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SOME COMMON FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS IN FUZZY METRIC SPACES

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ABSTRACT

In this paper we prove some common fixed point theorem for occasionally weakly compatible mapping in fuzzy metric spaces by taking average of some elements.

Keywords: *Occasionally weakly compatible (owc) mappings, fuzzy metric space.*

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1. INTRODUCTION:

Fuzzy set was defined by Zadeh [26]. Kramosil and Michalek [14] introduced fuzzy metric space, Grorge and Veermani [6] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Many researchers have obtained common fixed point theorems for mappings satisfying different types of commutativity conditions. Vasuki [25] proved fixed point theorems for R-weakly commutating mappings. Pant [18, 19, 20] introduced the new concept reciprocally continuous mappings and established some common fixed point theorems. Balasubramaniam er al. [4], have shoen that Rhoades [22] open problem on the existence of contractive definition which generates a fixed point but does not force the mappings to be continuous at the fixed point. Posses an affirmative answer. Pant and Jha [20] obtained some analogous results proved by Balasubramaniam et. Al. Recent literature in fixed point in fuzzy metric space can be viewed in [1, 2, 9, 16, 24].

This paper presents some common fixed point theorems for more general commutative condition i.e. occasionally weakly compatible mappings in fuzzy metric space by taking average of some elements.

2 PRELIMINARY NOTES:

Definition: 2.1 [12] A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition: 2.2 [4] A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norms if it satisfies the following conditions:

(i) *is associative and commutative;
(ii) *is continuous;
(iii) $a*1 = a$ for all $a \in [0, 1]$;
(iv) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition: 2.3 [2] A 3-tuples (X, M,*) is said to be a fuzzy metric space (shortly FM Space) if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set of $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $s, t > 0$;

(FM 1): $M(x, y, t) > 0$;
(FM 2): $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(FM 3): $M(x, y, t) = M(y, x, t)$;

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(FM 4): $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(FM 5): $M(x, y, .) : (0,\infty) \to (0,1]$ is left continuous.

(X, M,*) denotes a fuzzy metric space, M (x,y,t) can be thought of as degree of nearness between x and y with respect to t. We identify x = y with M(x, y, t) = 1for all t> 0. In the following example every metric induces a fuzzy metric.

Example: 2.4 (Induced fuzzy metric [6]) Let (X, d) be a metric space. Denote a * b = a.b & for all a,b $\in$ [0,1] and let M_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows.

$$M_d(x,y,t) = \frac{t}{t+d(x,y)}$$

Then (X, M,*) is a fuzzy metric space. We call this fuzzy metric induced by a metric **d** as the standard intuitionistic fuzzy metric.

Definition: 2.5 [8] Two self mappings f and g of a fuzzy metric space (X, M,*) are called compatible if $\lim_{n\to\infty} M(fg\, x_n, gf\, x_n, t) = 1$ wherever { x_n} is sequence in X such that

$\lim_{n\to\infty} f\, x_n = \lim_{n\to\infty} g\, x_n = x$ for some x in X

Definition: 2.6 [24] Two self maps f and g of a fuzzy metric space (X, M,*) are called reciprocally continuous on X if $\lim_{n\to\infty} f\, x_n = fx$ and $\lim_{n\to\infty} gf\, x_n = gx$ wherever, { x_n} is sequence in X such that

$\lim_{n\to\infty} f\, x_n = \lim_{n\to\infty} g\, x_n = x$ for some x in X.

Definition: 2.7 [6] Let (X, M,*) be a fuzzy metric space. Then

(a) A sequence $\{x_n\}$in X is said to converges to x in *X* if for each $\varepsilon > 0$ and each t > *0*, there exist $n_o \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_o$.
(b) A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\varepsilon > 0$ and each t > 0, there exist $n_o \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_o$.
(c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition: 2.8 Two self maps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute. A.Al-Thagafi and Naseer Shahzad [15] shown that occasionally weakly is weakly compatible but converse is not true.

Example: 2.9 Let R be the usual metric space. Define S, T: R → R by Sx = 3x and $Tx = x^2$ for all x $\in$ R. Then Sx = Tx for x = 0, 3 but ST0 = TS0, and ST3 ≠ TS3. Hence S and T are occasionally weakly compatible self maps but not weakly compatible.

