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ON THE C-ALGEBRA C^X

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ABSTRACT

It is known that in a Boolean algebra, every prime ideal is maximal. It is under investigation that whether every prime ideal is maximal in a C-algebra. In this paper we discuss an important example of a C-algebra namely C^X containing sufficiently many prime and maximal ideals and in which every prime ideal is maximal ideal.

Key words: *C-algebra, Ideal, Prime ideal, Maximal ideal.*

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INTRODUCTION:

In [1] Fernando Guzman and Craig C. Squier introduced the variety of C-algebras as the variety generated by the three element algebra C = {T, F, U} with the operations $\wedge$, $\vee$ and $'$of type (2, 2,1), which is the algebraic form of the three valued conditional logic. They proved that C and the two element Boolean algebra B = {T, F} are the only subdirectly irreducible C-algebras and that the variety of C-algebras is a minimal cover of the variety of Boolean algebras. In [4] U. M. Swamy et. al., have worked on three valued logic and introduced the concept of the Centre $\mathcal{B}(A)$ of a C-algebra A and proved that the centre of a C-algebra is a Boolean algebra. Later in [2], [3] S. Kalesha Vali et.al. Introduced the notion of an ideal, prime and maximal ideal of a C-algebra and discussed various properties of these. In [3] there is an open problem "whether every prime ideal is maximal in a C-algebra?". The answer is still under investigation. In the way of investigation we come across an example C^X, which contain sufficiently many prime ideals and maximal ideals and in which every prime ideal is maximal ideal.

1. C-ALGEBRA:

In this section we recall the definition of a C-algebra and some results from [1], [4] and [5]. Let us start with the definition of a C-algebra.

Definition 1.1: [1] By a C-algebra we mean an algebra of type (2, 2, 1) with binary operations $\wedge$ and $\vee$ and unary operation $'$ satisfying the following identities.

(1) $x'' = x$ (2) $(x \wedge y)' = x' \vee y'$
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ (4) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(5) $(x \vee y) \wedge z = (x \wedge z) \vee (x' \wedge y \wedge z)$ (6) $x \vee (x \wedge y) = x$
(7) $(x \wedge y) \vee (y \wedge x) = (y \wedge x) \vee (x \wedge y)$.

Example 1.2: [1] The three element algebra C= {T, F, U} with the operations given by the following

Tables are a C-algebra.

$\wedge$	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U

$\vee$	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U

x	x'
T	F
F	T
U	U

Note 1.3: [1] The identities 1.1(1), 1.1(2) imply that the variety of C-algebras satisfies all the dual statements of 1.1(2) to 1.1(7). $\wedge$ and $\vee$ are not commutative in C. The ordinary distributive law of $\wedge$ over $\vee$ fails in C. Every Boolean algebra is a C-algebra.

Now we recall some results on C-algebra collected from [1], [4] and [5].

Lemma 1.4: Every C-algebra satisfies the following identities:

(1) $x \wedge x = x$	(2) $x \wedge x' = x' \wedge x$
(3) $x \wedge y \wedge x = x \wedge y$	(4) $x \wedge x' \wedge y = x \wedge x'$
(5) $x \wedge y = (x' \vee y) \wedge x$	(6) $x \wedge y = x \wedge (y \vee x')$
(7) $x \wedge y = x \wedge (x' \vee y)$	(8) $x \wedge y \wedge x' = x \wedge y \wedge y'$
(9) $(x \vee y) \wedge x = x \vee (y \wedge x)$	(10) $x \wedge (x' \vee x) = (x' \vee x) \wedge x = (x \vee x') \wedge x = x.$

Duals of the statements in the above lemma are also true in a C-algebra.

Definition 1.5: [4] Let A be a C-algebra with T (T is the identity element for $\wedge$ in A). Then the Boolean centre of A is defined as the set $\mathcal{B}(A) = \{ a \in A \mid a \vee a' = T \}$. $\mathcal{B}(A)$ is known to be a Boolean algebra under the operations induced by those on A.

2. Ideals of a C-algebra:

In this section we recall the definition of an ideal, prime ideal and maximal ideal of a C-algebra and some results from [2], [3] which are useful in proving the results in the forthcoming sections. Let us start with the definition of an ideal of a C-algebra.

Definition 2.1: [2] A nonempty subset I of a C-algebra A is said to be an ideal of A if it satisfies

(i) $a, b \in I$ implies that $a \vee b \in I$ and
(ii) $a \in I$ implies that $x \wedge a \in I$, for each $x \in A$.

Theorem 2.2: [2] Let $A = A_1 \times A_2 \times \ldots \times A_n$ be the product of the C-algebras $A_1, A_2, \ldots, A_n$ and each C-algebra A_i is with T and $I \subseteq A$. Then I is an ideal of A if and only if I is of the form $I = I_1 \times I_2 \times \ldots \times I_n$, where I_i is an ideal of A_i.

