leftrightarrow a - b\in P$, is an anti-order relation on K compatible with the group operation on K.

2.2 ([19], Theorem 5.2) Let (K,=,≠,+,0,·,1) be a field and D be a cosubring of K. Then:

(1) The set $S = \{a\in K : a\in D \vee a^{-1}\in D\}$ is a strongly extensional cosubgroup of the multiplicative group $K^* = \{a\in K: a \neq 0\}$ compatible with the subgroup $S^C$ and we can construct the factor-group $G = K^*/(S^C,S)$;

(2) On the group G we define a relation Θ by $(aS^C,bS^C)\in \Theta \Leftrightarrow a^{-1}b\in D$. The relation Θ on G is an anti-order relation on G compatible with the group operation on G.

## 2 ORDERED SUBSTRUCTURES:

Our next notions in ordered set are order substructures. We follows classical definition of order ideal of ordered semigroup under a partially order. Here we doing with set ordered by a partial order and by an anti-order. Definitions of order ideal and anti-ideal are given in the following definitions:

Let (S, =,≠) be an ordered set with apartness under order relation ≤ and under anti-order relationΘ.

(1) An *order ideal* of S is a subset I of S such that

$$(\forall x,y)(x\in I \wedge y \leq x \Rightarrow y\in I)$$

(2) An *order anti-ideal* of S is a subset K of S such that

$$(\forall x,y)(y\in K \Rightarrow y\Theta x \vee x\in K)$$

For an order ideal I and order anti-ideal K we say that they are *compatible* if and only if $I \subseteq \neg K$.

**Example IV**: (1) The order ideal generated by an element x is the set $(a] = \{y\in S : y \leq a\}$ called a *principal ideal* generated by element a. (2) The subset $K(a) = \{z\in S : z\Theta a\}$ is an order anti-ideal called a *principal anti-ideal* generated by element a. In fact: Let z be an arbitrary element of K(a) and let y be an arbitrary element of S. Then, from $z\Theta a$ follows $z\Theta y$ or $y\Theta a$. So, the implication $z\in K(a) \Rightarrow y\in K(a) \vee z\Theta y$ holds. Therefore, set K(a) is an order anti-ideal of S.

Now, suppose that we have a function $\varphi : (S,=_S,\neq_S,\leq_S,\Theta_S) \to (T,=_T,\neq_T,\leq_T,\Theta_T)$ between two ordered set under order and anti-order relations. First let us remind oneself of some standard notions and notations about functions: A function $\varphi$ is *strongly extensional* if

$$(\forall x,x'\in S)(\varphi(x) \neq_T \varphi(x') \Rightarrow x \neq_S x');$$

$\varphi$ is an *embedding* if and only if

$$(\forall x,x'\in S)(x \neq_S x' \Rightarrow \varphi(x) \neq_T \varphi(x')).$$

Now, we need new kind of function between ordered sets

(1) A strongly extensional function $\varphi : (S,\leq_S) \to (T,\leq_T)$ of ordered sets under orders from S into T is an *order-isotone function* if and only if for every $x,y \in S$, $x \leq_S y$ implies $\varphi(x) \leq_T \varphi(y)$. If $x \leq_S y$ implies from $\varphi(y) \leq_T \varphi(x)$ we say that $\varphi$ is *order - reverse isotone function* of ordered sets.

(2) A strongly extensional function $\varphi : (S,\Theta_S) \to (T,\Theta_T)$ of ordered sets under anti-orders from S into T is an *anti-order isotone function* if and only if for every $x,y\in S$, $x\Theta_S y$ implies $\varphi(x)\ \Theta_T\ \varphi(y)$. If $\varphi(y)\ \Theta_T\ \varphi(x)$ implies $x\ \Theta_S\ y$, we say that $\varphi$ is *anti-order reverse isotone function* of ordered sets.

Let $\varphi : (S,=_S,\neq_S,\leq_S,\Theta_S) \to (T,=_T,\neq_T,\leq_T,\Theta_T)$ be a strongly extensional function of ordered sets. Then $\varphi^{-1}(\leq_T)$ is an order on set S, and $\varphi^{-1}(\Theta_T)$ is an anti-order on set S such that Anti-ker$\varphi = \{(x,x')\in S\times S : \varphi(x) \neq_T \varphi(x')\} \subseteq \varphi^{-1}(\Theta_T) \cup \varphi^{-1}(\Theta_T^{-1})$. Then:

$$(\varphi \text{ is order isotone function}) \Leftrightarrow\ \leq_S \subseteq \varphi^{-1}(\leq_T);$$

$$(\varphi \text{ is order reverse isotone function}) \Leftrightarrow\ \varphi^{-1}(\leq_T) \subseteq\ \leq_S\ ;$$

$$(\varphi \text{ is anti-order isotone function}) \Leftrightarrow \Theta_S \subseteq \varphi^{-1}(\Theta_T);$$

$$(\varphi \text{ is anti-order reverse isotone function}) \Leftrightarrow\ \varphi^{-1}(\Theta_T) \subseteq \Theta_S.$$

Binary relation 'to be order function of ordered sets under orders' is transitive. Symmetrically, the next lemma show that binary relation 'to be anti-order function of ordered sets under anti-orders' is transitive, too.

