International Journal of Mathematical Archive-2(6), June - 2011, Page: 910-918
Available online through www.ijma.info ISSN 2229 – 5046

SUFFICIENT CONDITIONS FOR A FUNCTION TO BE AN APPROXIMATE IDENTITY ON $\mathrm{L}^2(B_R(0))$

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(Received on: 12-05-11; Accepted on: 23-05-11)

ABSTRACT

In this paper, we investigate some properties of approximate identities and their Fourier coefficients with respect to some appropriate orthonormal bases of $\mathrm{L}^2(B_R(0))$. Sufficient conditions for a function in $\mathrm{L}^2(B_R(0)\times B_R(0))$ to be an approximate identity on $\mathrm{L}^2(B_R(0))$ are also proved.

Key Words: *Complete space, Jacobi Polynomials, orthonormal bases, Fourier coefficient, Approximate Identity, L^2-convergence.*

AMS(2000) Classification: *41A30, 41A35, 41A63, 44A35, 65N80, 86-08.*

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1. INTRODUCTION:

Since the interior of the Earth looks like a ball, therefore it becomes necessary to develop new methods and tools to approximate functions on the 3-dimensional ball. The study of the Earth's interior, such as the mass density, the speed of propagation of seismic P and S waves and other rheological quantities, is still a field of research. Similarly, if we look at the 3-dimensional image of a human brain, it has similarity to the 3-dimensional ball. Since biologists deal with the similarly simplified model of a living cell and they are very much satisfied by the results obtained from such a model ([10], [15]), therefore approximating tools on the 3-dimensional ball can also be used to study the human brain. A particular example is to study the electromagnetic potential of the Earth and also the electromagnetic potential in the human brain whether it is produced by the local environment of the brain or it is produced by an electrotherapy for the purpose of the treatment of a cancer patient.

In [2] M. Akram and V. Michel studied locally supported approximate identities on the unit ball, they further studied the regularization of the Helmholtz decomposition on 3d-ball in [3]. In this paper, we investigate some properties of approximate identities and their Fourier coefficients with respect to some appropriate orthonormal bases of $\mathrm{L}^2(B_R(0))$. Sufficient conditions for a function in $\mathrm{L}^2(B_R(0)\times B_R(0))$ to be an approximate identity on $\mathrm{L}^2(B_R(0))$ are also proved.

2. JACOBI POLYNOMIALS:

In this paper N denote the set of all positive integer, Where $\mathrm{N}_0 := \mathrm{N}\cup\{0\}$ and R represent the set of all real numbers. Here, we present definition and some properties of the Jacobi polynomials. For further details and proofs we refer to [11] and [17].

The functions $P_n^{(\alpha,\beta)}, n\in \mathrm{N}_0$, with $\alpha,\beta > -1$ fixed, are called Jacobi polynomials if they satisfy the following properties for all $n\in \mathrm{N}_0$:

(i) $P_n^{(\alpha,\beta)}$ is a polynomial of degree n, defined on $[-1,1]$.

(ii) $\int_0^1 (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)dx = 0$ for all $m\in \mathrm{N}_0 \setminus \{n\}$.

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(iii) $P_n^{(\alpha,\beta)}(1) = \dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} = \dfrac{(n+\alpha)!}{n!\alpha!}$, where Γ represents the Gamma function.

The Jacobi polynomials $P_n^{(\alpha,\beta)}, n \in \mathrm{N}_0$, with $\alpha, \beta > -1$ fixed satisfy the following Rodriguez's formula:

$$(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n \left\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\right\}.$$

Note that the Legendre polynomials P_n represent the special case of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ for each n with $\alpha = \beta = 0.$

The Jacobi polynomials $P_n^{(\alpha,\beta)}$ for $\alpha > -1, \beta > -1, x_0 = \dfrac{\beta-\alpha}{\alpha+\beta+1}$ have the following property

$$\text{(A)} \quad \max_{-1\le x\le +1}\left|P_n^{(\alpha,\beta)}(x)\right| = \begin{cases} \dbinom{n+q}{n} : \; n^q & if \quad q = \max(\alpha,\beta) \ge -\dfrac{1}{2}, \\ \left|P_n^{(\alpha,\beta)}(x')\right| : \; n^{\frac{-1}{2}} & if \quad q = \max(\alpha,\beta) < -\dfrac{1}{2}. \end{cases}$$

Here x' is one of the two maximum points nearest x_0.