Definition 2.3: [3] Let A be a C-algebra. A proper ideal P of A is called a prime ideal if, for any $a, b \in A$, $a \wedge b \in P$ implies that either $a \in P$ or $b \in P$.

Definition 2.4: [3] A proper ideal M of a C-algebra A is said to be a Maximal ideal of A if M is maximal among all the proper ideals of A.

Lemma 2.5: [3] Let A be a C-algebra. Every maximal ideal in A is a prime ideal.

The validity of the converse of the above theorem is not known. In Boolean algebras every prime ideal is maximal but in C-algebras, we do not know that every prime ideal is maximal, it is under investigation.

Theorem 2.6: [3] Let I be an ideal of a C-algebra and $a \in A \setminus I$. Then there exists a prime ideal P containing I and not containing a.

3. THE C-ALGEBRA C^X:

In the following we discuss an important example of a C-algebra containing sufficiently many prime ideals and maximal ideals.

Definition 3.1: Let X be a nonempty set and C = {T, F, U} be the three-element C-algebra. Let C^X be the set of all mappings of X into C. Then C^X is a C-algebra under the point wise operations. For any

$Y \subseteq X$, let $f_Y \in C^X$ be defined by $f_Y(x) = \begin{cases} T, & if\ x \in Y \\ F, & if\ x \notin Y \end{cases}$ and, for any $x \in X$, let $f_x = f_{\{x\}}$. Also, for any $f \in C^X$,

let $|f| = \{x \in X \mid f(x) = T\}$. $|f|$ is called the support of X.

Theorem 3.2: The following hold for any $g \epsilon C^X$ and $Y \subseteq X$.

(1) $f_Y{}' = f_{X \setminus Y}$ (2) $f_Y \wedge f_Y{}' = \overline{F}$, the constant map.
(3) $f_Y \epsilon \mathcal{B}(C^X)$ (4) $g \wedge f_{|g|} = g$.

Proof: Define $\overline{F}: X \to C$ by $\overline{F}(x) = F$, for all $x \epsilon X$.

(1) This follows from the facts that $T' = F$ and $f'(x) = f(x)'$ for all $x \epsilon X$ and for all $f \epsilon C^X$.
(2) If $z \epsilon Y$, $(f_Y \wedge f_Y{}')(z) = T \wedge F = F$ and if $z \notin Y$, $(f_Y \wedge f_Y{}')(z) = F \wedge T = F$.

Therefore $f_Y \wedge f_Y{}' = \overline{F}$.

(3) Since $f_Y \wedge f_Y{}' = \overline{F}$, $(f_Y \wedge f_Y{}')' = (\overline{F})'$, and hence $f_Y{}' \vee f_Y = \overline{T}$, so that $f_Y \vee f_Y{}' = \overline{T}$.

Therefore $f_Y \epsilon \mathcal{B}(C^X)$.

(4) For $x \in |g|$, $f_{|g|}(x) = T$ and $g(x) \wedge f_{|g|}(x) = g(x) \wedge T = T \wedge T = T = g(x)$.
For $x \notin |g|$, $f_{|g|}(x) = F$ and $g(x) \wedge f_{|g|}(x) = g(x) \wedge F = g(x)$, (since $g(x) = U$ or $F, g(x) \wedge F = g(x)$).

Therefore $g \wedge f_{|g|} = g$.

Theorem 3.3: Every prime ideal of C^X is a maximal ideal.

Proof: Let P be a prime ideal of C^X. Let Q be any ideal of C^X such that $P \subsetneq Q$.
Then there exists $g \epsilon Q$ such that $g \notin P$. Since $g \epsilon C^X$, $g \wedge f_{|g|} = g$. Then $f_{|g|} \notin P$ (since $g \notin P$). Since $f_{|g|} \wedge f'_{|g|} = \overline{F} \in P$ and P is a prime ideal, $f'_{|g|} \in P \subseteq Q$.

We shall prove that $f'_{|g|} \vee g = \overline{T}$.

If $x \in |g|$ then $g(x) = T$ and $f'_{|g|}(x) = F$ and hence $\left(f'_{|g|} \vee g\right)(x) = F \vee T = T = g(x)$.
If $x \notin |g|$ then $g(x) = U$ or F and $f'_{|g|}(x) = T$ and hence $\left(f'_{|g|} \vee g\right)(x) = T \vee U$ or $T \vee F = T$.

Therefore $f'_{|g|} \vee g = \overline{T}$. Since $f'_{|g|} \in Q$, and $g \epsilon Q$, $f'_{|g|} \vee g \in Q$ (since Q is an ideal).

Therefore $\overline{T} \in Q$ which implies that $Q = C^X$. Therefore P is maximal ideal of C^X.

