International Journal of Mathematical Archive-2(6), June - 2011, Page: 931-934
Available online through www.ijma.info ISSN 2229 – 5046

SOME PROPERTIES OF A CERTAIN CLASS OF ARITHMETICAL FUNCTIONS

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(Received on: 02-06-11; Accepted on: 15-06-11)

ABSTRACT

In this paper, we define a certain class of arithmetical multiplicative functions which are called R-multiplicative functions. Every R-multiplicative function is a multiplicative function but converse need not be true. A necessary and sufficient condition for the Dirichlet product of two R-multiplicative functions to be as R-multiplicative function is given. Some properties on R-multiplicative functions are derived.

Key words: *Multiplicative function, R-multiplicative function, completely multiplicative function.*

AMS Mathematics subject classification (2010): *11A25, 11N37.*

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1. INTRODUCTION:

An arithmetical function is a mapping from the set of all positive integers $\mathbb{Z}^+$ to set of all complex numbers $\mathbb{C}$. The set of all arithmetical functions is denoted by $\mathcal{A}$. An arithmetical function f is said to be multiplicative if $f(1) = 1$ and $f(m\,n) = f(m)f(n)$ whenever $(m, n) = 1$. f is said to be Completely multiplicative function if $f(1) = 1$ and $f(m\,n) = f(m)f(n)$ for all $m, n \in \mathbb{Z}^+$. The set of all multiplicative functions is denoted by $\mathcal{M}$ and the set of all Completely multiplicative functions is is denoted by $\mathcal{C}$. In this paper it is defined that Dirichlet $*_{l,k}$-multiplication of two arithmetical functions. It is proved that the Dirichlet $*_{l,k}$ multiplication of two multiplicative functions is again a multiplicative function. $*$ is a classical Dirichlet convolution that is $f, g \in \mathcal{A}$, $(f * g)(n) = \sum_{d/n} f(d)g(\frac{n}{d})$ or equivalently $\sum_{ab=n} f(a)g(b)$. It is known that $f, g \in \mathcal{M}$ then $f * g \in \mathcal{M}$. Also it is known that $f \in \mathcal{M}$ then $f \in \mathcal{C}$ if and only if $f(p^\alpha) = f(p)^\alpha$ for all prime p, for all positive integers α. We define a certain class of arithmetical multiplicative functions which are called R-multiplicative functions. Every R-multiplicative function is a multiplicative function but converse need not be true. A necessary and sufficient condition for the Dirichlet product of two R-multiplicative functions to be as R-multiplicative function is given. Some properties on R-multiplicative functions are derived.

2. DIRICHLET $*_{l,k}$-MULTIPLICATION OF ARITHMETICAL FUNCTIONS:

In this section we define $*_{l,k}$-multiplication of two arithmetical functions and prove that the Dirichlet Product $*_{l,k}$ two multiplicative functions is again a multiplicative function. Also prove some properties of completely multiplicative functions.

Theorem 2.1: $f, g \in \mu$ such that $f(p)g(p) = 0$ for every prime p then $(f * g)(p^i) = f(p)^i + g(p)^i$, for all $i \geq 2$ if and only if $f * g \in \mathcal{C}$

Proof: Suppose $f, g \in \mathcal{M}$ and $f(p)g(p) = 0$ for every prime p then we have $f * g \in \mathcal{M}$.
Assume that $(f * g)(p^i) = f(p)^i + g(p)^i$, for all $i \geq 2$, for all primes p

Now we have to show that $(f * g)(p^i) = (f * g)(p)^i$, for $i \geq 2$. Consider

$$\begin{aligned}(f * g)(p^i) &= f(p)^i + g(p)^i \\ &= f(p)^i + i_{C_1} f(p)^{i-1} g(p) + i_{C_2} f(p)^{i-2} g(p)^2 + \cdots + i_{C_{i-1}} f(p)^i g(p)^{i-1} + g(p)^i \text{(since } f(p)g(p) = 0) \\ &= (f(p) + g(p))^i \\ &= ((f * g)(p))^i \text{ (By Dirichlet Convolution)}\end{aligned}$$

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Conversely assume that $f * g \in \mathcal{C}$.