The Jacobi polynomials satisfies the following recurrence relation. For any $\alpha, \beta > -1$ and for all $x \in [-1,1]$

$$P_0^{(\alpha,\beta)}(x) = 1,\; P_1^{(\alpha,\beta)}(x) = \frac{\alpha-\beta}{2} + \frac{1}{2}(\alpha+\beta+2)x,$$

and for $n \ge 2$,

$$\begin{aligned} &2n(\alpha+\beta+n)(\alpha+\beta+2n-2)P_n^{(\alpha,\beta)}(x) \\ &= \left[(\alpha+\beta+2n-2)(\alpha+\beta+2n-1)(\alpha+\beta+2n)x + (\alpha^2-\beta^2)\right]P_{n-1}^{(\alpha,\beta)}(x) \\ &-2(\alpha+n-1)(\beta+n-1)(\alpha+\beta+2n)P_{n-2}^{(\alpha,\beta)}(x). \end{aligned}$$

3. SPHERICAL HARMONICS:

Spherical harmonics are the functions most commonly used to represent scalar fields on a spherical surface. In this section we present definitions and some well-known facts from the theory of spherical harmonics. For further details we refer to [6, 9, 14] and the references therein.

Let $D \subset \mathrm{R}^3$ be open and connected. A function $F \in \mathrm{C}^{(2)}(D)$ is called harmonic if and only if

$$\nabla_x^2 F(x) := \sum_{i=1}^{3} \frac{\partial^2 F}{\partial x_i^2}(x) = 0, \;\; for\ all\ x = (x_1, x_2, x_3)^{\mathrm{T}} \in D.$$

A polynomial P on R^3 is called homogeneous of degree n if $P(\lambda x) = \lambda^n P(x)$ for all $\lambda \in \mathrm{R}$, and all $x \in \mathrm{R}^3$. The set of all homogeneous harmonic polynomials on R^3 with degree $n \in \mathrm{N}_0$ is denoted by $Harm_n(\mathrm{R}^3)$, i.e.

$$\mathrm{Harm}_n(\mathrm{R}^3) := \left\{P \in \mathrm{Hom}_n(\mathrm{R}^3) \mid \nabla^2 P = 0\right\}, \;\; n \in \mathrm{N}_0.$$

A spherical harmonic of degree n is the restriction of a homogeneous harmonic polynomial on R^3 with degree $n \in \mathrm{N}_0$ to the unit sphere Ω. The collection of all spherical harmonics of degree n will be denoted by $Harm_n(\Omega)$, i.e.

$$\mathrm{Harm}_n(\Omega) = \left\{F|_{\Omega} \mid F \in \mathrm{Harm}_n(\mathrm{R}^3)\right\}, \;\; n \in \mathrm{N}_0.$$

Theorem 3.1 *(See* [9], *p. 38) For every* $Y_n \in \mathrm{Harm}_n(\Omega)$,

$$\sup_{\xi\in\Omega} |Y_n(\xi)| \le \left(\frac{2n+1}{4\pi}\right)^{\frac{1}{2}} \|Y_n\|_{\mathrm{L}^2(\Omega)}.$$

In particular,

$$\sup_{\xi\in\Omega} |Y_{n,j}(\xi)| \le \left(\frac{2n+1}{4\pi}\right)^{\frac{1}{2}},\ j=1,...,2n+1.$$

Theorem 3.2 *If* $m \ne n$ *then Harm* $_m(\Omega)$ *is orthogonal to Harm* $_n(\Omega)$ *in the sense of* $\mathrm{L}^2(\Omega)$, *i.e. if* $m \ne n$, *then for all* $Y_m \in \mathrm{Harm}_m(\Omega)$ *and all* $Y_n \in \mathrm{Harm}_n(\Omega)$, $\langle Y_m, Y_n\rangle_{\mathrm{L}^2(\Omega)} = 0.$

Theorem 3.3 *The dimension of Harm* $_n(\Omega),\ n\in \mathrm{N}_0$ *is equal to* $2n+1$, *i.e.*

$$\dim\left(\mathrm{Harm}_n(\Omega)\right) = 2n+1,\ \ n\in \mathrm{N}_0.$$

By $\{Y_{n,j}\}_{n\in\mathrm{N}_0, j=-n,...n}$ we will always denote a complete $\mathrm{L}^2(\Omega)$-orthonormal system in $\mathrm{Harm}_{0,...,\infty}(\Omega)$, such that $Y_{n,j} \in \mathrm{Harm}_n(\Omega)$ for all $j=-n,...,n.$ We call n the degree of $Y_{n,j}$ and j the order of $Y_{n,j}$. The evaluation of sums with spherical harmonics can be essentially simplified by the following theorem.