**Lemma 2.1** *If* $\varphi : (R,\Theta_R) \to (S,\Theta_S)$ *and* $\psi : (S,\Theta_S) \to (T,\Theta_T)$ *are anti-order isotone (anti-order reverse isotone) functions of ordered sets, then* $\psi \circ \varphi : (R,\Theta_R) \to (T,\Theta_T)$ *is an anti- order (anti-order reverse) isotone function of ordered sets.*

The notion of isomorphism of ordered sets is well-known: The order isotone and reverse isotone function must be strongly extensional and embedding bijection. In the next definition we give a notion of anti-order isomorphism between ordered sets under anti-orders: For the strongly extensional function $\varphi : (S,=,\neq,\cdot,\Theta_S) \to (T,=,\neq,\cdot,\Theta_T)$ of ordered sets under anti-orders is an *anti-order isomorphism* if and only if it is injective, embedding and surjective anti-order isotone and anti-order reverse isotone function.

The following propositions show that order anti-ideals are preserved under union, intersection and reverse inverse functions.

**Lemma 2.2** *Let* $(S,=,\neq,\Theta)$ *be an ordered set under an antiorder* $\Theta$ *and let* $\mathfrak{R} = \{K_j : j\in\Gamma\}$ *be a family of order anti-ideals of* S. *Then* $\cap\mathfrak{R}$ *and* $\cup\mathfrak{R}$ *are order anti-ideals of* S.

**Proof**: (1) Let $y\in\cup\mathfrak{I}$ . Then there exists $j \in\Gamma$, $y\in K_j$ and thus $y\Theta x$ or $x\in K_i$. It follows that $y\Theta x$ or $x\in\cup\mathfrak{R}$ and thus $\cup\mathfrak{R}$ is an order anti-ideal of S.

(2) Let $y\in\cap\mathfrak{R}$. Then for all $j\in\Gamma$, $y\in K_j$ and thus $y\Theta x$ or $x\in K_j$. In the Constructive logic we know exactly which of formula in previous disjunction holds for all singly $j\in\Gamma$. If $\neg(y\Theta x)$ for all $j\in\Gamma$ holds, then $x\in\cap\mathfrak{R}$. So, $y\in\cap\mathfrak{R}$ implies $y\Theta x$ or $x\in\cap\mathfrak{R}$. Therefore, $\cap\mathfrak{R}$ is an order anti-ideal of S.

**Lemma 2.3** *Let* $\varphi : (S,=,\neq,\Theta) \to (T,=,\neq,\Theta)$ *be a reverse isotone anti-order function of ordered sets. If* W *is an anti-ideal of* T, *then* $\varphi^{-1}(W)$ *is an anti-ideal of* S.

**Proof**: Let $y\in\varphi^{-1}(W)$ and let x be an arbitrary element of S. Then $\varphi(y)\in W$. Thus $\varphi(y)\Theta_T\varphi(x)$ or $\varphi(x)\in W$. If $\varphi(y)\Theta_T\varphi(x)$, then $y\Theta_S x$ because $\varphi$ is reverse-isotone antiorder- function. If $\varphi(x)\in W$, then $x\in\varphi^{-1}(W)$. So, $\varphi^{-1}(W)$ is an anti-ideal of S.

## 3 PRELIMINARY RESULTS:

Following classical definitions, we give a few new notions in the following definitions:

We say that $(S,=_S,\neq_S,\leq_S,\Theta_S)$ is an *ordered substructure* of $(T,=_T,\neq_T,\leq_T,\Theta_T)$ if S is a subset of T and the order on S is the restriction to S of the order on T.

Let $(V, =,\neq, \leq,\Xi)$ be an ordered set. The following lemmas show some basic properties of ordered sets:

**Lemma 3.1** *Each nonempty subset* Z *of an ordered set* $(V,=,\neq,\leq,\Xi)$ *with the relations* $=_Z$, $\neq_Z$ , $\leq_Z$ ,$\xi_Z$ *on* Z *defined by*

$$=_Z\, = \,=_V \cap(Z\times Z),\ \neq_Z\, = \,\neq_V \cap( Z\times Z),\ \leq_Z\, = \,\leq_V \cap(Z \times Z),\ \xi_Z = \Xi_V \cap (Z\times Z),$$

*is an ordered set*.

*In the following, each subset* Z *of an ordered set* $(V,=,\neq,\leq,\Xi)$ *is considered as an ordered set* .

**Proof:**

It is clear that relations $=_Z\, = \,=_V \cap(Z\times Z)$, $\neq_Z\, = \,\neq_V \cap( Z\times Z)$, $\leq_Z\, = \,\leq_V \cap(Z \times Z)$ are well-defined. We will show the proof for the anti-order relation $\xi_Z = \Xi \cap (Z\times Z)$ only.

(1) $\xi_Z = \Xi \cap (Z \times Z) \subseteq \neq_V \cap (Z \times Z) = \neq_Z$ ;

(2) $\neq_Z = \neq_V \cap (Z \times Z) \subseteq (\Xi \cup \Xi^{-1}) \cap (Z \times Z) = (\Xi \cap (Z\times Z)) \cup (\Xi^{-1}\cap (Z\times Z)) = (\Xi \cap (Z\times Z)) \cup (\Xi \cap (Z\times Z))^{-1} = \xi_Z \cup (\xi_Z)^{-1}$ .

(3) $\xi_Z = \Xi \cap (Z\times Z) \subseteq (\Xi*\Xi)\cap (Z\times Z) \subseteq (\Xi \cap (Z\times Z))*(\Xi \cap (Z\times Z)) = \xi_Z * \xi_Z$ .

(4) Let $a \leq_Z b$ and $a\, \xi_Z\, b$. Then $a\in Z$, $b\in Z$ and $a \leq_V b$ and $a\Xi b$. It is impossible. Thus $\neg( a \leq_Z b \wedge a\, \xi_Z\, b)$. So, the relation $\leq_Z$ and $\xi_Z$ are compatible if the relation $\leq_V$ and $\Xi$ are such.