Now, $(f * g)(p^i) = ((f * g)(p))^i$
$= (f(p) + g(p))^i$
$= f(p)^i + g(p)^i$ (since $f(p)g(p) = 0$)

Definition 2.2: Let f, g be arithmetical functions and l, k be positive integers. Then we define $(f *_{l,k} g)(m) = \sum_{d/n}(f(d^l)g\left(\frac{n}{d}\right)^k)$. This is called Dirichlet $*_{l,k}$-multiplication.

Now, we prove that Dirichlet $*_{l,k}$-multiplication of two arithmetical functions are again multiplicative.

Theorem 2.3: f, g are multiplicative functions, then $f *_{l,k} g$ is also multiplicative function

Proof: Let f, g are multiplicative functions, clearly $(f *_{l,k} g)(1)$ =1.
Let m, n be positive integers such that $(m, n) = 1$. Then every devisor c of mn is in the form $c = ab$ where ${}^{a}/_{m}, {}^{b}/_{n}$.
We observe that if $(a, b) = 1, \left(\frac{m}{a}, \frac{n}{b}\right) = 1$ then $(a^l, b^l) = 1$, $\left(\left(\frac{m}{a}\right)^k, \left(\frac{n}{b}\right)^k\right) = 1$.

$(f *_{l,k} g)(mn) = \sum_{a/m, b/n} f((ab)^l)g(\left(\frac{mn}{ab}\right)^k)$.
$= \sum_{a/m,b/n} f(a^l)f(b^l)g(\left(\frac{m}{a}\right)^k)(g\left(\frac{n}{b}\right)^k)$
$= \sum_{a/m,b/n} f(a^l)f(b^l)g(\left(\frac{m}{a}\right)^k)(g\left(\frac{n}{b}\right)^k)$
$= \sum_{a/m} f(a^l)g(\left(\frac{m}{a}\right)^k) \sum_{b/n} f(b^l)(g\left(\frac{n}{b}\right)^k)$
$= (f *_{l,k} g)(m)\,(f *_{l,k} g)(n)$

Therefore $f *_{l,k} g$ is also multiplicative.

The Dirichlet $*_{l,k}$ product of two completely functions is need not be completely multiplicative. In fact, if $l = 1, k = 1$ then $*_{l,k}$ is a classical Dirichlet multiplication.

3. R-MULTIPLICATIVE FUNCTIONS:

In this section we define a certain class of arithmetical functions which are called R-multiplicative functions. The set of all R-multiplicative functions is denoted by $\mathcal{R}$. Every R-multiplicative function is multiplicative but converse need not be true.

Definition 3.1: An arithmetical function f is said to be R-multiplicative function if

(i) $f(1) = 1$
(ii) $f(n) = f(p_1)f(p_2)\ldots\ldots f(p_n)$ where $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots\ldots p_n^{\alpha_n}, p_1, p_2 \ldots., p_n$ are distinct primes and $\alpha_1, \alpha_2 \ldots., \alpha_n$ are positive integers.

Note: Every R-multiplicative function satisfies $f(p^\alpha) = f(p)$ for all p, for all positive integers α.

The set of all R-multiplicative functions is denoted by $\mathcal{R}$.

Lemma 3.2: Every R-multiplicative function is multiplicative but converse need not be true.

Proof: Let f be an R-multiplicative function.

Let $(m, n) = 1, m = p_1^{\alpha_1}p_2^{\alpha_2}\ldots\ldots p_k^{\alpha_k}, n = q_1^{\beta_1}q_2^{\beta_2}\ldots\ldots q_l^{\beta_l}$, where p_i's , q_i's are primes, $\alpha, \beta's$ are positive integers. There are no common prime factors of m, n.