Theorem 3.4 *(Addition Theorem for Spherical Harmonics) For all* $\xi,\eta \in \Omega$ *we have*

$$\sum_{j=-n}^{n} Y_{n,j}(\xi) Y_{n,j}(\eta) = \frac{2n+1}{4\pi} P_n(\xi\cdot\eta),$$

where P_n is the Legendre polynomial of degree n.

4. COMPLETE ORTHONORMAL SYSTEM IN HILBERT SPACE $\mathrm{L}^2(B_R(0))$:

The following systems of orthogonal polynomials on a three-dimensional ball $B_R(0) := \{x\in \mathrm{R}^3 : |x| \le R\}$ are known (see , for example [4], [7], [12], [18]).

Theorem: 4.1 *Two complete orthonormal systems in the Hilbert space* $\mathrm{L}^2(B_R(0))$ *are given by*

$$G^{\mathrm{I}}_{n,j,m}(x) := \sqrt{\frac{4m+2n+3}{R^3}}\ P_m^{(0,n+1/2)}\left(2\frac{|x|^2}{R^2}-1\right)\left(\frac{|x|^n}{R}\right) Y_{n,j}\left(\frac{x}{|x|}\right);$$

$n,m\in \mathrm{N}_0;\ j=1,...,2n+1.$

$$G^{\mathrm{II}}_{n,j,m}(x) := \sqrt{\frac{2m+3}{R^3}}\ P_m^{(0,2)}\left(2\frac{|x|}{R}-1\right) Y_{n,j}\left(\frac{x}{|x|}\right);$$

$x\in B_R(0)\setminus\{0\};\ n,m\in \mathrm{N}_0;\ j=1,...,2n+1.$ In both cases, $\{P_m^{(a,b)}\}_{m\in\mathrm{N}_0}$ represents the Jacobi polynomials (see Section 2 for further details) and $\{Y_{n,j}\}_{n\in\mathrm{N}_0; j=1,...,2n+1}$ stands for a spherical harmonics orthonormal basis (see, for instance, Section 3) in $\mathrm{L}^2(\Omega)$, where Ω is the unit sphere in R^3.

If simply $G_{n,j,m}$ is written, then system I as well as II could be chosen.

The kernel $\Phi_\delta \in L^2(B_R(0)\times B_R(0))$ can be expressed in the form

$$\Phi_\delta(x,y)=\sum_{n_1,m_1=0}^{\infty}\sum_{j_1=0}^{2n_1+1}\sum_{n_2,m_2=0}^{\infty}\sum_{j_2=0}^{2n_2+1}\Phi_\delta^\wedge(n_1,j_1,m_1,n_2,j_2,m_2)G_{n_1,j_1,m_1}(x)G_{n_2,j_2,m_2}(y)$$

for all $x,y\in B_R(0)$, where the series converges in the $L^2(B_R(0)\times B_R(0))$-sense, that is

$$\sum_{n_1,m_1=0}^{\infty}\sum_{j_1=0}^{2n_1+1}\sum_{n_2,m_2=0}^{\infty}\sum_{j_2=0}^{2n_2+1}\left(\Phi_\delta^\wedge(n_1,j_1,m_1,n_2,j_2,m_2)\right)^2<+\infty.$$

Here $\Phi_\delta^\wedge(n_1,j_1,m_1,n_2,j_2,m_2)$ are the Fourier coefficients for Φ_δ defined as

$$\begin{aligned}\Phi_\delta^\wedge(n_1,j_1,m_1,n_2,j_2,m_2) &:= \left\langle \Phi_\delta, G_{n_1,j_1,m_1}G_{n_2,j_2,m_2}\right\rangle_{L^2(B_R(0)\times B_R(0))}\\ &= \int_{B_R(0)}\int_{B_R(0)}\Phi_\delta(x,y)G_{n_1,j_1,m_1}(x)G_{n_2,j_2,m_2}(y)dxdy.\end{aligned}$$

