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UNCERTAINTY ANALYSIS USING POLYNOMIAL CHAOS THEORY CASE STUDY- TRANSPORT OF CONTAMINANT THROUGH GROUND WATER AND TEMPERATURE PROFILE OF A RECTANGULAR BAR

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ABSTRACT

This paper presents the soft computing technique on computational mathematics for current research. Current research on computational mathematics basically needs to address the issues of uncertainty for reliability analysis and optimization process. Research on reliability modeling can not be addressed completely by probability distribution of the governing parameters because some of the parameters may be of imprecise due to lack of knowledge or information about them. It is also true that the imprecise information can not be translated or transformed into probability distribution. Hence, it is mandatory to treat the imprecise nature of parameters of a reliability model in a different manner. The methodology of handling the uncertainty of such parameters is basically carried out by using evidence theory which provides the belief and plausibility – lower and upper bounds of the epistemic uncertainty of the model. Traditional probability exists between belief and plausibility. The body of evidence for the model to be tested for its reliability is gathered by expert's opinion by assigning a basic probability mass to each of the focal sets comprising the evidence. Soft computing technique also includes the chaos theory so called as polynomial chaos which also can be applied to quantify the uncertainty. Polynomial chaos theory is an efficient version of traditional Monte Carlo method of handling aleatory uncertainty. Potential capability of polynomial chaos theory in handling aleatory or stochastic uncertainty is illustrated with a case study of contaminant transport through groundwater.

1.0 INTRODUCTION:

Soft computing techniques emerged from the studies of natural systems. In the past bi-inspired methods were difficult to be implemented because of the limitations in computational power. Examples of bio-inspired techniques are Artificial Immune systems (AIS), Artificial Neural Networks (ANN), Fuzzy systems (FS), Evolutionary computing (EC) and swarm intelligence (SI). Such techniques are also known as Computational Intelligence (CI) techniques and are part of the so-called Artificial Intelligence (AI) research area. They can be combined among themselves and with stochastic methods in order to develop more effective methods to solve complex engineering problems. In summary, soft Computing is a complex of methodologies that includes artificial neural networks, genetic algorithms, fuzzy logic, Bayesian networks, and their hybrids. It admits approximate reasoning, imprecision, uncertainty and partial truth in order to mimic the remarkable human capability of making decisions in real-life, ambiguous environments. Soft Computing has therefore become popular in developing systems that encapsulate human expertise. Chaos belongs to the soft computing group. Chaos in the polynomial form is very important research application on computational mathematics. Stochastic differential equation can be solved by using the polynomial chaos. Not only that, one can very easily quantify the aleatory uncertainty which is only due to the stochasticity or randomness uncertainty of the parameter associated with the governing equation of the system. In this regard, it is required to introduce the types of uncertainties. This article addresses the polynomial chaos component of the soft computing for research on computational mathematics. Computational mathematics basically dictates the knowledge of science.

The paper describes the utility of polynomial chaos for quantifying aleatory uncertainty efficiently. Before going to polynomial chaos theory it is always better to introduce a brief description of uncertainty. Uncertainties are typically classified as aleatory and epistemic [1]. Aleatory uncertainty (also called probabilistic uncertainty) arises from randomness in the system whereas epistemic uncertainty arises due to the lack of knowledge (or ignorance). Epistemic uncertainties may also arise from assumptions introduced in the mathematical models and it can be possible to reduce them using inference from experimental observations. Uncertainty that is explicitly recognized by a stochastic model is categorized as aleatory. Uncertainty of the model parameters and the model itself is epistemic. Hence, the aleatory/epistemic split of the total uncertainty is model-dependent [2]. The steps involved in the uncertainty quantification of a model generally include (a) estimation of uncertainties of model inputs, (b) estimation of uncertainty of the model output, and (c) propagation of uncertainty in the model output. Monte Carlo methods are the most widely

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used techniques for statistical/probabilistic uncertainty analysis, with diverse applications. Given the input uncertainty distributions (frequency or probability density data), these methods involve repeated generation of pseudo-random instantiations (sampling) of inputs followed by application of the model to these instantiations to yield a set of model responses. These model outputs are then analyzed statistically.