$f(mn) = f(p_1^{\alpha_1}p_2^{\alpha_2}\ldots\ldots p_k^{\alpha_k}.q_1^{\beta_1}q_2^{\beta_2}\ldots\ldots q_l^{\beta_l})$
$= f(p_1)f(p_2)\ldots\ldots f(p_k)f(q_1)f(q_2)\ldots\ldots f(q_l)$
$= f(m)f(n)$

Therefore every R-multiplicative function is multiplicative.

Definition 3.3[1]: $\mu: \mathbb{N} \to \mathbb{C}$ (mobious function) defined by

$\mu(1) = 1$, if $n > 1$ write $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}$.
$\mu(n) = (-1)^k$, if $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1$
$\mu(n) = 0$ otherwise.

μ is multiplicative function but it is not R-multiplicative because $\mu(3^2) = 0$, but $\mu(3)$= -1 , μ is not completely multiplicative function.

Theorem 3.4: f, g are R-multiplicative functions, then Dirichlet product $f * g$ is also R-multiplicative function if and only if $f(p)\ g(p) = 0$ for all primes p.

Proof: Let f, g are R-multiplicative functions then f, g are multiplicative and hence $f * g$ is also multiplicative function.

Suppose $f * g$ is R-multiplicative function. Let $\alpha \geq 2$

$$\begin{aligned} \text{Now, } (f * g)(p^\alpha) &= \textstyle\sum_{d/p^\alpha} f(d) g\left(\left(\frac{p^\alpha}{d}\right)\right) \\ &= g(p^\alpha) + f(p)g(p^{\alpha-1}) + f(p^2)g(p^{\alpha-2}) + \cdots + f(p^{\alpha-1})g(p) + f(p^\alpha) \\ &= g(p) + f(p)\,g(p) + f(p)\,g(p) + \ldots. + f(p)\,g(p) + g(p) \end{aligned}$$

$$\begin{aligned} (f * g)(p) &= g(p) + f(p)\,g(p) + f(p)\,g(p) + \ldots. + f(p)\,g(p) + g(p) \\ f(p) + g(p) &= g(p) + (\alpha - 1)\,f(p)\,g(p) + g(p) \\ &\Rightarrow (\alpha - 1)\,f(p)\,g(p) = 0 \\ &\Rightarrow f(p)\,g(p) = 0 \end{aligned}$$

Therefore $f(p)g(p) = 0$ for all primes p . Conversely assume that $f(p)g(p) = 0$ for all primes p. Since f, g are multiplicative functions then $f * g$ is also multiplicative. Therefore $(f * g)(1) = 1$.

First we prove $(f * g)(p^\alpha) = (f * g)(p)$

$$\begin{aligned} (f * g)(p^\alpha) &= \textstyle\sum_{d/p^\alpha} f(d) g\left(\left(\frac{p^\alpha}{d}\right)\right) \\ &= f(1)g(p^\alpha) + f(p)g(p^{\alpha-1}) + f(p^2)g(p^{\alpha-2}) + \cdots + f(p^{\alpha-1})g(p) + f(p^\alpha)g(1) \\ &= g(p) + f(p)\,g(p) + f(p)\,g(p) + \ldots. + f(p)\,g(p) + f(p) \quad (\text{since } (p^\alpha) = f(p)\ ) \\ &= f(p) + g(p) \quad (\text{since } f(p)g(p) = 0) \\ &= (f * g)(p) \end{aligned}$$

Now for $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_n^{\alpha_n}$

$$\begin{aligned} (f * g)(m) &= (f * g)(p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_n^{\alpha_n}) \\ &= (f * g)(p_1^{\alpha_1})(f * g)(p_2^{\alpha_2}) \ldots.(f * g)(p_n^{\alpha_n}) \\ &= (f * g)(p_1)(f * g)(p_2) \ldots.(f * g)(p_n) \end{aligned}$$

Hence $f * g$ is R-multiplicative function.