5. PROPERTIES OF THE FOURIER COEFFICIENTS OF THE APPROXIMATE IDENTITIES ON $L^2(B_R(0))$:

Lemma: 5.1 *Let* $\Phi_\delta \in L^2(B_R(0)\times B_R(0))$ *be an approximate identity in* $L^2(B_R(0))$ *such that* $\int_{B_R(0)}\Phi_\delta(x,y)dy=1$ *for all* $x\in B_R(0)$ *and* $\Phi_\delta^\wedge(n_1,j_1,m_1,n_2,j_2,m_2)$ *be its Fourier coefficients as defined above, then*

$$\Phi_\delta^\wedge(0,1,0,0,1,0)=1.$$

Proof: Using the definition of $\Phi_\delta^\wedge(n_1,j_1,m_1,n_2,j_2,m_2)$ and the addition theorem, we get

$$\begin{aligned}\Phi_\delta^\wedge(0,1,0,0,1,0) &= \int_{B_R(0)}\int_{B_R(0)}\Phi_\delta(x,y)G_{0,1,0}(x)G_{0,1,0}(y)dxdy\\ &= \int_{B_R(0)}\int_{B_R(0)}\Phi_\delta(x,y)\sqrt{\frac{3}{R^3}}\,Y_{0,1}\left(\frac{x}{|x|}\right)\sqrt{\frac{3}{R^3}}\,Y_{0,1}\left(\frac{y}{|y|}\right)dxdy\\ &= \frac{3}{R^3}\int_{B_R(0)}\int_{B_R(0)}\Phi_\delta(x,y)\frac{1}{4\pi}P_0\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)dxdy.\end{aligned}$$

Applying Fubini's theorem (see [16]) and the fact that $P_0\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)=1$, we get

$$\begin{aligned}\Phi_\delta^\wedge(0,1,0,0,1,0) &= \frac{3}{4\pi R^3}\int_{B_R(0)}\int_{B_R(0)}\Phi_\delta(x,y)dydx\\ &= \frac{3}{4\pi R^3}\int_{B_R(0)}dx=1\end{aligned}$$

because $\int_{B_R(0)}\Phi_\delta(x,y)dy=1$.

Theorem: 5.2 *Let* $\Phi_\delta \in L^2(B_R(0)\times B_R(0))$ *be an approximate identity in* $L^2(B_R(0))$ *then*

$$\lim_{\delta\to 0+}\Phi_\delta^\wedge(n_1,j_1,m_1,n_2,j_2,m_2)=\delta_{n_1,n_2}\delta_{m_1,m_2}\delta_{j_1,j_2}.$$

Proof: From the information given above, we can write

$$\begin{aligned}
&\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2) - \delta_{n_1,n_2}\delta_{m_1,m_2}\delta_{j_1,j_2} \\
&= \int_{B_R(0)}\int_{B_R(0)} \Phi_\delta(x,y) G_{n_1,j_1,m_1}(x) G_{n_2,j_2,m_2}(y)\,dxdy - \int_{B_R(0)} G_{n_1,j_1,m_1}(x) G_{n_2,j_2,m_2}(x)\,dx \\
&= \int_{B_R(0)} \left(\left(\Phi_\delta \mathring{a}\, G_{n_2,j_2,m_2}\right)(x) - G_{n_2,j_2,m_2}(x)\right) G_{n_1,j_1,m_1}(x)\,dx.
\end{aligned}$$

Taking the absolute values of both hand sides, we have

$$\begin{aligned}
&\left|\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2) - \delta_{n_1,n_2}\delta_{m_1,m_2}\delta_{j_1,j_2}\right| \\
&= \left|\int_{B_R(0)} \left(\left(\Phi_\delta \mathring{a}\, G_{n_2,j_2,m_2}\right)(x) - G_{n_2,j_2,m_2}(x)\right) G_{n_1,j_1,m_1}(x)\,dx\right| \\
&\leq \int_{B_R(0)} \left|\left(\left(\Phi_\delta \mathring{a}\, G_{n_2,j_2,m_2}\right)(x) - G_{n_2,j_2,m_2}(x)\right) G_{n_1,j_1,m_1}(x)\right| dx.
\end{aligned}$$