A traditional "uncertainty analysis or error analysis" typically focuses on uncertainty present in the data itself labeled as the "data uncertainty". The traditional method consists of (a) the characterization of uncertainty in model parameters/inputs via their probability density functions (pdfs) and (b) the propagation of these pdfs through model equations to obtain the pdf of selected output metrics [3]. A large number of sample realizations (10^6 or more) of model inputs are required to achieve an acceptable level of confidence in the model output uncertainty. The large numbers of realizations reduce the efficiency of the simulation even though it involves standard or Latin Hypercube sampling. In case of computationally intensive models, the time and resources required by these methods will be prohibitively expensive. However, the number of simulations for adequate estimation of uncertainty of the model output can be substantially reduced as compared to conventional simulation, if the model uncertain inputs and output are expressed in the form of a series expansion of standard normal random variable (chaotic expansion). The output of the model then contains the coefficients which are calculated from a limited number of model simulations. The net result is to create a statistically equivalent polynomial approximation to the model outputs. This efficient simulation method presented in this paper is called as "Polynomial Chaos Expansion" (PCE) [3]. PCE is applied for quantification and propagation of the uncertainty of the model output with a limited number of model runs. This article is organized in the following way. Section 2 presents the theory of PCE (mathematical details of PCE). Section 3 illustrates the implementation of PCE through the problems. Section 4 draws the conclusion about the PCE.

2.0 THEORY OF POLYNOMIAL CHAOS EXPANSION:

The PCE approach has its foundation in the work of Wiener [4], who represented a Gaussian process as an infinite series of Hermite polynomial that takes a vector of random variables as arguments. Ghanem and Spanos [5] used this representation to develop the stochastic finite element method. Xiu and Karniadakis [6] extended the theoretical framework to non-Gaussian process by using different polynomial basis functions. This generalized polynomial chaos approach was used to address the problem of heat transfer with random material properties by Wan et al. [7].

The PCE is the representation of a random variable, more generally a stochastic process, with an infinite series of orthogonal polynomials that take a vector of independent and identically distributed (iid) random variables as arguments. Mathematically, the PCE of a random process can be represented by

$$y_j = a_{j,0} + \sum_{i_1=1}^{n} a_{j,i_1}\Gamma_1\left(\xi_{i_1}\right) + \sum_{i_1=1}^{n-1}\sum_{i_2>i_1}^{n} a_{j,i_1 i_2}\Gamma_2\left(\xi_{i_1},\xi_{i_2}\right) + \ldots\ldots \tag{1}$$

where $\{a_{j,k} \mid k = 0, 1,\ldots, n-1\}$ are unknown coefficients to be determined with respect to the specified model used for the uncertainty analysis, n represents the number of uncertain model inputs and $\Gamma_p(\xi)$'s are defined to be multivariate Hermite polynomials in the p – dimensional sequence of uncorrelated standard normal random variables, $\{\xi_i\}$. The multivariate Hermite polynomials can be written as:

$$\Gamma_p\left(\xi_{i_1},\ldots,\xi_{i_p}\right) = (-1)^p e^{\frac{1}{2}\xi^T\xi}\frac{\partial^p}{\partial\xi_{i_1}\ldots\ldots\partial\xi_{i_p}}e^{-\frac{1}{2}\xi^T\xi} \tag{2}$$

The inputs are represented as functions of identically independently distributed normal random variables $\{\xi_i \mid i=1,n\}$ and each ξ_i has zero mean and unit variance. These random variables are referred to as "Standard Random Variables (srvs)". Once the inputs are expressed as functions of these srvs, the output metrics can be represented as functions of the same set of srvs [8]. The minimum number of srvs needed to represent the inputs is defined as the "number of degrees of freedom" in the input uncertainty. In practice, in the theory of PCE, the minimum number of simulations required for generating the sample points of the uncertain inputs from the respective pdf depends on the order of the Hermite polynomial and the number of uncertain inputs. Therefore, if n is the number of uncertain inputs and r be the order of the polynomial, the number of simulations required can be formulated as:

$$N = \frac{(n+r)!}{n!\,r!} \tag{3}$$

Since the model outputs are deterministic functions of model inputs, they have at most the same number of degrees of freedom in uncertainty. So, the number of unknown coefficients to be determined for the fitted polynomial that represents the model output can be explicitly written using Eq. (3) as:

$$N_2 = 1 + 2n + \frac{n(n-1)}{2}, \qquad \text{for } 2^{nd} \text{ order Hermite polynomial} \tag{4}$$

$$N_3 = 1 + 3n + \frac{3n(n-1)}{2} + \frac{n(n-1)(n-2)}{6}, \text{ for } 3^{rd} \text{ order Hermite polynomial} \tag{5}$$

