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BOUND FOR THE COMPLEX WAVE VELOCITY OF AN UNSTABLE PERTURBATION WAVE OF AN INVISCID HETEROGENEOUS PARALLEL SHEAR FLOWS

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ABSTRACT

The present paper concerns with the study of the upper bound for the complex wave velocity of an unstable perturbation wave of an inviscid heterogeneous parallel shear flows. We have obtained a necessary condition of instability of the flow which leads to the reduction of Howard's {1} semi circle for the bound of the complex wave velocity

Keywords: *Heterogeneous parallel shear flows, perturbation, linear stability, Howard's semi circle.*

MSC 2000 classification: *76E05, 76F45.*

INTRODUCTION:

The fundamental governing of instability of inviscid parallel shear flow confined within two rigid horizontal boundaries in the concept of linear stability theory is the Taylor Goldstein equation is given by

$$\left(D^2 - k^2\right)w - \frac{U'' w}{(U - c)} + \frac{g\beta w}{(U - c)^2} = 0, \ with \ \ c_i \neq o \tag{1}$$

with the boundary condition $w\left(z_1\right) = w\left(z_2\right) = 0$ (2)

Where D = $\frac{d}{dz}$ z is real independent variable such that $z_1 \le z \le z_2$, $w(z)$ is the z dependence of stream function perturbation and stand for dependent variable, $U(z)$ is basic velocity field, $c = c_r + ic_i$ is the complex wave velocity such that c_r and c_i are respectively the real and imaginary part of c which is constant , k^2 is the square wave number which is constant and satisfy the inequality $0 < k^2 < \infty$, $\beta(z) = -\frac{1}{\rho}\frac{d\rho}{dz}$ denotes the non-homogeneity field and is non negative everywhere in the flow domain and ρ denotes the density field.

The requirement of non trivial solution of equation (1) satisfying equation (2) posses a double eigen value problem for c_r and c_i for prescribed value of k^2 and the flow unstable if such solution exist for which the imaginary part c_i of c is greater than zero.

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Howard's {1} have proved that the phase velocity c_r and the amplification factor $c_i > 0$ must lie in the upper half of the $c_r c_i$ - plane bounded by the semi circle $\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 \le \left(\frac{b-a}{2}\right)^2$ (3)

where $U_{\min} = a$ and $U_{\max} = b$.

Theorem: A necessary condition for the existence of non-trivial non-singular solution (w, c) of the double eigen-value problem for c_r and c_i, for given $U(z)$, ρ and k^2 and described by equations (1) and (2) with $U''(U - U_s) \ge 0 \ \forall z \in [z_1, z_2]$ and $c_i > 0$ is that

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 \left[1 + \frac{U''(U - U_s)}{g\beta}\right]_{[z_1,z_2]}^{\min} \le \left(\frac{b-a}{2}\right)^2,$$

provided β vanishes at the point of inflexion $z = z_s \in [z_1, z_2]$ and $\frac{U''(U - U_s)}{g\beta}$ remains well defined $\forall z \in [z_1, z_2]$.

Proof: Multiplying equation (1) by complex conjugate of w i.e. w^* throughout and integrating the resulting equation over the range of z using boundary conditions (2), we have

$$\int_{z_1}^{z_2} w^* \left[\left(D^2 - k^2\right) w - \frac{U'' w}{(U - c)} + \frac{g\beta |w|}{(U - c)^2} \right] dz = 0 \tag{4}$$

$$\Rightarrow \int_{z_1}^{z_2} \left[|Dw|^2 + k^2 |w|^2 \right] dz + \int_{z_1}^{z_2} \frac{U'' |w|^2}{(U - c)} dz - \int_{z_1}^{z_2} \frac{g\beta |w|^2}{(U - c)^2} dz = 0$$

$$\Rightarrow \int_{z_1}^{z_2} \left[|Dw|^2 + k^2 |w|^2 \right] dz + \int_{z_1}^{z_2} \frac{U'' \left(U - c^*\right) |w|^2}{|U - c|^2} dz - \int_{z_1}^{z_2} \frac{g\beta \left(U - c^*\right)^2 |w|^2}{|U - c|^4} dz = 0 \tag{5}$$

Equating real and imaginary parts of equation (5), we have

$$\int_{z_1}^{z_2} \left[|Dw|^2 + k^2 |w|^2 \right] dz + \int_{z_1}^{z_2} \frac{U'' \left(U - c_r\right) |w|^2}{|U - c|^2} dz - \int_{z_1}^{z_2} \frac{g\beta \left[(U - c_r)^2 - c_i^2 \right] |w|^2}{|U - c|^4} dz = 0 \tag{6}$$

and
$$\int_{z_1}^{z_2} \frac{U'' |w|^2}{|U - c|^2} dz = 2 \int_{z_1}^{z_2} \frac{g\beta \left(U - c_r\right) |w|^2}{|U - c|^4} dz \tag{7}$$