Now using Hölder's inequality (see [5]) we get

$$\begin{aligned}
&\left|\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2) - \delta_{n_1,n_2}\delta_{m_1,m_2}\delta_{j_1,j_2}\right| \\
&\leq \left(\int_{B_R(0)} \left|\left(\Phi_\delta \mathring{a}\, G_{n_2,j_2,m_2}\right)(x) - G_{n_2,j_2,m_2}(x)\right|^2 dx\right)^{\frac{1}{2}} \left(\int_{B_R(0)} |G_{n_1,j_1,m_1}(x)|^2\, dx\right)^{\frac{1}{2}} \\
&= \mathrm{P}\Phi_\delta \mathring{a}\, G_{n_2,j_2,m_2} - G_{n_2,j_2,m_2}\, \mathrm{P}_{\mathrm{L}^2(B_R(0))} \mathrm{P} G_{n_1,j_1,m_1}\, \mathrm{P}_{\mathrm{L}^2(B_R(0))} \\
&= \mathrm{P}\Phi_\delta \mathring{a}\, G_{n_2,j_2,m_2} - G_{n_2,j_2,m_2}\, \mathrm{P}_{\mathrm{L}^2(B_R(0))}.
\end{aligned}$$

Taking the limit $\delta \to 0+$ and using the fact that Φ_δ is an approximate identity in $\mathrm{L}^2(B_R(0))$ we get

$$\begin{aligned}
&\lim_{\delta\to 0+} \left|\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2) - \delta_{n_1,n_2}\delta_{m_1,m_2}\delta_{j_1,j_2}\right| \\
&\leq \lim_{\delta\to 0+} \mathrm{P}\Phi_\delta \mathring{a}\, G_{n_2,j_2,m_2} - G_{n_2,j_2,m_2}\, \mathrm{P}_{\mathrm{L}^2(B_R(0))} \\
&= 0.
\end{aligned}$$

This proves the result.

Lemma: 5.3 *Let* $\Phi_\delta \in \mathrm{L}^2(B_R(0)\times B_R(0))$ *be given such that* $\int_{B_R(0)} |\Phi_\delta(x,y)|\,dy = 1$ *for all* $x \in B_R(0)$ *and* $\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2)$ *be its Fourier coefficients as defined above. If the system* $G_{n,j,m}^{\mathrm{I}}$ *is used in the definition of the Fourier coefficients then*

$$\left|\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2)\right| \leq \Pi_{n_1 m_1}\Pi_{n_2 m_2}, \tag{1}$$

with

$$\Pi_{n_1 m_1} = \frac{\sqrt{4m_1 + 2n_1 + 3}\binom{m_1 + n_1 + 1/2}{m_1}\sqrt{2n_1 + 1}}{\sqrt{3}},$$

and

$$\Pi_{n_2 m_2} = \frac{\sqrt{4m_2 + 2n_2 + 3}\binom{m_2 + n_2 + 1/2}{m_2}\sqrt{2n_2 + 1}}{\sqrt{3}},$$

where $n_1, n_2, m_1, m_2 \in \mathrm{N}_0$, $j_1 = 1, \ldots, 2n_1 + 1$ and $j_2 = 1, \ldots, 2n_2 + 1$.

If the system $G_{n,j,m}^{\mathrm{II}}$ is used in the definition of the Fourier coefficients then

$$\left|\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2)\right| \le \frac{(2m_1+3)^{\frac{5}{2}}(2m_2+3)^{\frac{5}{2}}(2n_1+1)^{\frac{1}{2}}(2n_2+1)^{\frac{1}{2}}}{12}, \tag{2}$$

where $n_1, n_2, m_1, m_2 \in \mathrm{N}_0$, $j_1 = 1,...,2n_1+1$ and $j_2 = 1,...,2n_2+1$.

Proof: Using the definition of $\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2)$, we get

$$|\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2)| \le \int_{B_R(0)}\int_{B_R(0)} |\Phi_\delta(x,y) G_{n_1,j_1,m_1}(x) G_{n_2,j_2,m_2}(y)|\, dxdy. \tag{3}$$

Taking system I of Theorem 4.1 in Equation (3) we have

$$|\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2)| \le \sqrt{\frac{4m_1+2n_1+3}{R^3}}\sqrt{\frac{4m_2+2n_2+3}{R^3}}\int_{B_R(0)}\int_{B_R(0)}\left|\Phi_\delta(x,y)\right|$$
$$\times\left(\left|P_{m_1}^{(0,n_1+1/2)}\left(2\frac{|x|^2}{R^2}-1\right)\right|\left(\frac{|x|}{R}\right)^{n_1}\left|Y_{n_1,j_1}\left(\frac{x}{|x|}\right)\right|\left|P_{m_2}^{(0,n_2+1/2)}\left(2\frac{|y|^2}{R^2}-1\right)\right|\left(\frac{|y|}{R}\right)^{n_2}\right.$$
$$\left.\left|Y_{n_2,j_2}\left(\frac{y}{|y|}\right)\right|\right)dxdy.$$