**Corollary: 3.1.1** Let $\Xi$ *is a cotransitive relation on* V*, and* Z *be a subset of* V*. Then the relation* $\xi_Z = \Xi_V \cap (Z\times Z)$ *is a cotransitive relation on* Z*.*

**Lemma: 3.2** *Let* $(X, \leq_X )$, $(Y,\leq_Y)$ *be ordered sets such that* $X \cap Y = \emptyset$. *Let* $\theta \subseteq X \times Y$ *and* $V = X \cup Y$. *Define a relation* "$\leq$" *on* V *as follows:*

$$\leq\, = \,\leq_X \cup \leq_Y \cup\, \Theta(\theta) \subseteq (X\cup Y)\times(X\cup Y).$$

*where* $\Theta(\leq) = \{(a,b)\in X \times Y \mid (\exists(x,y)\in \theta \subseteq X\times Y)(a \leq_X x \wedge y \leq_Y b\}$. *Then* $(V,\leq)$ *is an ordered set under order relation* $\leq$.

**Proof:** (1) Let $a\in V$. If $a\in X$, then $(a,a)\in \leq_X \subseteq \leq$. If $a\in Y$, then $(a,a)\in \leq_Y \subseteq \leq$. So, the relation $\leq$ is reflexive.

(2) Let $(a,b)\in\leq$ and $(b,c)\in\leq$. Then $(a,c)\in\leq$.Indeed we consider the following cases:

(a) $(a,b)\in\leq_X \wedge (b,c)\in \leq_X \Rightarrow (a,c)\in \leq_X$ ;
(b) $(a,b)\in\leq_X \wedge (b,c)\in \leq_Y$ is impossible because $b\in X \cap Y = \emptyset$;
(c) $(a,b)\in \leq_X \wedge (b,c)\in \Theta(\theta) \Rightarrow (\exists(b',c')\in \theta)(b,b')\in \leq_X \wedge (c',c)\in \leq_Y )$
$\Rightarrow (\exists(b',c')\in \theta)((a,b')\in \leq_X \wedge (c',c)\in \leq_Y )$
$\Rightarrow (a,c)\in \Theta(\theta)$;
(d) $(a,b)\in \leq_Y \wedge (b,c)\in \leq_X$ is impossible because $b\in X \cap Y = \emptyset$;
(e) $(a,b)\in \leq_Y \wedge (b,c)\in \leq_Y \Rightarrow (a,c)\in \leq_Y$ ;
(f) $(a,b)\in \leq_Y \wedge (b,c)\in \Theta(\theta)$ is impossible because $b\in X \cap Y = \emptyset$;
(g) $(a,b)\in \Theta(\theta) \wedge (b,c)\in \leq_X$ is impossible because $b\in X \cap Y = \emptyset$,
(h) $(a,b)\in \Theta(\theta) \wedge (b,c)\in\leq_Y \Rightarrow (\exists(a',b')\in \theta)((a.a')\in \leq_X \wedge (b',b)\in\leq_Y )$
$\Rightarrow (\exists(a',b')\in \theta)((a,a')\in \leq_X \wedge (b',c)\in\leq_Y )$
$\Rightarrow (a,c)\in \Theta(\theta)$;
(i) $(a,b)\in \Theta(\theta) \wedge (b,c)\in \Theta(\theta)$ is impossible because $b\in X\cap Y = \emptyset$.

Therefore, the relation $\leq$ is transitive.

(3) Let $(a,b)\in\leq$ and $(b,a)\in\leq$. Then $a = b$. In fact: We put a instead of c in conditions (a) – (i) above.

(a) If $(a,b)\in\leq_X \wedge (b,a)\in\leq_X$, then $a =_X b$;
(b) If $(a,b)\in\leq_X \wedge (b,a)\in \leq_Y$, then $a,b \in X\cap Y = \emptyset$. The case is impossible.
(c) Let $(a,b)\in\leq_X \wedge (b,a)\in \Theta(\theta)$. Since $\Theta(\theta)\in X\times Y$, we have $a,b \in X \cap Y = \emptyset$ The case is impossible.

(d) $(a,b)\in \leq_Y \wedge (b,c)\in \leq_X$ is impossible because $b\in X \cap Y = \emptyset$;
(e) If $(a,b)\in\leq_Y \wedge (b,a)\in\leq_Y$, then $a = b$.
(h) $(a,b)\in \Theta \wedge (b,c)\in\leq_Y \Rightarrow (\exists(x,y)\in \theta)( a \leq_X x \wedge y \leq_Y b) \wedge (b,c)\in\leq_Y$

$\Rightarrow (\exists(x,y)\in \theta)( a \leq_X x \wedge y \leq_Y c)$

$\Rightarrow (a,c)\in \Theta$;

The cases (f), (g) (i) are also impossible.

Therefore, the relation $\leq$ is an anti-symmetric.

**NOTES:** (1) $\theta \subseteq \Theta(\theta)$.

(2) $\Theta(\theta) = \cup\{(a] \times [b) : (a,b)\in \theta\}$.

In fact: $(x,y)\in \Theta(\theta)$ if and only if there exists $(a,b)\in \theta$ such that $x \leq_X a$ and $b \leq_Y y$. Thus $x\in (a]$ and $y\in [b)$ for some $(a,b)\in \theta$. Opposite, if $(u,v)\in \cup\{(a] \times [b) : (a,b)\in \theta\}$. Then there exists an element $(a,b)\in \theta$ such that $u\in (a]$ and $v\in [b)$. So, $u \leq_X a$ and $b \leq_Y v$. We conclude that $(u,v)\in \Theta(\theta)$.