So, an explicit representation of 2nd order polynomial chaos expansion for two and three uncertain inputs can be written using Eqs.(1-5) as:

$$\begin{aligned} y_2 &= a_0 + a_1\xi_1 + a_2\xi_2 + a_3\left(\xi_1^2 - 1\right) + a_4\left(\xi_2^2 - 1\right) + a_5\xi_1\xi_2 \\ y_2 &= a_0 + a_1\xi_1 + a_2\xi_2 + a_3\xi_3 + a_4\left(\xi_1^2 - 1\right) + \\ & a_5\left(\xi_2^2 - 1\right) + a_6\left(\xi_3^2 - 1\right) + a_7\xi_1\xi_2 + a_8\xi_1\xi_3 + a_9\xi_2\xi_3 \end{aligned} \tag{6}$$

So, according to the number of uncertain model inputs, n = 2, 3, and 4, the number of unknown coefficients to be determined in the polynomial chaos expansion can be obtained using Eqs.(4) and (5) as {6, 10 and 15} and {10, 20 and 35} respectively. The number of unknown coefficients will guide the number of simulations. Thus for two uncertain model inputs, the second order polynomial chaos expansion needs six simulations to estimate the unknown coefficients. For reference, the first few Hermite polynomials are given by:

$$H_0(\xi) = 1,\ H_1(\xi) = 2\xi,\ H_2(\xi) = 2(\xi^2 - 1) \tag{7}$$

and the higher order Hermite polynomials can be generated using the recurrence relation given by:

$$H_{k+1}(\xi) = 2\xi\, H_k(\xi) - 2k\, H_{k-1}(\xi) \tag{8}$$

Polynomial chaos theory is not limited to the Hermite polynomials. Generalized polynomial chaos (otherwise known as Wiener-Askey Polynomial Chaos) expanded the theory to use all the polynomials from the Askey scheme of orthogonal polynomials [9]. In reference with the most common distribution we have: Hermite polynomials are associated with the Gaussian distribution, Legendre polynomials are associated with the uniform distribution, and Laguerre polynomials are associated with the exponential distribution [9]. The use of Hermite, Legendre, and Laguerre polynomials will from now on be referred to as Hermite-Chaos, Legendre-Chaos, and Laguerre-Chaos respectively [10]. When all of the variables in the examined system have been expanded onto the basis of choice, a Galerkin projection onto the basis is applied. The Galerkin projection is realized by the integration of each component of the examined system with the polynomial basis. The projection takes the form of an integral because the chosen polynomial bases are all continuous. The limits of the integral correspond to the region where the chosen polynomials are valid. The limits for Hermite-Chaos, Legendre-Chaos, and Laguerre-Chaos are $-\infty$ to ∞, -1 to 1, and 0 to ∞ respectively. This region is represented by the symbol Ω.

2.1 TRANSFORMATION OF MODEL INPUTS:

The number of sample values for the model outputs will have to be generated on the basis of the number of unknown coefficients. Therefore, for six unknown coefficients, six model outputs are to be generated for the specified model. Sampling points for the generation of these outputs will be obtained from the model uncertain inputs for which inputs are to be transformed into standard normal random variables (srvs) [8]. In the PCE, approach for transforming model uncertain inputs is based on the principle that random variables with well-behaved (square-integrable) probability density functions can be represented as functions of a set of srvs [8, 9]. Standard transformation of the uniform, normal, lognormal and gamma pdfs of model inputs in terms of srvs can be written as:

$$\begin{aligned} &\text{Uniform [a,b]}: \quad a + (b-a)\left\{\frac{1}{2} + \frac{1}{2}\,erf\left(\frac{\xi}{\sqrt{2}}\right)\right\} \\ &\text{Normal } (\mu,\sigma): \quad \mu + \sigma\xi \\ &\text{Lognormal } (\mu,\sigma): \quad \exp(\mu + \sigma\xi) \\ &\text{Gamma (a, b)} = ab\left(\xi\sqrt{\frac{1}{9a}} + 1 - \frac{1}{9a}\right)^2 \end{aligned} \tag{9}$$