By using equation (A) and Theorem 3.1 we get

$$|\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2)| \le \frac{1}{R^3}\sqrt{4m_1+2n_1+3}\sqrt{4m_2+2n_2+3}\int_{B_R(0)}\int_{B_R(0)}\left|\Phi_\delta(x,y)\right|$$
$$\times\binom{m_1+n_1+1/2}{m_1}\left(\frac{2n_1+1}{4\pi}\right)^{\frac{1}{2}}\binom{m_2+n_2+1/2}{m_2}\left(\frac{2n_2+1}{4\pi}\right)^{\frac{1}{2}}dxdy.$$

Applying Tonelli's theorem (see [16],) we get

$$|\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2)| \le \sqrt{4m_1+2n_1+3}\binom{m_1+n_1+1/2}{m_1}\sqrt{\frac{2n_1+1}{4\pi}}$$
$$\times\sqrt{4m_2+2n_2+3}\binom{m_2+n_2+1/2}{m_2}\sqrt{\frac{2n_2+1}{4\pi}}\frac{1}{R^3}\int_{B_R(0)}\int_{B_R(0)}\left|\Phi_\delta(x,y)\right|dydx.$$

After integrating we obtain

$$|\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2)| \le \Pi_{n_1 m_1}\Pi_{n_2 m_2}.$$

Taking system II of Theorem 4.1 in Equation ((3)) we have

$$|\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2)| \le \frac{1}{R^3}\sqrt{2m_1+3}\sqrt{2m_2+3}\times$$
$$\int_{B_R(0)}\int_{B_R(0)}\left|\Phi_\delta(x,y)\right|\left|P_{m_1}^{(0,2)}\left(2\frac{|x|}{R}-1\right)\right|\left|Y_{n_1,j_1}\left(\frac{x}{|x|}\right)\right|\left|P_{m_2}^{(0,2)}\left(2\frac{|y|}{R}-1\right)\right|\left|Y_{n_2,j_2}\left(\frac{y}{|y|}\right)\right|dxdy.$$

By using equation (A) and Theorem 3.1, we get

$$|\Phi_\delta^\wedge(n_1,j_1,m_1,n_2,j_2,m_2)| \le \sqrt{2m_1+3}\sqrt{2m_2+3}$$

$$\times \frac{1}{R^3}\int_{B_R(0)}\int_{B_R(0)} \left|\Phi_\delta(x,y)\right| \binom{m_1+2}{m_1}\left(\frac{2n_1+1}{4\pi}\right)^{\frac{1}{2}}\binom{m_2+2}{m_2}\left(\frac{2n_2+1}{4\pi}\right)^{\frac{1}{2}} dydx$$

$$\le \sqrt{2m_1+3}\sqrt{2m_2+3}\,\frac{(m_1+2)(m_1+1)}{2}\,\frac{(m_2+2)(m_2+1)}{2}\,\frac{\sqrt{2n_1+1}\sqrt{2n_2+1}}{4\pi}$$

$$\times \frac{1}{R^3}\int_{B_R(0)}\int_{B_R(0)} |\Phi_\delta(x,y)|\,dydx \le \frac{(2m_1+3)^{\frac{5}{2}}(2m_2+3)^{\frac{5}{2}}(2n_1+1)^{\frac{1}{2}}(2n_2+1)^{\frac{1}{2}}}{16\pi R^3}\int_{B_R(0)} dx.$$

Finally, we obtain

$$|\Phi_\delta^\wedge(n_1,j_1,m_1,n_2,j_2,m_2)| \le \frac{(2m_1+3)^{\frac{5}{2}}(2m_2+3)^{\frac{5}{2}}(2n_1+1)^{\frac{1}{2}}(2n_2+1)^{\frac{1}{2}}}{12}.$$