**Lemma: 3.3** *Let* $(X,=_X,\neq_X)$ *and* $(Y,=_Y,\neq_Y)$ *be sets with apartness such that* $X \rhd\lhd Y$. *If define a relation* "$\neq$" *on* $V = X \cup Y$ *as follows:*

$$\neq \; = \; \neq_X \cup \neq_Y \cup (X\times Y)\cup(Y\times X) \subseteq (X\cup Y)\times(X\times Y).,$$

*then* $(V,=,\neq)$ *is a set with apartness.*

**Proof:** (1) Let $a\in V$ be an arbitrary element. If $a\in X$, then $\neg(a \neq_X a)$ holds. If $a\in Y$, then $\neg(a \neq_Y a)$ holds. So, the relation "$\neq$" is a consistent relation on V.

(2) Let $a \neq b$. If $a\in X$ and $b\in X$, then $a \neq_X b$. Thus, $b \neq_X a$. If $a\in Y$ and $b\in Y$, then $a \neq_Y b$ holds. Thus, $b \neq_Y a$. If $a\in X$ and $b\in Y$, then $b\in Y$ and $a\in X$. Therefore, relation "$\neq$" is a symmetric relation.

(3) Let $a \neq c$ and let b be an arbitrary element of V. Then:

$a\in X \wedge c\in X \wedge b\in X \wedge a \neq_X c \Rightarrow (a \neq_X b \vee b \neq_X c)$;
$a\in X \wedge c\in X \wedge b\in Y \wedge a \neq_X c \Rightarrow (a \neq b \wedge b \neq c)$;
$a\in X \wedge c\in Y \wedge b\in X \wedge a \neq c \Rightarrow b \neq c$;
$a\in X \wedge c\in Y \wedge b\in Y \wedge a \neq c \Rightarrow a \neq b$ ;
$a\in Y \wedge c\in X \wedge b\in X \wedge a \neq c \Rightarrow a \neq b$;
$a\in Y \wedge c\in X \wedge b\in Y \wedge a \neq c \Rightarrow b \neq c$;
$a\in Y \wedge c\in Y \wedge b\in X \wedge a \neq_Y c \Rightarrow (a \neq b \wedge b \neq c)$;
$a\in Y \wedge c\in Y \wedge c\in Y \wedge a \neq_Y c \Rightarrow (a \neq_Y b \vee b \neq_Y c)$.
So, the relation "$\neq$" is cotransitive relation.

Therefore, the relation $\neq$ on V is coequality relation.

**Lemma: 3.4** *Let* $(X,=_X,\neq_X,\leq_X,\alpha_X)$ *and* $(Y,=_Y,\neq_Y,\leq_Y,\alpha_Y)$ *be ordered sets under antiorders* $\alpha_X$ *and* $\alpha_Y$ *respectively such that* $X \rhd\lhd Y$. *Let* $\theta \subseteq X \times Y$ *and* $V = X \cup Y$. *If define a relation* "$\Xi$" *on* V *as follows:*

$$\Xi = (\alpha_X\cup\alpha_Y)\cap\Omega(\theta), \quad \Omega(\theta) = c((\Theta(\theta)^C) \cap ((X \times Y)\cup(Y\times X)),$$

*then* $(V,=,\neq,\Xi)$ *is an ordered set under anti-order relation* $\Xi$..

**Proof:** (1) $\Xi$ is a consistent relation. Indeed: From $\alpha_X \subseteq \neq_X$ , $\alpha_Y \subseteq \neq_Y$ and $\Omega(\theta) \subseteq (X\times Y)\cup(Y\times X)$ follows $\Xi = \alpha_X \cup \alpha_Y \cup \Omega(\theta) \subseteq \neq$ .

(2) $c((\Theta(\theta)^C)$ is cotransitive relation on set $X\cup Y$. So, the relation $\Omega(\theta) = c((\Theta(\theta)^C)\cap ((X\times Y)\cup(Y\times X))$ is a cotransitive relation on set $((X\times Y)\cup(Y\times X))$ by Lemma 3.1.

(3) $\Xi$ is linear relation. Indeed: Let a and b be arbitrary element of $V = X \cup Y$ such that $a \neq b$. Then $a \neq_X b$ or $a \neq_Y b$ or $(a,b)\in (X\times Y)\cup(Y\times X)$. Then

$(a,b)\in \neq_X \subseteq \alpha_X \cup (\alpha_X)^{-1} \subseteq \Xi\cup\Xi^{-1}$ ;
$(a,b)\in \neq_Y \subseteq \alpha_Y \cup (\alpha_Y)^{-1} \subseteq \Xi\cup\Xi^{-1}$ ;
$(a,b)\in (X\times Y)\cup(Y\times X) \subseteq (\Omega(\theta))^{-1} \cup \Omega(\theta) \subseteq \Xi \cup \Xi^{-1}$ .