Definition 3.5[1]: An arithmetical function I is given by $I(n) = \begin{cases} 1, & if\ n = 1 \\ 0, & if\ n > 1 \end{cases}$.

Then $I(n)$ is both R-multiplicative and completely multiplicative.

Definition 3.6[1]: The arithmetical function $U(n) = 1$ for all $n \in \mathbb{Z}^+$ then U is both R-multiplicative and completely multiplicative.

Example 3.7: Example of a R-multiplicative function which is not completely multiplicative is given below.
An arithmetical function h is defined by $h(1) = 1, h(p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_n^{\alpha_n}) = 2^n$ then

$h(p_{1)},\ h(p_2) \ldots., h(p_n) =$ 2.2.....2(n times) = 2^n

Therefore h is R-multiplicative function.

But h is not completely multiplicative function.

$h(2^2.3.2^5.3^2.7^4) = h(2^7.3^3.7^4) = 2^3$ and $h(2^2.3).h(2^5.3^2.7^4) = 2^2.2^3 = 2^5$

Therefore $h(mn) \neq h(m)h(n)$ for all $m, n \in \mathbb{Z}^+$.
Therefore h is not completely multiplicative function.

Theorem 3.8: An arithmetical function f which is both R-multiplicative and completely multiplicative functions then $f(n) = 0$ or 1 for all positive integers n.

Proof: Let f be an arithmetical function which is both R-multiplicative and completely multiplicative.

Therefore $f(1) = 1$ for any prime p and $n \in \mathbb{Z}^+$.

$f(p^a) = f(p)^a$ (since f completely multiplicative)
$\Rightarrow f(p) = f(p)^a$ (since f is R-multiplicative)
$\Rightarrow f(p)\ (f(p)^{a-1} - 1) = 0$
$\Rightarrow f(p) = 0$ or $f(p)^{a-1} = 1$ for all positive integers $a \geq 2$.
$\Rightarrow f(p) = 0$ or $f(p) = 1$ for all primes.

However for $n > 1, n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\ldots\ldots{p_n}^{\alpha_n}$

$f(n) = f(p_1)f(p_2)\ldots\ldots f(p_n)$
If one $f(p_i)$ =0 then $f(n) = 0$
If all $f(p_i)$ =1 then$f(n) = 1$.

Lemma 3.9: A multiplicative function is R-multiplicative if and only if $f(p^n) = f(p)$ for all primes p, all positive integers n.

Proof: Let f is multiplicative function.

Suppose f is R-multiplicative then $f(p^n) = f(p)$ for all primes p

Conversely assume that $f(p^n) = f(p)$ for all primes $p,\ n \in \mathbb{Z}^+$.

Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\ldots\ldots{p_n}^{\alpha_n}$

$f(n) = f({p_1}^{\alpha_1})f({p_2}^{\alpha_2})\ldots.f({p_n}^{\alpha_n})$ (since f is multiplicative)
$f(n) = f(p_1)f(p_2)\ldots\ldots f(p_n)$

Hence f is R-multiplicative function.

Corollary 3.10: $g, f * g$ are R-multiplicative then f is also R-multiplicative if and only if
$f(p^n) = f(p)$ for all primes $p,\ n \in \mathbb{Z}^+$.

Proof: Let $g, f * g$ are R-multiplicative functions then $g, f * g$ are multiplicative functions. Then f is also multiplicative function.

Therefore by above lemma 3.9, f is R-multiplicative if and only if $f(p^n) = f(p)$for all primes $p,\ n \in \mathbb{Z}^+$.

Example 3.11[1]: A completely multiplicative but not R-multiplicative. Consider $N(n) = n$ for all $n \in \mathbb{Z}^+$.
Clearly $\mathbb{N}$ is completely multiplicative but not R-multiplicative.

Note: The set of all R-multiplicative function does not form a semi group under Dirchilet convolution.

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