Sample values of the output metrics [Eq. (6)] and the corresponding polynomial chaos expansion are finally arranged in the matrix form as $[\xi]\{a\} = y$, from which the coefficient vector, $\{a\}$ can be solved using singular value decomposition.

3.0 PROBLEM 1: ONE DIMENSIONAL SOLUTE TRANSPORT THROUGH GROUNDWATER:

Solute transport in groundwater is used for computing the concentration of a dissolved chemical species (contaminant) in an aquifer at any time and at any specified distance from the point of release of the chemical. The measured parameters associated with the representing model are seepage velocity and longitudinal dispersivity. In case of one – dimensional case, the governing equation is as given below:

$$\frac{\partial C}{\partial t} = D_L \frac{\partial^2 C}{\partial x^2} - u \frac{\partial C}{\partial x} \tag{10}$$

where,

$C = concentration\,of\ the\ dissolved\,chemical\,species(\text{contaminant})\,(M/L^3)$

$D_L = longitudinal\,dispersion\,coefficient = \alpha_L u(L^2/T)$

$u = seepage\ velocity\ in\ the\ x-direction\,(L/T)$

The boundary conditions for the above one-dimensional solute transport equations are:

$$\begin{aligned} C(0,t) &= C_0 \ , t \geq 0 \\ C(\infty,t) &= 0 \ , t \geq 0 \end{aligned} \tag{11}$$

and for the initial condition, we assume that,

$$C(x,0) = 0 \ , x \geq 0 \tag{12}$$

For the present problem, we have taken the initial concentration, C_0 = 100 mg/L. In the deterministic calculations, both the longitudinal dispersivity, α_L and seepage velocity, u will be specified exactly. But this assumption may not be always correct. Hence these parameters must be treated as random parameters with specific probability distribution. In the present case study, these parameters are assumed to be normally distributed with the mean, μ and the standard deviation, σ as specified in Eq. (13)

$$\begin{aligned} &for\ u\ ,\ \mu_u = 2.5\,m/day\ and\ \sigma_u = 0.2\,m/day \\ &for\ \alpha_L, \mu_\alpha = 15.3\,m\ and\ \sigma_\alpha = 3.0\,m \end{aligned} \tag{13}$$

The analytical expression of the concentration of the solute can be written as [12]:

$$C/C_0 = \frac{1}{2}\left[erfc\left\{ \frac{x - ut}{\sqrt{4\alpha_L ut}} \right\} + \exp\left(\frac{x}{\alpha_L} \right) erfc\left\{ \frac{x + ut}{\sqrt{4\alpha_L ut}} \right\} \right] \tag{14}$$

The second term in the square bracket of Eq. (14) has a little effect compared to the first term and that is why the model is simplified into the following form:

$$C/C_0 = \frac{1}{2}\left[erfc\left\{ \frac{x - ut}{\sqrt{4\alpha_L ut}} \right\} \right]$$

SOLUTION OF PROBLEM:

The number of uncertain inputs are two (seepage velocity, u and longitudinal dispersivity, α_L) and their distributions are presented in Table 1. We have selected the 2nd order polynomial as our response surface from the point of ease of computation. Following Eq. (4), six simulations are required for solving six unknown coefficients. Two sets (one for u