## 4. THE DEFINITION AND THE MAIN RESULTS:

**Definition:** Let $(X,=_X,\neq_X,\leq_X,\alpha_X)$, $(Y,=_Y,\neq_Y,\leq_Y,\alpha_Y)$ be ordered sets and $X \rhd\lhd Y$. An ordered set $(V,=_V,\neq_V,\leq_V,\Xi)$ is called an *extension* of X by Y if there exists an ideal A and an anti-ideal B of V such that

$$(X,=_X,\neq_X,\leq_X,\alpha_X) \cong (A,=_A,\neq_A,\leq_A,\alpha),\ (B,=_B,\neq_B\leq_B,\beta) \cong (Y,=_Y,\neq_Y,\leq_Y,\alpha_Y)$$

Where

$$=_A \,=\, =_V \cap(A\times A),\ \neq_A = \neq_V \cap(A\times A),\ \leq_A = \leq_V \cap(A \times A),\ \alpha = \Xi_V \cap (A\times A), \text{ and}$$

$$=_B \,=\, =_V \cap(B\times B),\ \neq_B = \neq_V \cap(B\times B),\ \leq_B = \leq_V \cap(B \times B),\ \beta = \Xi_V \cap (B\times B).$$

If $(V,=,\neq,\leq_V,\Xi)$ is an extension of X by Y, we always denote by $\varphi$ and $\psi$ the isomorphisms

$$\varphi : (X,=_X,\neq_X,\leq_X,\alpha_X) \rightarrow (A,=_A,\neq_A,\leq_A,\alpha),$$

$$\psi : (Y,=_Y,\neq_Y,\leq_Y,\alpha_Y) \rightarrow (B,=_B,\neq_B\leq_B,\beta).$$

We always denote by $\Theta$ and $\Omega$ sets defined by

$$\Theta = \{(a,b)\in X\times Y : \varphi(a) \leq \psi(b)\} \text{ and } \Omega = \{(a,b)\in X\times Y : \varphi(a)\Xi\psi(b)\}.$$

The following theorem gives our first result on extension of ordered sets:

**Theorem 4.1** *Let* $(V,=_V,\neq_V,\leq_V,\Xi)$ *be an extension of* $(X,=_X,\neq_X,\leq_X,\alpha_X)$ *by* $(Y,=_Y,\neq_Y,\leq_Y,\alpha_Y)$*. Then the set* $X \cup Y$, *endowed with the relations* "=", "≠", "≤" and "Σ" *defined by*

$$\neq\, = \neq_X \cup \neq_Y \cup (X\times Y)\cup(Y\times X),\quad \leq\, = \leq_X \cup \leq_Y \cup \Theta,\quad \Sigma = \alpha_X \cup \alpha_Y \cup \Omega,$$

*is an ordered set and there exists strongly extensional, embedding, injective, order isotone and reverse isotone, anti-order isotone and reverse isotone function*

$$f : (X \cup Y, =, \neq, \leq, \Sigma) \rightarrow (V,=_V, \neq_V, \leq_V, \Xi).$$

**Proof:** Let $(X,=_X,\neq_X,\leq_X,\alpha_X)$ and $(Y,=_Y,\neq_Y,\leq_Y,\alpha_Y)$ be ordered sets, $X \rhd\lhd Y$, and $(V, =_V, \neq_V, \leq_V, \Xi)$ an extension of X by Y. Then there exist an ideal A and an anti-ideal B of V and isomorphisms:

$$\varphi : (X, =_X, \neq_X, \leq_X, \alpha_X) \rightarrow (A, =_A, \neq_A, \leq_A, \alpha),$$
$$\psi : (Y, =_Y, \neq_Y, \leq_Y, \alpha_Y) \rightarrow (B, =_B, \neq_B, \leq_B, \beta).$$

The set $X \cup Y$ endowed with the relations: $=, \neq, \leq$ and $\Sigma$ as above, is an ordered set.

(1) By Lemma 3.3 the relation "≠", defined by $\neq\, = \neq_X \cup \neq_Y \cup (X\times Y)\cup(Y\times X)$ is an apartness on $X\cup Y$.

(2) Let $a\in V$. If $a\in X$, then $(a,a)\in \leq_X \subseteq \leq$. If $a\in Y$, then $(a,a)\in \leq_Y \subseteq \leq$. So, the relation $\leq$ is reflexive.

Let $(a,b)\in\leq$ and $(b,c)\in\leq$. Then $(a,c)\in\leq$.Indeed we consider the following cases:

(a) $(a,b)\in\leq_X \wedge (b,c)\in \leq_X \Rightarrow (a,c)\in \leq_X$ ;
(b) $(a,b)\in\leq_X \wedge (b,c)\in \leq_Y$ is impossible because $b\in X \cap Y = \emptyset$;
(c) $(a,b)\in \leq_X \wedge (b,c)\in\Theta \Rightarrow a \leq_X b \wedge \varphi(b) \leq_V \psi (c)$
$\Rightarrow \varphi(a) \leq_V \varphi(b) \wedge \varphi(b) \leq_V \psi (c)$
$\Rightarrow \varphi(a) \leq_V \psi (c)$
$\Rightarrow (a,c)\in\Theta$;
(d) $(a,b)\in \leq_Y \wedge (b,c)\in \leq_X$ is impossible because $b\in X \cap Y = \emptyset$;
(e) $(a,b)\in \leq_Y \wedge (b,c)\in \leq_Y \Rightarrow (a,c)\in \leq_Y$ ;
(f) $(a,b)\in \leq_Y \wedge (b,c)\in\Theta$ is impossible because $b\in X \cap Y = \emptyset$;
(g) $(a,b)\in\Theta \wedge (b,c)\in \leq_X$ is impossible because $b\in X \cap Y = \emptyset$,
(h) $(a,b)\in \Theta \wedge (b,c)\in\leq_Y \Rightarrow \varphi(a) \leq_V \psi (b) \wedge b \leq_Y c$
$\Rightarrow \varphi(a) \leq_V \psi (b) \wedge \psi(b) \leq_V \psi(c)$
$\Rightarrow \varphi(a) \leq_V \psi (c)$
$\Rightarrow (a,c)\in\Theta$;
(i) $(a,b)\in \Theta(\theta) \wedge (b,c)\in \Theta(\theta)$ is impossible because $b\in X\cap Y = \emptyset$.