In theorem 3.2 we have proved that $f_Y \epsilon \mathcal{B}(C^X)$ for any subset Y of X. In fact, every element of $\mathcal{B}(C^X)$ must be of the form f_Y for a suitable $Y \subseteq X$. This is proved in the following.

Theorem 3.4: For any $a \epsilon C^X$, the following are equivalent

(1) $a(x) \neq U$, for all $x \in X$ (2) $a \vee a' = \overline{T}$; that is, $a \epsilon \mathcal{B}(C^X)$
(3) $a = f_{|a|}$ (4) $a = f_{|Y|}$ for some $Y \subseteq X$.

Proof: (1) ⇒(2): Suppose that $a(x) \neq U$, for all $x \in X$. Then $a(x) = T$ or F, for all $x \in X$ and hence $a'(x) = F$ or T. Now, $(a \vee a')(x) = T \vee F$ *or* $F \vee T = T$. Therefore $a \vee a' = \overline{T}$.
(2) ⇒ (3): Suppose $a \vee a' = \overline{T}$. If $x \in |a|$, then $f_{|a|}(x) = T = a(x)$.

If $x \notin |a|$ then $a(x) = U$ or F and hence $a(x) = F$ (otherwise if $a(x) = U, (a \vee a')(x) = U \neq \overline{T}(x)$, a contradiction). Therefore $f_{|a|}(x) = F = a(x)$, if $x \notin |a|$. Thus $a = f_{|a|}$.

(3) ⇒ (4) is trivial, since $|a| \subseteq X$.
(4) ⇒ (1): $a = f_{|Y|}$ for some $Y \subseteq X$. Since $f_{|Y|}(x) = T$ or F according as $x \in Y$ or $x \notin Y$, $a(x) = T$ or F for any $x \in X$. Thus $a(x) \neq U$ for any $x \in X$.

Theorem 3.5: The map $\phi: \wp(X) \to \mathcal{B}(C^X)$, defined by $\phi(Y) = f_Y$ is an isomorphism of Boolean algebras, where $\wp(X)$ is the Boolean algebra of all subsets of X.

Proof: If $Y \subseteq X$, then $f_Y(z) = \begin{cases} T, & if\ z \epsilon Y \\ F, & if\ z \notin Y \end{cases}$

We shall prove that $f_{Y \cap Z} = f_Y \wedge f_Z$, $f_{Y \cup Z} = f_Y \vee f_Z$ and $f_Y^{'} = f_{X \setminus Y}$ for any subsets Y and Z of X , so that ϕ becomes a homomorphism of Boolean algebras.

Let Y and $Z \subseteq X$. Let $x \epsilon Y \cap Z$. Then $x \in Y$ and $x \in Z$ and $f_{Y \cap Z}(x) = T = T \wedge T = f_Y(x) \wedge f_Z(x) = (f_Y \wedge f_Z)(x)$.

Suppose $x \notin Y \cap Z$. Then $x \notin Y$ or $x \notin Z$

$$x \notin Y \Rightarrow f_Y(x) = F \ and \ x \notin Z \Rightarrow f_Z(x) = F$$

Since, $x \notin Y \cap Z$, we have that $f_{Y \cap Z}(x) = F = f_Y(x) \wedge f_Z(x)$ (since at least one of $f_Y(x)$ and $f_Z(x)$ is F).

Therefore $f_{Y \cap Z} = f_Y \wedge f_Z$.

Let $x \in Y \cup Z$. Then $x \in Y$ or $x \in Z$.

$f_{Y \cup Z}(x) = T = (f_Y \vee f_Z)(x)$ $(since\ at\ least\ one\ of\ f_Y(x)\ and\ f_Z(x)\ is\ T)$

Suppose $x \notin Y \cup Z$. Then $x \notin Y\ and\ x \notin Z$.

Therefore $f_Y(x) = F$ and $f_Z(x) = F$

Now $f_{Y \cup Z}(x) = F = F \vee F = f_Y(x) \vee f_Z(x)$.

Therefore $f_{Y \cup Z} = f_Y \vee f_Z$.

We have, by theorem 4.2(1), $f_Y^{'} = f_{X \setminus Y}$.

Thus, ϕ is a homomorphism of Boolean algebras.

Let $Y, Z \epsilon \wp(X)$ such that $\phi(Y) = \phi(Z)$ that is $f_Y = f_Z$.

Let $x \in Z$. Then $T = f_Y(x) = f_Z(x)$ and hence $x \in Y$.

Therefore $Z \subseteq Y$. Similarly, $Y \subseteq Z$. Thus Y = Z. Therefore ϕ is an injection map.

By theorem 3.4, the map $\phi: \wp(X) \rightarrow \mathcal{B}(C^X)$ is a surjection map too.

Thus, $\phi: \wp(X) \rightarrow \mathcal{B}(C^X)$ is an isomorphism.