Lemma: 5.4 *Let* $\Phi_\delta \in \mathrm{L}^2(B_R(0)\times B_R(0))$ *and let there exist an integrable function* $\tilde{\Phi}_\delta$ *defined on* $B_{2R}(0)$ *such that for each* $F \in \mathrm{L}^2(B_R(0))$, $\Phi_\delta \,\text{å}\, F = \tilde{\Phi}_\delta * F$ *and* $\int_{B_{2R}(0)} |\tilde{\Phi}_\delta(y)|\,dy = 1$, *then*

$$|\Phi_\delta^\wedge(n_1,j_1,m_1,n_2,j_2,m_2)| \le 1.$$

Proof: As we know,

$$\Phi_\delta^\wedge(n_1,j_1,m_1,n_2,j_2,m_2) = \int_{B_R(0)}\int_{B_R(0)} \Phi_\delta(x,y)G_{n_1,j_1,m_1}(x)G_{n_2,j_2,m_2}(y)dxdy.$$

Applying Fubini's theorem, we get

$$\begin{aligned}\Phi_\delta^\wedge(n_1,j_1,m_1,n_2,j_2,m_2) &= \int_{B_R(0)}\int_{B_R(0)} \Phi_\delta(x,y)G_{n_2,j_2,m_2}(y)dy\; G_{n_1,j_1,m_1}(x)dx \\ &= \int_{B_R(0)} \left(\Phi_\delta \,\text{å}\, G_{n_2,j_2,m_2}\right)(x)G_{n_1,j_1,m_1}(x)dx.\end{aligned}$$

This implies

$$|\Phi_\delta^\wedge(n_1,j_1,m_1,n_2,j_2,m_2)| \le \int_{B_R(0)} \left|\left(\Phi_\delta \,\text{å}\, G_{n_2,j_2,m_2}\right)(x)G_{n_1,j_1,m_1}(x)\right| dx$$

Using Hölder's inequality, we have

$$\begin{aligned}&\left|\Phi_\delta^\wedge(n_1,j_1,m_1,n_2,j_2,m_2)\right| \\ &\le \left(\int_{B_R(0)} \left|\left(\Phi_\delta \,\text{å}\, G_{n_2,j_2,m_2}\right)(x)\right|^2 dx\right)^{\frac{1}{2}} \left(\int_{B_R(0)} \left|G_{n_1,j_1,m_1}(x)\right|^2 dx\right)^{\frac{1}{2}} \\ &\le \left(\int_{B_R(0)} \left|\left(\tilde{\Phi}_\delta * G_{n_2,j_2,m_2}\right)(x)\right|^2 dx\right)^{\frac{1}{2}} \left(\int_{B_R(0)} \left|G_{n_1,j_1,m_1}(x)\right|^2 dx\right)^{\frac{1}{2}} \\ &= \|\tilde{\Phi}_\delta * G_{n_2,j_2,m_2}\|_{\mathrm{L}^2(B_R(0))} \|G_{n_1,j_1,m_1}\|_{\mathrm{L}^2(B_R(0))}.\end{aligned}$$

After applying Young's inequality, we obtain

$$\left|\Phi_\delta^\wedge(n_1,j_1,m_1,n_2,j_2,m_2)\right| \le \|\tilde{\Phi}_\delta\|_{\mathrm{L}^1(B_{2R}(0))}\|G_{n_2,j_2,m_2}\|_{\mathrm{L}^2(B_R(0))}\|G_{n_1,j_1,m_1}\|_{\mathrm{L}^2(B_R(0))}.$$

Since G_{n_1,j_1,m_1} and G_{n_2,j_2,m_2} belongs to an orthonormal system in $\mathrm{L}^2(B_R(0))$, therefore their L^2-norms are equal to 1. Also by hypothesis, $\|\tilde{\Phi}_\delta\|_{\mathrm{L}^1(B_{2R}(0))} = 1$. In view of the argument given above, we have

$$|\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2)| \le 1.$$

Next theorem gives the sufficient conditions for a function in $L^2(B_R(0)\times B_R(0))$ to be an approximate identity in $L^2(B_R(0))$.

Theorem: 5.5 *Let* $\Phi_\delta \in L^2(B_R(0)\times B_R(0))$ *and the Fourier coefficients* $\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2)$ of Φ_δ *have the properties*

$$(i)\, |\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2)| \le 1,$$

$$(ii)\, \lim_{\delta\to 0}\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2) = \delta_{n_1,n_2}\delta_{m_1,m_2}\delta_{j_1,j_2},$$

$$(iii)\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2) = 0 \, for\ n_1 \ne n_2, m_1 \ne m_2, j_1 \ne j_2$$

then, Φ_δ is an approximate identity in $L^2(B_R(0))$.