and the other for α_L) of six standard random variables are sampled using standard standardized normally distributed random number generator. The polynomial chaos matrix of order 6 x 6 is constructed using the first one of Eq. (6). Simple random sampling (Monte Carlo) technique is used to simulate the random values of the model output [$(C/C_0) \equiv y_2$]. Finally, the unknown coefficients of the response surface (keeping time fixed at t = 400 days) are solved using singular value decomposition and their values are as shown in Table 2. Statistics of this response surface is generated using 5000 Monte Carlo samples (convergence criteria followed) and is tabulated in Table 3. The uncertainty plot of the ratio of steady state derived concentration to the initial concentration (C/C_0) for varying downstream distance at time t = 400 days is presented in Fig. 1. The mean value of dilution factor at downstream distance, x = 1220 m is calculated using this response surface constructed as 7.86. Table 4 represents the coefficients of the polynomial chaos expansion which are used to construct the transient response surface by using the similar procedure. The transient response surface of concentration of contaminant is also constructed using the polynomial chaos and the statistics of this response surface at different times for a specific downstream distance (400 m) is tabulated in Table 5. The uncertainty plot of the ratio of the derived concentration to the initial concentration (C/C_0) for varying time at downstream distance, x = 400 m is shown in Fig. 2. The ratio of the derived concentration of contaminant to the original concentration at time 300 (days) is computed as 0.99. This indicates that the response surface can be used to predict the concentration of the contaminant at any time.

Table 1: Uncertain input parameters of solute transport problem

Parameter	Distribution	Mean	Standard deviation
Seepage velocity, v	Normal	2.50	0.20
Longitudinal dispersivity, α_L	Normal	15.30	3.00

Table 2: Coefficients of response polynomial for varying downstream distance

Downstream distance, x (m)	a_0	a_1	a_2	a_3	a_4	a_5
1000	0.4898	0.1794	-0.0001	-0.0106	0.0004	-0.0221
1100	0.2938	0.1620	0.0159	0.0084	0.0011	-0.0140
1200	0.1494	0.1056	0.0205	0.0214	0.0011	0.0027
1300	0.0636	0.0496	0.0153	0.0206	-0.0003	0.0106
1400	0.0221	0.0168	0.0078	0.0121	-0.0011	0.0084

Table 3: Statistics of the response surface of the concentration at time t = 400 days

Downstream distance, x (m)	Mean	Standard deviation	Skewness	Kurtosis	5^{th} percentile C/C_0	95^{th} percentile C/C_0
1000	0.4923	0.1840	-0.3590	3.1433	0.1763	0.7698
1100	0.2973	0.1664	0.1417	2.8964	0.0334	0.5704
1200	0.1491	0.1130	1.1619	4.6479	0.0093	0.3725
1300	0.0627	0.0577	2.0072	9.0895	0.0109	0.1787
1400	0.0209	0.0240	1.9749	7.4161	0.0003	0.0717

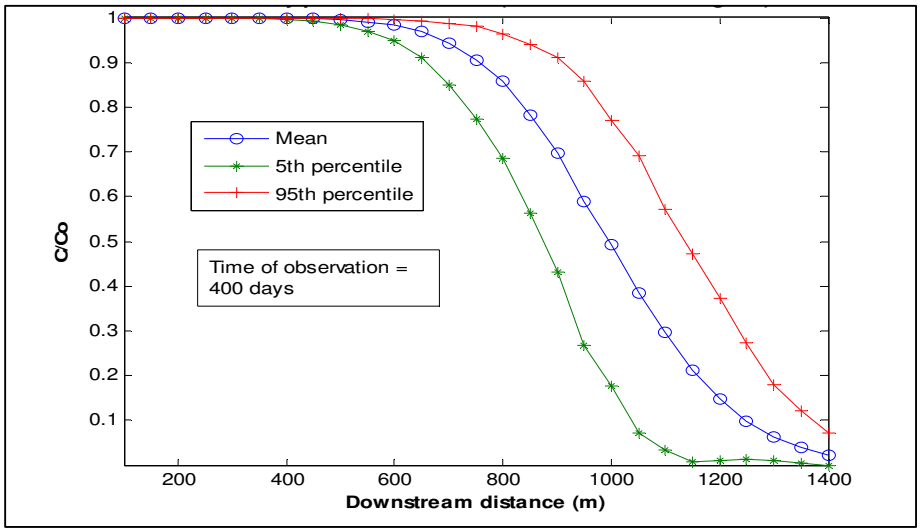

Fig 1: Variation of mean value, 5^{th} and 95^{th} percentiles of C/C_0 at time, t = 400 days