Therefore, the relation $\leq$ is transitive.

Let $(a,b)\in\leq$ and $(b,a)\in\leq$. Then $a = b$. In fact: We put a instead of c in conditions (a) – (i) above.

(a) If $(a,b)\in\leq_X \wedge (b,a)\in\leq_X$, then $a =_X b$;
(b) If $(a,b)\in\leq_X \wedge (b,a)\in \leq_Y$, then $a,b \in X\cap Y = \emptyset$. The case is impossible.
(c) Let $(a,b)\in\leq_X \wedge (b,a)\in \Theta$. Since $\Theta(\theta)\in X\times Y$, we have $a,b \in X \cap Y = \emptyset$ The case is impossible.
(d) $(a,b)\in \leq_Y \wedge (b,c)\in \leq_X$ is impossible because $b\in X \cap Y = \emptyset$;
(e) If $(a,b)\in\leq_Y \wedge (b,a)\in\leq_Y$, then $a = b$.

The cases (f), (g) (i) are also impossible.

Therefore, the relation $\leq$ is an anti-symmetric.

(3) We consider the mapping $f : X\cup Y \rightarrow V$ defined by $f(a) =_V \varphi(a)$ if $a\in X$ and $f(a) =_V \psi(a)$ if $a\in Y$.

The mapping f is a strongly extensional function:

3.1 The mapping f is a function: Let a and b be elements of $X\cup Y$ such that $a = b$. Then:
$a\in X \wedge b\in X \wedge a =_X b \Rightarrow f(a) =_V \varphi(a) =_A \varphi(b) =_V f(b)$;
$a\in Y \wedge b\in Y \wedge a =_Y b \Rightarrow f(a) =_V \psi(a) =_B \psi(b) =_V f(b)$;
Cases $a\in X \wedge b\in Y$ and $a\in Y \wedge b\in X$ are impossible because $X\cap Y = \emptyset$.
3.2. The mapping f is an embedding function: Let a and b be elements of $X\cup Y$ such that $a \neq b$. Then:
$a\in X \wedge b\in X \wedge a \neq_X b \Rightarrow f(a) =_V \varphi(a) \neq_A \varphi(b) =_V f(b)$;
$a\in Y \wedge b\in Y \wedge a \neq_Y b \Rightarrow f(a) =_V \psi(a) \neq_B \psi (b) =_V f(b)$;
$a\in X \wedge b\in Y \wedge a \neq b \Rightarrow f(a) =_V \varphi(a) \neq_V \psi(b) =_V f(b)$ because $A\cap B = \emptyset$;
$a\in Y \wedge a\in Y \wedge a \neq b \Rightarrow f(a) =_V \psi(a) \neq_V \varphi(b) =_V f(b)$ because $A\cap B = \emptyset$.
3.3 The mapping is an injective function. Let a and b be elements of $X\cup Y$ such that $f(a) = f(b)$. Then:

$$a\in X \wedge b\in X \wedge f(a) = f(b) \Rightarrow \varphi(a) =_V f(a) = f(b) =_V \varphi(b)$$
$$\Rightarrow a =_X b;$$
$$a\in Y \wedge b\in Y \wedge f(a) = f(b) \Rightarrow \psi(a) =_V f(a) = f(b) =_V \psi(b)$$
$$\Rightarrow a =_Y b \,;$$

The case $a\in X \wedge b\in Y \wedge \varphi(a) =_V f(a) \wedge f(b) =_V \psi(b)$ and $\varphi(a) = \psi(b)$ is impossible because $A\cap B = \emptyset$;

The case $a\in Y \wedge b\in X \wedge \psi(b) =_V f(b) \wedge f(a) \neq_V \varphi(b)$ and $\varphi(b) = \psi(a)$ is impossible also because $A\cap B = \emptyset$.

3.4 f is strongly extensional function**.** Indeed: Let a and b be arbitrary elements of $V = X\cup Y$ such that $f(a) \neq f(b)$.

Then

$$a\in X \wedge b\in X \wedge f(a) \neq f(b) \Rightarrow a\in X \wedge b\in X \wedge \varphi(a) \neq_A \varphi(b)$$
$$\Rightarrow a \neq_X b \quad (\varphi \text{ is a strongly extensional})$$
$$\Rightarrow a \neq b.$$
$$a\in Y \wedge b\in Y \wedge f(a) \neq f(b) \Rightarrow a\in Y \wedge b\in Y \wedge \psi(a) \neq_B \psi(b)$$
$$\Rightarrow a \neq_Y b \quad (\psi \text{ is a strongly extensional })$$
$$\Rightarrow a \neq b.$$
$$a\in X \wedge b\in Y \wedge f(a) \neq f(b) \Rightarrow a \neq b.$$
$$a\in Y \wedge b\in X \wedge f(a) \neq f(b) \Rightarrow a \neq b.$$

3.5 f is order isotone. Let $a,b\in X\cup Y$, $a \leq b$. If $a \leq_X b$, then $\varphi(a) \leq_A \varphi(b)$ since $\varphi$ is order isotone function. Since $a\in X$ and $b\in Y$, we have $f(a) = \varphi(a)$ and $f (b) = \varphi(b)$. Then $f (a) \leq_A f (b)$ that is, $f(a) \leq f(b)$. Let $a \leq_Y b$. Since $\psi$ is isotone,

we have $\psi(a) \leq_B \psi(b)$. Since $a\in Y$ and $b\in Y$, we have $f(a) = \psi(a)$, $f(b) = \psi(b)$. Then $f (a) \leq_B f (b)$, that is, $f(a) \leq f(b)$.