Example: 2.10 [3] Let R be the usual metric space. Define S, T: R → R by Sx = 2x and $Tx = x^2$ for all x $\in$ R. Then Sx = Tx for x = 0, 2, but ST0 = TS0, and ST2 ≠ TS2. Hence S and T are occasionally weakly compatible self maps but not weakly compatible.

Lemma: 2.11 [12] Let X be a set and f, g owc self maps of X. If f and g have a unique point of coincidence, w = fx = gx, then w is the unique common fixed point of f and g.

Lemma: 2.12 Let (X, M,*) be a fuzzy metric space. If then exist q $\in$ (0, 1) such that

$M(x,y,qt) \geq M(x,y,t)$ for all x, y $\in$ X & t > 0 then x = y.

3 MAIN RESULTS:

Theorem: 3.1
Let (X, M, *) be a compete fuzzy metric space and let P, Q, Sand T are self–mapping of X. Let the pairs {P, S} and {Q, T} be owc . If there exists q $\in$ (0, 1) such that

$$M(Px,Qy,qt) \geq \min\{\frac{M(Sx,Ty,t)+M(Px,Ty,t)+M(Qy,Sx,t)}{3}, M(Sx,Px,t), M(Qy,Ty,t)\} \quad (1)$$

for all x, y ∈X and for all t > 0, then there exists a unique point w ∈ X such that Pw = Sw = w and a unique point z ∈ X such that Qz = Tz = z. Moreover, z = w, so that there is a unique common fixed point of P, Q, S and T.

Proof: Let the pairs {P, S} and {Q, T} be owc so there are points x,y ∈ X such that Px = Sx and Qy = Ty. We claim that Px = Qy. If not by inequality (1)

$$\begin{aligned} M(Px,Qy,qt) &\geq \min\{\frac{M(Sx,Ty,t)+M(Px,Ty,t)+M(Qy,Sx,t)}{3}, M(Sx,Px,t), M(Qy,Ty,t)\} \\ &= \min\{\frac{M(Px,Qy,t)+M(Px,Qy,t)+M(Qy,Px,t)}{3}, M(Px,Px,t), M(Qy,Qy,t)\} \\ &= \min\{\frac{M(Px,Qy,t)+M(Px,Qy,t)+M(Qy,Px,t)}{3}, 1,1\} \\ &= M(Px,Qy,t) \end{aligned}$$

Therefore Px = Qy , i.e. Px = Sx = Qy = Ty. Suppose that z such that Pz = Sz then by (1) we have Pz = Sz = Qy = Ty so Px = Pz and w = Px = Sx is the unique point of coincidence of P and S.

Similarly there is a unique point z ∈ X such that z = Qz = Tz.

Assume that w ≠ z .We have

$$M(w,z,qt) = M(Pw,Qz,qt)$$

$$\begin{aligned} M(Px,Qy,qt) &\geq \min\{\frac{M(Sx,Tz,t)+M(Pw,Tz,t)+M(Qz,Sw,t)}{3}, M(Sw,Pz,t), M(Qz,Tz,t)\} \\ &= \min\{\frac{M(w,z,t)+M(w,z,t)+M(w,z,t)}{3}, M(w,z,t), M(z,z,t)\} \\ &= M(w,z,t) \end{aligned}$$

Therefore we have z = w by Lemma 2.14 and z is a common fixed point of P, Q, S and T. The uniqueness of fixed point holds from (1)

Theorem: 3.2
Let (X, M, *) be complete fuzzy metric space and let P,Q,S and T be self mappings of X .Let the pairs {P, S} and {Q, T} be owc . If there exists q ∈ (0, 1) such that

$$M(Px,Qy,qt) \geq \emptyset[\min\{\frac{M(Sx,Ty,t)+M(Px,Ty,t)+M(Qy,Sx,t)}{3}, M(Sx,Px,t), M(Qy,Ty,t)\}] \quad (2)$$

for all x , y ∈ X and Ø : [0,1] ⟶ [0,1] such that Ø(t) > t for all 0 < t < 1 ,then there exist a unique common fixed point of P, Q,S and T

Proof:

$$\begin{aligned} M(Px,Qy,qt) &\geq \emptyset[\min\{\frac{M(Sx,Ty,t)+M(Px,Ty,t)+M(Qy,Sx,t)}{3}, M(Sx,Px,t), M(Qy,Ty,t)\}] \\ &\geq \emptyset[M(Px,Qy,t)] \quad \text{from theorem 3.1} \\ &\geq \emptyset[M(Px,Qy,t)] \end{aligned}$$

Now proof fallows by (3.1)

Theorem: 3.3
Let (X, M,*) be a complete fuzzy metric space and let P, Q, S and T are self mappings of X.