Table 4: Coefficients of response polynomial for different time at downstream distance, x = 400 m

Time, t (days)	a_0	a_1	a_2	a_3	a_4	a_5
200	0.7814	0.0838	-0.0223	-0.0125	0.0035	-0.0028
220	0.8651	0.0601	-0.0221	-0.0141	0.0022	0.0025
240	0.9190	0.0400	-0.0180	-0.0132	0.0008	0.0055
260	0.9523	0.0253	-0.0130	-0.0110	-0.0002	0.0061
280	0.9722	0.0153	-0.0086	-0.0084	-0.0006	0.0054

Table 5: Statistics of the response surface of C/C_0 for different times at downstream 400 m

Time, t (days)	Mean value	Standard Deviation	Skewness	Kurtosis	5th percentile	95th percentile
200	0.7830	0.0879	-0.6857	3.7536	0.6301	0.9074
220	0.8654	0.0690	-1.1590	5.2942	0.7411	0.9512
240	0.9223	0.0445	-1.4492	5.9535	0.8379	0.9721
260	0.9522	0.0338	-1.7708	6.8369	0.8859	0.9850
280	0.9719	0.0216	-1.9609	9.0094	0.9272	0.9920

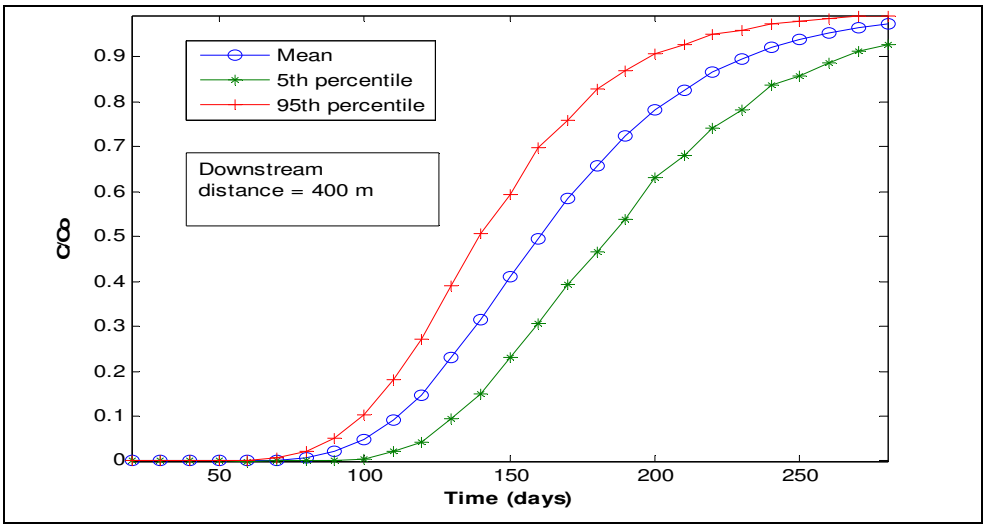

Fig 2: Variation of mean value, 5th and 95th percentiles of C/C_0 at downstream, x = 400 m

3.1 PROBLEM 2: ONE DIMENSIONAL HEAT CONDUCTION:

Consider a rectangular slab of length, say 1 m. Heat transfer takes place through this slab. Problem is to estimate the uncertainty of the temperature at 20 cm along the length of the slab.

SOLUTION OF PROBLEM 2:

Governing equation is

$$\frac{1}{k}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

where, k = Thermal diffusivity = $\frac{K}{\rho C}$

K = Thermal conductivity

ρ = Density of material and (15)

C = Specific heat

Boundary conditions are given as: $T(x=0)=0, \quad \frac{\partial T}{\partial x}(x=1)=0$ (16)

Initial condition is given as: $T\,(t\,=\,0)\,=\,T_0\,=100\,^{0}\,C$ (17)

Analytical solution of the equation (16) with the initial and boundary conditions can be written as

$$T(x,t)=T_0\left[1-erf\left(\frac{x}{2\sqrt{\frac{1}{\kappa t}}}\right)\right] \tag{18}$$

In this problem, we have considered thermometric conductivity (k) and initial temperature (T_0) as uncertain and their uncertainty is selected as stochastic in nature. Uncertainty (stochastic) of the input parameters is as given in Table 6.