Let $(a,b)\in \Theta$. By hypothesis, $(a,b)\in X\cup Y$ and $\varphi(a) \leq_V \psi(b)$. Since $a\in X$, $b\in Y$, we have $f(a) = \varphi(a)$, $f(b) = \psi(b)$. Then $f(a) \leq_V f(b)$.

3.6 f is order reverse isotone function. If $a\in X$ and $b\in X$ and $f(a) \leq f(b)$. Then $\varphi(a) = f(a) \leq f(b) = \varphi(b)$, and $\varphi(a) \leq_A \varphi(b)$. Since is reverse isotone, we have $a \leq_X$ , so $a \leq b$.

 

Let $a\in X$ and $b\in Y$ and $f(a) \le f(b)$. Then $f(a) = \varphi(a)\in A$, $f(b) = \psi(b)\in B$, $\varphi(a) \le_V \psi(b)$. Since $(a,b)\in X\times Y$ and $\varphi(a) \le_V \psi(b)$, we have $(a,b)\in \Theta \subseteq \le$.

Let $a\in Y$ and $b\in X$ and $f(a) \le f(b)$. Then $f(a) = \psi(a)\in B$, $f(b) = \varphi(b)\in A$. Since V _ $f(a) \le_V f(b)$ and $f(b)\in A$ and A is an ideal of V, we have $f(a)\in A$. The case is impossible.

Suppose that $a\in Y$ and $b\in Y$ and $f(a) \le f(b)$. Then $f(a) = \psi(a)\in B$, $f(b) = \psi(b)\in B$, and $\psi(a) \le_V \psi(b)$. Since $\psi$ is reverse isotone, we have $a \le_X b$ and $a \le b$.

(4) 4.1 Firstly, we conclude $\alpha \subseteq \neq_X \subseteq \neq$ and $\beta \subseteq \neq_Y \subseteq \neq$. Secondly, if $(a,b)\in \Omega$, i.e. if $(\varphi(a),\psi(b))\in \Xi$, then $\varphi(a)\in A$ and $\psi(b)\in B$. So, $a\in X$ and $b\in Y$. Thus, $(a,b)\in X\times Y \subseteq \neq$. Therefore, the relation $\Sigma$ is a consistent relation on $X\cup Y$.

4.2 Let $a,b\in X\cup Y$ and $a \neq b$. If $a\in X$ and $b\in X$, then $(a,b)\in \alpha_X$ or $(b,a)\in \alpha_X$. So, $(a,b)\in \Sigma$ or $(b,a)\in \Sigma$.

If $a,b\in Y$, then $(a,b)\in \alpha_Y$ or $(b,a)\in \alpha_Y$. Thus $(a,b)\in \Sigma$ or $(b,a)\in \Sigma$.

If $a\in X$ and $b\in Y$, then $\varphi(a)\in A$ and $\psi(b)\in B$. Since $A \rhd\lhd B$, we have $\varphi(a)\Xi\psi(b)$ or $\psi(b)\Xi\varphi(a)$. Hence $(a,b)\in \Sigma$ or $(a,b)\in \Sigma^{-1}$ .

If $a\in Y$ and $b\in X$, then $\varphi(b)\in A$ and $\psi(a)\in B$. Since $A \rhd\lhd B$, we have $\varphi(a)\Xi\psi(b)$ or $\psi(b)\Xi\varphi(a)$. Hence $(b,a)\in \Sigma$ or $(a,b)\in \Sigma^{-1}$ . Therefore, the relation $\Sigma$ is linear.

4.3 The function f is anti-order isotone. If $(a,b)\in \Sigma$, i.e. if $(a,b)\in \alpha_X$ or $(a,b)\in \alpha_Y$ or $(a,b)\in \Omega$, then:

$(a,b)\in \alpha_X \Rightarrow (\varphi(a),\varphi(b))\in \alpha = \Xi \cap (A\times A)$
$\Rightarrow (f(a),f(b))\in \Xi$ because $\varphi(a) = f(a)$ and $\varphi(b) = f(b)$;
$(a,b)\in \alpha_Y \Rightarrow (\psi(a),\psi(b))\in \beta = \Xi \cap (B\times B)$
$\Rightarrow (f(a),f(b))\in \Xi$ because $\psi(a) = f(a)$ and $\psi(b) = f(b)$;
$(a,b)\in \Omega \Leftrightarrow (\varphi(a),\psi(b))\in \Xi$
$\Rightarrow (f(a),f(b))\in \Xi$ because $\varphi(a) = f(a)$ and $\psi(b) = f(b)$.

4.4 f is anti-order reverse isotone function. Let $a,b\in X\cup Y$ such that $(f(a),f(b))\in \Xi$. Then:

$a\in X \wedge b\in X \wedge (f(a),f(b))\in \Xi \Rightarrow$
$f(a) = \varphi(a)\in A \wedge f(b) = \varphi(b)\in A \wedge (\varphi(a),\varphi(b))\in \Xi \cap (A\times A) = \alpha$
$\Rightarrow (a,b)\in \alpha$ $\varphi$ is anti-order reverse isotone
$\Rightarrow (a,b)\in \Sigma$ ;
$a\in X \wedge b\in Y \wedge (f(a),f(b))\in \Xi \Rightarrow$
$f(a) = \varphi(a)\in A \wedge f(b) = \psi(b)\in B \wedge (\varphi(a),\psi(b))\in \Xi$
$\Rightarrow (a,b)\in \Omega \subseteq \Sigma$ ;
If $a\in Y \wedge b\in X \wedge (f(a),f(b))\in \Xi$ then $f(b) = \varphi(b)\in A \wedge f(a) = \psi(a)\in B \wedge (\psi(a),\varphi(b))\in \Xi$ which is impossible.
$a\in Y \wedge b\in Y \wedge (f(a),f(b))\in \Xi \Rightarrow$
$f(a) = \psi(a)\in B \wedge f(b) = \psi(b)\in B \wedge (\psi(a),\psi(b))\in \Xi \cap(B\times B) = \beta$
$\Rightarrow (a,b)\in \alpha_Y \subseteq \Sigma$ .