Let the pairs {P, S}and {Q,T} be owc . If there exists q ∈ (0, 1) such that

$$M(Px,Qy,qt) \geq Ø[\min\{\frac{M(Sx,Ty,t)+M(Px,Ty,t)+M(Qy,Sx,t)}{3}, M(Sx,Px,t), M(Qy,Ty,t)\}] \quad (3)$$

For all x, y ∈ X and Ø: $[0,1]^4 \to [0,1]$ such that Ø (t,1,t,t) > t for all 0 < t < 1 then there exists a unique common fixed point of P, Q, S and T.

Proof: Let the pairs {P, S}and {Q, T} are owc ,there are points x, y ∈ X such that Px = Sx and Qy = Ty are claim that Px = Qy. By inequality (3) we have

$$M(Px,Qy,qt) \geq Ø[\min\{\frac{M(Sx,Ty,t)+M(Px,Ty,t)+M(Qy,Sx,t)}{3}, M(Sx,Px,t), M(Qy,Ty,t)\}]$$

$$= Ø[\min\{\frac{M(Px,Qy,t)+M(Px,Qy,t)+M(Qy,Px,t)}{3}, M(Px,Px,t), M(Qy,Qy,t)\}]$$

$$= Ø[\min\{M(Px,Qy,t),1,1\}], [\because M(Px,Px,t) = 1, M(Qy,Qy,t) = 1]$$

$$> M(Px,Qy,t)$$

a contradiction , therefore Px = Qy i.e. Px = Sx = Qy = Ty suppose that there is another point z such that Pz = Sz then by (3) we have

Pz = Sz = Qy = Ty so Px = Pz and w = Px = Tx is unique point of coincidence of P and T. Qy Lemma 2.14 w is a unique common fixed point of P and S, similarly there is a unique point z ∈ X such that z = Qz = Tz..Thus z is common fixed point of P, Q, S, and T. The uniqueness of fixed point holds from (3).

Theorem: 3.4
Let (X,M,∗) be complete fuzzy metric space and let P,Q,S and T be self mappings of X, let the pairs {P,S} and {Q,T} are owc. If there exists a points q ∈ (0,1) for all x, y ∈ X and t > 0

$$M(Px,Qy,qt) > \{\frac{M(Sx,Ty,t)+M(Px,Ty,t)+M(Sx,Qy,t)}{3}, M(Px,Sx,t), M(Qy,Ty,t)\} \quad (4)$$

then there exists a unique common fixed points of P, Q, S and T.

Proof: Let the points {P, S} and {Q, T} are owc and there are points x, y ∈ X such that Px = Sx and Qy = Ty and claim that Px = Qy By inequality (4)

We have

$$M(Px,Qy,qt) > \{\frac{M(Sx,Ty,t)+M(Px,Ty,t)+M(Sx,Qy,t)}{3}, M(Px,Sx,t), M(Qy,Ty,t)\}$$

$$= \{\frac{M(Px,Qy,t)+M(Px,Qy,t)+M(Px,Qy,t)}{3}, M(Px,Px,t), M(Qy,Qy,t)\}$$

$$\geq M(Px,Qy,t)*1*1$$

$$\geq M(Px,Qy,t)$$

Thus we have Px = Qy i.e. Px = Sx = Qy = Ty. Suppose that there is another point z such that Pz = Sz then by (4) we have Pz = Sz = Qy=Ty so Px = Pz and w = Px = Sx is unique point of coincidence of P and S.

Similarly there is a unique point z ∈ X such that z = Qz = Tz. Thus w is a common fixed point of P, Q, S and T.