Table 6: Distribution of Input Parameters

Parameter	Distribution	Mean	Standard deviation
k	Normal	10	0.2
T_0	Normal	100	0.5

Order of the polynomial chaos matrix in this case is again 6 x 6 due to the fact that the number of uncertain parameters is 2 and order of the polynomial selected as response surface is also 2. Elements of the 6 x 6 polynomial chaos matrix given in Table 7 are expressed in terms of standardized random variables (ξ_1 and ξ_2).

Table 7: Elements of Polynomial Chaos Matrix

Const	ξ_1	ξ_2	ξ_1^2 -1	ξ_2^2 -1	ξ_1 *ξ_2
1.0	0.1654	- 0.4587	- 0.9727	-0.7895	-0.0759
1.0	0.5191	0.5019	- 0.7305	-0.7481	0.2605
1.0	- 0.1941	- 1.3347	- 0.9623	0.7815	0.2591
1.0	- 1.0320	- 0.1039	0.0649	-0.9892	0.1072
1.0	- 0.4783	0.0048	- 0.7712	-1.0000	-0.0023
1.0	1.0548	1.3162	0.1126	0.7324	1.3883

Using this polynomial chaos matrix (A), representation of T(x,t) and right hand side of equation (18) at different sampled values of the model corresponding to the uncertain parameters, coefficients of response polynomial {a_k| k = 0,1,2,3,4,5} at different spatial points varying from x = 1 to 100 m with an increment of x = 5 cm at a specific time, t = 50 seconds are evaluated and they are tabulated in Table 8.

Table 8: Coefficients of the response polynomial

Distance, x (m)	a_0	a_1	a_2	a_3	a_4	a_5
1	97.48	0.0252	0.4874	-0.0004	-0.0000	0.0002
6	84.94	0.1487	0.4247	-0.0023	-0.0001	0.0009
11	72.79	0.2612	0.3639	-0.0039	-0.0001	0.0016
16	61.28	0.3552	0.3064	-0.0050	-0.0001	0.0021
21	50.66	0.4250	0.2532	-0.0056	-0.0001	0.0025
26	41.09	0.4679	0.2054	-0.0056	-0.0001	0.0026
31	32.69	0.4837	0.1634	-0.0050	-0.0001	0.0026
36	25.49	0.4751	0.1275	-0.0041	0.0000	0.0025
56	7.66	0.2946	0.0383	0.0003	0.0001	0.0013
76	1.63	0.1068	0.0081	0.0015	0.0000	0.0005
96	0.24	0.0242	0.0012	0.0007	0.0000	0.0002

Statistics of the response surface of the temperature at a specific time, t = 50 secs for those spatial points, x (Table 8) are generated using traditional Monte Carlo simulations of the response surface. Statistical results are presented in Table 9. Uncertainty plot of the temperature for varying distance (x = 1 to 100 m) at a fixed time t = 50 seconds is as shown in Fig 3.

Table 9: Statistics of the Response surface

Distance, x m	Average	Standard deviation	skewness	Kurtosis	5^{th} percentile	95^{th} percentile
1	97.46	0.4667	0.1891	3.3217	96.72	98.26
6	84.95	0.4444	0.0482	2.7250	84.23	85.69
11	72.83	0.4356	-0.0579	2.9864	72.11	73.49
16	61.27	0.4745	-0.0508	2.7811	60.48	62.04
21	50.66	0.4922	-0.1857	2.8505	49.86	51.43
26	41.06	0.5166	-0.0076	3.0952	40.23	41.88
31	32.68	0.5267	-0.1453	3.1166	31.76	33.50
36	25.51	0.4988	-0.1214	3.0739	24.67	26.28
56	7.66	0.2971	0.0455	2.8463	7.17	8.17
76	1.62	0.1096	0.0487	2.6648	1.45	1.80
96	0.24	0.0241	0.1618	3.0981	0.20	0.28