From equation (6), we have

$$\int_{z_1}^{z_2} \left[|Dw|^2 + k^2 |w|^2 \right] dz + \int_{z_1}^{z_2} \frac{U'' \left(U - U_s\right) |w|^2}{|U - c|^2} dz - \int_{z_1}^{z_2} \frac{U'' \left(c_r - U_s\right) |w|^2}{|U - c|^2} dz$$
$$- \int_{z_1}^{z_2} \frac{g\beta \left[(U - c_r)^2 - c_i^2 \right] |w|^2}{|U - c|^4} dz = 0$$

$$\Rightarrow \int_{z_1}^{z_2}\left[|Dw|^2 + k^2|w|^2\right]dz + \int_{z_1}^{z_2}\frac{U''(U-U_s)|w|^2}{|U-c|^2}dz - 2\int_{z_1}^{z_2}\frac{(U-c_r)(c_r-U_s)g\beta|w|^2}{|U-c|^4}dz - \int_{z_1}^{z_2}\frac{g\beta\left[(U-c_r)^2-c_i^2\right]|w|^2}{|U-c|^4}dz = 0$$

$$\Rightarrow \int_{z_1}^{z_2}\left[|Dw|^2 + k^2|w|^2\right]dz + \int_{z_1}^{z_2}\frac{U''(U-U_s)|w|^2}{|U-c|^2}dz$$

$$-2\int_{z_1}^{z_2}\frac{g\beta|w|^2\{(U-U_s)-(c_r-U_s)\}(c_r-U_s)}{|U-c|^4}dz - \int_{z_1}^{z_2}\frac{g\beta|w|^2\left[\{(U-U_s)-(c_r-U_s)\}^2-c_i^2\right]}{|U-c|^4}dz = 0$$

$$\Rightarrow \int_{z_1}^{z_2}\left[|Dw|^2 + k^2|w|^2\right]dz + \int_{z_1}^{z_2}\frac{U''(U-U_s)|w|^2}{|U-c|^2}dz - 2\int_{z_1}^{z_2}\frac{g\beta|w|^2(U-U_s)(c_r-U_s)}{|U-c|^4}dz$$
$$+2\int_{z_1}^{z_2}\frac{g\beta|w|^2(c_r-U_s)^2}{|U-c|^4}dz - \int_{z_1}^{z_2}\frac{g\beta|w|^2\left[\{(U-U_s)^2+(c_r-U_s)^2-2(U-U_s)(c_r-U_s)\}-c_i^2\right]}{|U-c|^4}dz = 0$$

$$\Rightarrow \int_{z_1}^{z_2}\left[|Dw|^2 + k^2|w|^2\right]dz + \int_{z_1}^{z_2}\frac{U''(U-U_s)|w|^2}{|U-c|^2}dz - \int_{z_1}^{z_2}\frac{g\beta|w|^2}{|U-c|^4}\left\{(U-U_s)^2-(c_r-U_s)^2-c_i^2\right\}dz = 0$$

$$\Rightarrow \int_{z_1}^{z_2}\left[|Dw|^2 + k^2|w|^2\right]dz + \int_{z_1}^{z_2}\frac{U''(U-U_s)|w|^2}{|U-c|^2}dz - \int_{z_1}^{z_2}\frac{g\beta|w|^2}{|U-c|^4}\left\{\left(\frac{b-a}{2}\right)^2-\left(c_r-\frac{a+b}{2}\right)^2-c_i^2\right\}dz \le 0$$

where $U_{\min} = a$, $U_{\max} = b$ and $U_s = \dfrac{a+b}{2}$

$$\Rightarrow \int_{z_1}^{z_2}\left[|Dw|^2 + k^2|w|^2\right]dz + \int_{z_1}^{z_2}\frac{U''(U-U_s)|w|^2 c_i^2}{|U-c|^4}dz - \int_{z_1}^{z_2}\frac{g\beta|w|^2}{|U-c|^4}\left\{\left(\frac{b-a}{2}\right)^2-\left(c_r-\frac{a+b}{2}\right)^2-c_i^2\right\}dz \le 0$$

$$\Rightarrow \int_{z_1}^{z_2}\left[|Dw|^2 + k^2|w|^2\right]dz - \int_{z_1}^{z_2}\frac{g\beta|w|^2}{|U-c|^4}\left[\left(\frac{b-a}{2}\right)^2-\left(c_r-\frac{a+b}{2}\right)^2-c_i^2\left\{1+\frac{U''(U-U_s)}{g\beta}\right\}\right]dz \le 0$$

$\Rightarrow \exists$ a point $z = z_p \in (z_1, z_2)$ such that

$$\left(\frac{b-a}{2}\right)^2 - \left(c_r - \frac{a+b}{2}\right)^2 - c_i^2\left[1 + \frac{U''(U-U_s)}{g\beta}\right]\min_{[z_1,z_2]} \geq 0$$

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2\left[1 + \frac{U''(U-U_s)}{g\beta}\right]\min_{[z_1,z_2]} \leq \left(\frac{b-a}{2}\right)^2 \tag{8}$$

Hence prove the theorem.

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