Corollary: 3.5

Let (X, M,*) be a complete fuzzy metric space and let P, Q, S and T be self mappings of X. Let the pairs {P, S} and {Q, T} are owc. If there exists a point q ∈ (0, 1) for all x, y ∈ X and t > 0

$$M(Px,Qy,qt) \geq [\{\frac{M(Sx,Ty,t)+M(Qy,Sx,2t)+M(Px,Ty,t)}{3}\} * M(Px,Sx,t) * M(Qy,Ty,t)] \quad (5)$$

then there exists a unique common fixed point of P, Q, S and T.

Proof: We have

$$M(Px,Qy,qt) \geq [\{\frac{M(Sx,Ty,t)+M(Qy,Sx,2t)+M(Px,Ty,t)}{3}\} * M(Px,Sx,t) * M(Qy,Ty,t)]$$

$$\geq [\{\frac{M(Sx,Ty,t)+M(Sx,Ty,t)*M(Ty,Qy,t)+M(Px,Ty,t)}{3}\} * M(Px,Sx,t) * M(Qy,Ty,t)]$$

$$\geq [\{\frac{M(Sx,Ty,t)+M(Sx,Ty,t)*M(Qy,Qy,t)+M(Px,Ty,t)}{3}\} * M(Px,Px,t) * M(Qy,Ty,t)]$$

$$\geq [\{\frac{M(Sx,Ty,t)+M(Sx,Ty,t)*1+M(Px,Ty,t)}{3}\} *1*1]$$

$$\geq [\{\frac{M(Sx,Ty,t)+M(Sx,Ty,t)+M(Px,Ty,t)}{3}\} *1*1]$$

$$\geq M(Px,Px,t), \ [\because Px = Sx \text{ and } Qy = Ty]$$

and therefore from Theorem 3.4 , P, Q, S and T have common fixed point.

Corollary: 3.6
Let (X, M,*) be complete fuzzy metric space and let P, Q, S and T be self-mapping of X. Let the pairs {P, S} and {Q, T} are owc. If there exist point q ∈ (0, 1) for all x, y ∈ X and t > 0

$$M(Px,Qy,qt) \geq M(Sx,Ty,t) \quad (6)$$

then there exists a unique common fixed point of P, Q, S and T.

Proof: The proof follows from Corollary 3.5

Theorem:3.7
Let (X, M,*) be complete fuzzy metric space. Then continues self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping P of X such that the following conditions are satisfied

(i) PX ⊂ TX ∩ SX
(ii) pairs {P, S} and {P, T} are weakly compatible,
(iii) there exists a point q ∈ (0,1) such that for every x, y ∈ X and t > 0

$$M(Px,Qy,qt) \geq \{\frac{M(Sx,Ty,t)+M(Px,Sx,t)+M(Px,Ty,t)}{3} * M(Py,Ty,t)\} \quad (7)$$

then P, S and T have a unique common fixed point.

Proof: Since compatible implies owc , the result follows from 3.4

Theorem: 3.8
Let (X, M,*) be a complete fuzzy metric space and let P and Q be self mapping of X .Let P and Q are owc. If there exists a point q ∈ (0,1) for all x, y ∈ X and t > 0

$$M(Sx,Sy,qt) \geq \alpha M(Px,Py,t) + \beta\left\{\frac{M(Px,Py,t)+M(Sy,Px,t)+M(Sx,Py,t)}{3}\right\} \quad (8)$$

for all x,y ∈ X , where α, β > 0 , α+β >1. Then P and S have a unique common fixed point.

Proof: Let the pairs {P, S} be owc, so there is a point x ∈ X such that Px = Sx. Suppose that there exist another point y ∈ X for which Py = Sy We claim that Sx = Sy by equation (8) we have

$$M(Sx,Sy,qt) \geq \alpha M(Px,Py,t) + \beta\left\{\frac{M(Px,Py,t)+M(Sy,Px,t)+M(Sx,Py,t)}{3}\right\}$$

$$M(Sx,Sy,qt) \geq \alpha M(Sx,Sy,t) + \beta\left\{\frac{M(Sx,Sy,t)+M(Sy,Sx,t)+M(Sx,Sy,t)}{3}\right\}$$

$$= (\alpha+\beta)\, M(Sx,Sy,t)$$

A contradiction, since (α+β) > 1therefore Sx = Sy. Therefore Px = Py and Px is unique. From lemma 2.14, P and S have a unique fixed point.

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