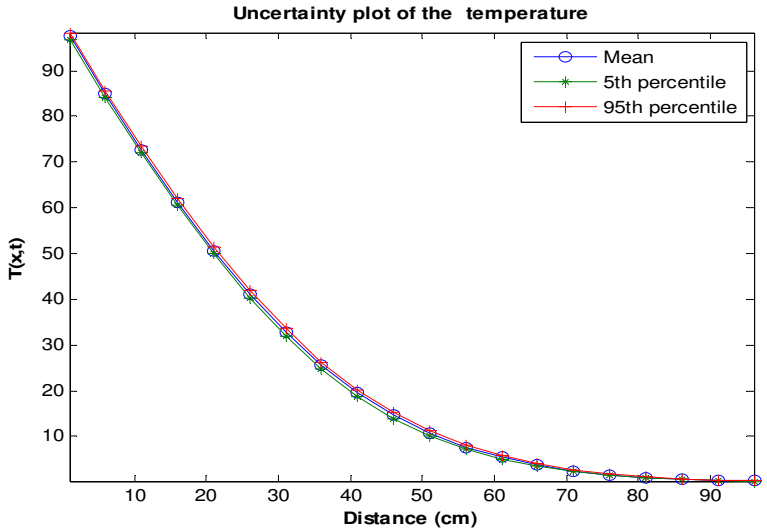

Fig 3: Variation of Temperature, T(x,t) with distance, x at time t = 20 secs

Mean value of the temperature evaluated at 20 (cm) distance = 52.70^{o} C. In a similar way stochastic response surface of the temperature for varying time with a fixed distance x = 20 cm is generated. Uncertainty plot of the generated temperatures for different time varying from 0 to 100 seconds is as shown in Fig 4. Mean value of temperature at time 100 (seconds) at x = 20 cm is estimated 65.66^{0}C.

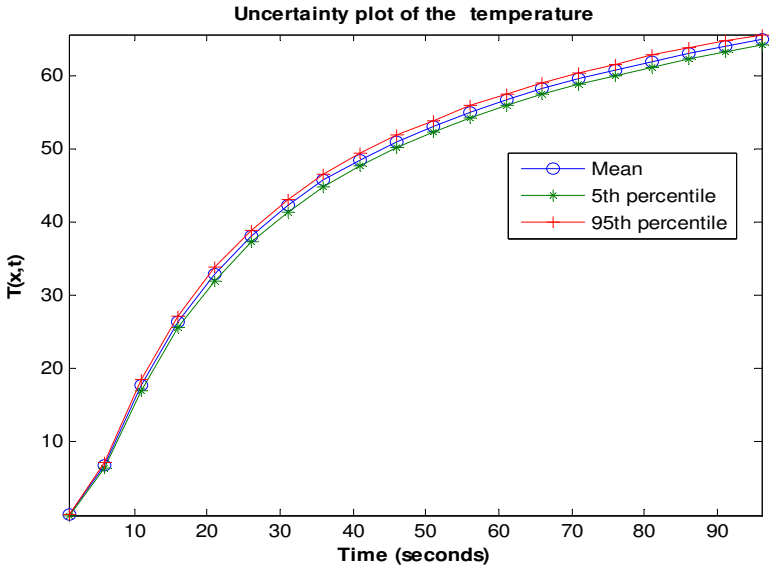

Fig 4: Time variation of temperature, T(x,t) at x = 20 cm

4.0 CONCLUSIONS:

In this work, uncertainty of concentration of contaminant at time, t = 400 days for varying downstream distances has been quantified using modified polynomial chaos. The same is applied to compute the uncertainties of the transient response surface for which downstream distance has been fixed at 400 m for convenience of calculation. Velocity of the water body and longitudinal dispersivity are considered as uncertain input parameters of the model. Hermite polynomials are considered as basis of the polynomial expansion and input parameters of the model are simulated in terms of standard random variable using the transformation mentioned. Hermite polynomials being represented as Gaussian, the expansion scheme can be named as Hermite (polynomial) chaos. Coefficients of the expansion are considered as time dependent and accordingly standard Hermite polynomials are modified. Modification of the Hermite polynomial justifies the present treatment of uncertainty as Modified chaos

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