Proof: For any F in $L^2(B_R(0))$ we have

$$\left(\Phi_\delta \text{ å } F\right)(x) = \left\langle \Phi_\delta(x,.), F\right\rangle_{L^2(B_R(0))}$$

$$= \left\langle \sum_{n_1,m_1=0}^{\infty}\sum_{j_1=1}^{2n_1+1}\sum_{n_2,m_2=0}^{\infty}\sum_{j_2=1}^{2n_2+1}\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2)G_{n_1,j_1,m_1}(x)G_{n_2,j_2,m_2}, F\right\rangle_{L^2(B_R(0))}$$

$$= \sum_{n_1,m_1=0}^{\infty}\sum_{j_1=1}^{2n_1+1}\sum_{n_2,m_2=0}^{\infty}\sum_{j_2=1}^{2n_2+1}\Phi_\delta^\wedge(n_1, j_1, m_1, n_2, j_2, m_2)G_{n_1,j_1,m_1}(x)\left\langle G_{n_2,j_2,m_2}, F\right\rangle_{L^2(B_R(0))}$$

$$= \sum_{n_1,m_1=0}^{\infty}\sum_{j_1=1}^{2n_1+1}\Phi_\delta^\wedge(n_1, j_1, m_1)G_{n_1,j_1,m_1}(x)\left\langle G_{n_1,j_1,m_1}, F\right\rangle_{L^2(B_R(0))},$$

where $\Phi_\delta^\wedge(n_1, j_1, m_1) := \Phi_\delta^\wedge(n_1, j_1, m_1, n_1, j_1, m_1)$.

Let us consider

$$\left(\Phi_\delta \text{ å } F\right) - F \;=\; \sum_{n_1,m_1=0}^{\infty}\sum_{j_1=1}^{2n_1+1}\Phi_\delta^\wedge(n_1, j_1, m_1)G_{n_1,j_1,m_1}\left\langle G_{n_1,j_1,m_1}, F\right\rangle_{L^2(B_R(0))}$$

$$-\sum_{n_1,m_1=0}^{\infty}\sum_{j_1=1}^{2n_1+1}G_{n_1,j_1,m_1}\left\langle F, G_{n_1,j_1,m_1}\right\rangle_{L^2(B_R(0))}$$

$$= \sum_{n_1,m_1=0}^{\infty}\sum_{j_1=1}^{2n_1+1}G_{n_1,j_1,m_1}\left(\Phi_\delta^\wedge(n_1, j_1, m_1) - 1\right)\left\langle G_{n_1,j_1,m_1}, F\right\rangle_{L^2(B_R(0))}.$$

By Parseval's relation (see [19]), we have

$$\text{P}\Phi_\delta \text{ å } F - F\,\text{P}^2_{L^2(B_R(0))} = \sum_{n_1,m_1=0}^{\infty}\sum_{j_1=1}^{2n+1}\left(\Phi_\delta^\wedge(n_1, j_1, m_1) - 1\right)^2\left\langle G_{n_2,j_2,m_2}, F\right\rangle^2_{L^2(B_R(0))}.$$

Taking the limit $\delta \to 0+$ on both hand sides, we get

$$\lim_{\delta\to 0+}\text{P}\Phi_\delta \text{ å } F - F\,\text{P}^2_{L^2(B_R(0))} = \lim_{\delta\to 0+}\sum_{n_1,m_1=0}^{\infty}\sum_{j_1=1}^{2n_1+1}\left(\Phi_\delta^\wedge(n_1, j_1, m_1) - 1\right)^2\left\langle G_{n_2,j_2,m_2}, F\right\rangle^2_{L^2(B_R(0))}.$$

As it is given in (i) the bounds of $\Phi_\delta^\wedge(n_1, j_1, m_1)$ are independent of δ, therefore we can take the limit inside the sum. After simplifying, we obtain

$$\lim_{\delta\to 0+} \|\Phi_\delta * F - F\|^2_{L^2(B_R(0))} = \sum_{n_1,m_1=0}^{\infty} \sum_{j_1=1}^{2n_1+1} \lim_{\delta\to 0+} \left(\Phi_\delta^\wedge(n_1,j_1,m_1)-1\right)^2 \left\langle G_{n_2,j_2,m_2}, F\right\rangle^2_{L^2(B_R(0))} = 0.$$

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