Theorem 3.6: For any $g \epsilon C^X$ there exists a smallest $a \epsilon \mathcal{B}(C^X)$ such that $g = g \wedge a$ and $a' \vee g = \overline{T}$

Proof: Let $g \epsilon C^X$. We know that $f_Y \epsilon \mathcal{B}(C^X)$, for any $Y \subseteq X$.

Put $a = f_{|g|}$. Then $f_Y \epsilon \mathcal{B}(C^X)$.

By theorems 3.2, 3.4, we have $g = g \wedge a$ and $a^{'} \vee g = \overline{T}$.
Further, let $b \epsilon \mathcal{B}(C^X)$, such that $g = g \wedge b$ and $b' \vee g = \overline{T}$ Then $|g| \subseteq |b|$ and,
by theorem 3.5, we have that $f_{|g|} \wedge f_{|b|} = f_{|g| \cap |b|} = f_{|g|}$ so that $a = f_{|g|} \leq f_{|b|} = b$. Thus a is the smallest element of $\mathcal{B}(C^X)$ such that $g = g \wedge a$.

Theorem 3.7: For any $x \in X$, let $P_x = \{ f \in C^X \mid f(x) \neq T \}$. Then P_x is a prime (maximal) ideal of C^X .

Proof: Clearly P_x is a nonempty subset of C^X . Let $f, g \in P_x$. Then

$(f \vee g)(x) = f(x) \vee g(x) \neq T$ (Since, if $f(x) = U$ then $f(x) \vee g(x) = U$ and if $f(x) = F$, $g(x) = F$, then $f(x) \vee g(x) = F$ also, if $f(x) = F$, $g(x) = U$, then $f(x) \vee g(x) = U$).

Therefore $f \vee g \in P_x$.

Let $f \in P_x\ and\ h \in C^X$.

Now $(h \wedge f)(x) = h(x) \wedge f(x) = \begin{cases} f(x), & \text{if } h(x) = T \\ h(x), & \text{if } h(x) \neq T \end{cases}$

Therefore $(h \wedge f)(x) \neq T$ and hence $h \wedge f \in P_x$. Thus P_x is an ideal of C^X.

Let $f, g \in C^X$ such that $f \wedge g \in P_x$. Then $f(x) \wedge g(x) \neq T$.

If $f(x) = T$ and $g(x) = T$ then $f(x) \wedge g(x) = (f \wedge g)(x) = T$.

Therefore $f(x) \neq T$ or $g(x) \neq T$. Therefore $f \in P_x$ or $g \in P_x$.

Thus P_x is a prime ideal and hence a maximal ideal of C^X.

Note that all the prime ideals of C^X need not be of the form P_x. For this, consider the following example

Example 3.8: Let X be an infinite set and $I = \{ f \in C^X \mid |f| \text{ is finite}\}$. Then I is a proper ideal of C^X and hence, by theorem 3.4, there exists a prime ideals P of C^X containing I. We observe that $P \neq P_x$ for all $x \in X$; for any $x \in X$, the element f_x of C^X defined in the definition 3.1 is in I and hence in P but not in P_x.
However, when X is a finite set, we do prove that every prime ideal of C^X of the form P_x for some $x \in X$.

Theorem 3.9: Let X be a nonempty finite set and P a prime ideal of C^X. Then there exists unique $x \in X$ such that $P_x = P$.

Proof: Let X = $\{ x_1, x_2, \ldots\ldots, x_n \}$ where n is a positive integer. First we observe that

$C^X \cong C_1 \times C_2 \times \ldots\ldots \times C_n$. Let P be a prime ideal of C^X. Then by theorem 2.2, we can assume that $P = I_1 \times I_2 \times \ldots\ldots \times I_n$ where each I_i is an ideal of A. Recall that each I_i is either {U, F} or C (since these two are the only ideals of C).

We argue that $I_i = \{U, F\}$, for unique i. Suppose $i \neq j$ such that $I_i = \{U, F\} = I_j$. Then consider the elements a and b defined by $a = (F, \ldots, F, T, F, \ldots, F)$ (ith place is T and F elsewhere) and $b = (F, \ldots, F, T, F, \ldots, F)$ (jth place is T and F elsewhere).

Now, $a \wedge b = (F, \ldots F, F)$ (since $i \neq j$) and hence $a \wedge b$ is a left zero which is in P. But neither a nor b belongs to P which is a contradiction to the prime ideal of P.

Also, if each $I_i \neq \{U, F\}$, then $I_i = C$ for all i and hence $P \cong C_1 \times C_2 \times \ldots\ldots \times C_n$, which is again a contradiction, since P is a prime ideal. Thus there is exactly one $i, (1 \leq i \leq n)$ such that I_i = {U, F}. From this it follows that $P = P_{x_i}$ for some $x_i \in X$.

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