So, the function f is anti-order reverse isotone function.

We give the main theorem of extensions: If $(X,=_X,\neq_X,\le_X,\alpha_X)$ and $(Y,=_Y,\neq_Y,\le_Y,\alpha_Y)$ are two aparted ordered sets, $\theta$ an arbitrary subset of $X \times Y$

$$\Theta(\theta) = \{(a,b)\in X \times Y \mid (\exists(x,y)\in \theta \subseteq X\times Y)(a \le_X x \wedge y \le_Y b\},$$

and

$$\Omega(\theta) = c((\Theta(\theta)^C) \cap ((X \times Y)\cup(Y\times X)),$$

then set $V = X \cup Y$, endowed with the order "$\le$", defined by $\le = \le_X \cup \le_Y \cup \Theta$, and with the antiorder "$\Sigma$", defined by $\Sigma = (\alpha_X\cup\alpha_Y) \cup \Omega(\theta)$, is an ordered set and it is an extension of X by Y.

**Theorem: 4.2** *Let* $(X,=_X,\neq_X,\le_X,\alpha_X)$ *and* $(Y,=_Y,\neq_Y,\le_Y,\alpha_Y)$ *be ordered sets such that* $X \rhd\lhd Y$. *Let* $\theta \subseteq X \times Y$ *and* $V = X \cup Y$. *Define relations* "$=$", "$\neq$", "$\le$" *and* "$\Sigma$"*on* V *by*

$$\neq \, = \, \neq_X \cup \neq_Y \cup (X\times Y)\cup(Y\times X), \quad \leq \, = \, \leq_X \cup \leq_Y \cup \Theta, \quad \Sigma = \alpha_X \cup \alpha_Y \cup \Omega,$$

*Then* $(V, =_V, \neq_V, \leq_V, \Sigma)$ *is an ordered set and it is an extension of* X *by* Y.

**Proof:** (I) Set V is an ordered set under partial order $\leq$, by Lemma 3.2, and it is ordered set under anti-order $\Sigma$, by Lemma 3.4.

(II) The set X is an ideal of V. In fact, let $a\in X$ and $b \leq a$. Thus, we have $b \leq_X a$, $b \leq_Y a$ or $(b,a)\in \Theta(\theta)$. If $b \leq_X a$, then $b\in X$. If $b \leq_Y a$, then $a\in X\cap Y = \emptyset$. The case is impossible. If $(b,a)\in \Theta(\theta) \subseteq X \times Y$, we have $a\in X\cap Y = \emptyset$. The case is impossible.

The set Y is an anti-ideal of V. Indeed: Let $b\in Y$ and $a\in V = X\cup Y$. From $b\in Y \wedge a\in Y$ we conclude $a\in Y$. Let $a\in X$.

Then $a \neq b$. Thus $a\ \Sigma\ b$ or $b\ \Sigma\ a$. So, last means

$$(a\ \alpha_X\ b \ \vee\ a\ \alpha_Y\ b \vee (a,b)\in \Omega) \vee (b\ \alpha_X\ a \ \vee\ b\ \alpha_Y\ a \vee (b,a)\in \Omega).$$

The case $(a,b)\in \Omega$ is only impossible. Thus, we have $(a,b)\in \Omega \subseteq \Sigma$. Therefore, the implication

$$b\in Y \wedge a\in V \Rightarrow a\Sigma b \vee a\in Y$$

holds. So, the set Y is an anti-ideal of V.

(III) The identity mappings

$$I_X : (X, =_X, \neq_X, \leq_X, \alpha_X) \rightarrow (X, =\cap X^2, \neq\cap X^2, \leq \cap X^2, \Sigma \cap X^2)$$

$$I_Y : (Y, =_Y, \neq_Y, \leq_Y, \alpha_Y) \rightarrow (Y, = \cap Y^2, \neq \cap Y^2, \leq \cap Y^2, \Sigma \cap Y^2)$$

are strongly extensional, injective, embedding and onto functions. Moreover, we have

$$=_X \, = \, = \cap (X\times X),\ \neq_X \, = \, \neq \cap (X\times X), \leq_X \, = \, \leq \cap (X\times X) \text{ and } \alpha_X = \Sigma \cap (X\times X),$$
$$=_Y \, = \, = \cap (Y\times Y),\ \neq_Y \, = \, \neq \cap (Y\times Y), \leq_Y \, = \, \leq \cap (Y\times Y) \text{ and } \alpha_Y = \Sigma \cap (Y\times Y).$$

By above equalities, the mappings $I_X$ and $I_Y$ are order-isotone and reverse isotone, and antiorder-isotone and reverse isotone functions. Thus, we have

$$(X, =_X, \neq_X, \leq_X, \alpha_X) \cong (X, = \cap X^2, \neq \cap X^2, \leq \cap X^2, \Sigma \cap X^2)$$

and

$$(Y, =_Y, \neq_Y, \leq_Y, \alpha_Y) \cong (Y, = \cap Y^2, \neq \cap Y^2, \leq \cap Y^2, \Sigma \cap Y^2).$$

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