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$(n_{1b_1}, n_{2b_2}, n_{3b_3})$ - OPTIMAL BURST ERROR CORRECTING CODES OVER GF (2)

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ABSTRACT

This paper explores the possibilities of the existence of block-wise burst error correcting (BBEC) optimal codes where the burst is considered in the sense of Chein and Tang and the code length n is partitioned into sub-blocks. An $(n_{1b_1}, n_{2b_2}, n_{3b_3})$ optimal code is defined as a block-wise burst error correcting linear code that can correct all bursts of length b_1 (fixed), in the first n_1 components, all burst of length b_2 (fixed) in the next n_2 components and all burst of length b_3 (fixed) in the last n_3 components ($n = n_1 + n_2 + n_3$) and no other error pattern. When burst length is same i.e. $b_1 = b_2 = b_3$, the codes are treated as byte correcting codes.

Key words: *Burst of length b (fixed), optimal codes, byte correcting codes, block-wise, error correcting codes.*

INTRODUCTION:

In most memory and storage system, the information is stored in various parts of the code length, known as sub-blocks or bytes (sub-blocks are treated as bytes when burst length is same). So, when error occurs in such a system, it does in a few places of the same sub-block and the pattern of errors is known. Thus, when we consider error correction, we correct errors which occur within the same sub-block. Hence there is a need to study block-wise error correcting (BEC) codes.

Burst is the most common error in many communication systems. In sub-block oriented codes, burst can occur only within the same sub-block. The treatment for correcting such burst errors sometimes result into new type of perfect codes or perfect type codes.Though, the problem of the existence of perfect codes was settled by Titavanien and Perko [1] by showing that there is no perfect codes other than the single error correcting Hamming [8], double and triple error correcting Golay [6] and the repetitive codes, there have been attempts to find out codes which are not perfect in the usual sense but are of the type that for a given set of error patterns, these codes correct all such errors and no other (Perfect type codes). Most of the studies, under this class have been carried out for constructing BBEC perfect codes w. r. t. usual definition of burst according to which,

"*A burst of length b, in an (n, k) linear code, is a vector whose all the non zero components are confined to some b-consecutive positions, the first and the last of which are non zero*."

With respect to this definition of burst, some byte oriented codes have been studied by Tuvi Etzion [9]. He has shown with the help of an example, a perfect (9, 5) code which correct a single burst of length 2 or less with in bytes of size 3. The parity check matrix of this code is given as

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In general, the number of different burst of length b or less with in a byte of length β is bβ-1. The total number of non-zero vectors of length r (= n-k) is q^r-1. Thus the total number of bytes in such a code with redundancy r can be

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Calculated as $\dfrac{q^r - 1}{b\beta - 1}$.

In many other such communication systems, it has been observed that errors do occur but not near the end of the code words. This unique situation has brought the definition of burst due to Chein and Tang [7] with a modification due to Dass [2] into active consideration. According to this definition, '*A burst of length b is a vector whose all the non zero components are confined to some b-consecutive positions, the first of which is non zero and its starting positions in a vector of length n is the first n-b+1 positions'*. Accordingly, (100000) is considered as a burst of length at most 6. This definition has been found very useful in error analysis experiments on telephone lines (Alexander et al.[5]), in some space channel models in which an amplitude modulated carrier is generated aboard a satellite and transmitted to an earth antenna, and in systems where error do not occur near the end of the code words with burst length very large. Using this definition of burst, Dass and Tyagi [3] have studied BBEC-linear codes for two sub-blocks of length n_1 and n_2; ($n_1+n_2 = n$) and obtained necessary and sufficient bounds over the number of parity check digits. Dass and Tyagi [4] also proved that for a fixed burst length $b_1=1$ (to be corrected with in first n_1-digits) and $b_2=2$ (fixed) (to be corrected in next n_2 digits) ($n=n_1+n_2$), the BBEC – linear codes turn out to be optimal and have shown that for $n_1+n_2 \leq 50$, optimal codes exist for all possible values of the parameters. Such codes are named as (1, 2) optimal codes [4]. For example, the parity check matrix for a (9, 5) = (1+8, 5) code with n_1=1, n_2=8, b_1=1, b_2=2 may be given as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

It can be verified that the code under discussion is (1, 2) optimal. In many other such communication systems, it has been observed that errors do occur but not near the end of the code words. This unique situation has brought the definition of burst due to Chein and Tang [7] with a modification due to Dass [2] into active consideration. According to this definition, '*A burst of length b is a vector whose all the non zero components are confined to some b-consecutive positions, the first of which is non zero and its starting positions in a vector of length n is the first n-b+1 positions'*. Accordingly, (100000) is considered as a burst of length at most 6. This definition has been found very useful in error analysis experiments on telephone lines (Alexander et al.[5]), in some space channel models in which an amplitude modulated carrier is generated aboard a satellite and transmitted to an earth antenna, and in systems where error do not occur near the end of the code words with burst length very large. Using this definition of burst, Dass and Tyagi [3] have studied BBEC-linear codes for two sub-blocks of length n_1 and n_2; ($n_1+n_2 = n$) and obtained necessary and sufficient bounds over the number of parity check digits. Dass and Tyagi [4] also proved that for a fixed burst length $b_1=1$ (to be corrected with in first n_1-digits) and $b_2=2$ (fixed) (to be corrected in next n_2 digits) ($n=n_1+n_2$), the BBEC – linear codes turn out to be optimal and have shown that for $n_1+n_2 \leq 50$, optimal codes exist for all possible values of the parameters. Such codes are named as (1, 2) optimal codes [4]. . The utility of these codes can be enhanced if the number of sub-blocks can be increased to three or more in a code of length n and the stored information in each sub-block can be transmitted without error as shown by Tuvi[9] in case of three sub-blocks w.r.t. usual definition of burst. Tyagi and Sethi [11] have studied lower and upper bounds for codes that can correct different bursts of length b_1 (fixed), b_2 (fixed) and b_3 (fixed) in three different sub-blocks of length n_1, n_2 and n_3; ($n = n_1+n_2+n_3$) and have shown the existence (n_{1b_1}, n_{2b_2}, n_{3b_3}) linear codes.

In this correspondence, we construct (n_{1b_1}, n_{2b_2}, n_{3b_3}) optimal burst correcting codes in different cases based on three different sizes of sub-blocks and three different lengths of the burst. The five types of byte oriented codes given by Tuvi [10] viz.

(1) Bytes having the same size;
(2) One byte is of size b_1 and the other bytes are of size b_2;
(3) Each byte is of either size b_1 or size b_2;
(4) The size of each byte is a power of 2;
(5) All the other cases are also included.

The necessary bound proved by Tyagi and Sethi [11] is as follows:

Theorem: The number of parity check digits required for an (n_{1b_1}, n_{2b_2}, n_{3b_3}) linear (n, k) code is at least

$$\log_q \left[1+\sum_{i=1}^{3}(n_i - b_i + 1)\, q^{b_i - 1}\, (q-1)\right]. \tag{1}$$

Whenever one obtains a bound, it is desirable to examine as to for which values of the parameters, the bound is tight. We will show that for different values of b_i, n_i, i = 1, 2, 3 and k, one can obtain (n_{1b_1}, n_{2b_2}, n_{3b_3}) optimal codes some of which are also byte oriented.

OPTIMAL CODES:

Consider the inequality in (1) as equality for q=2, i.e.

$$2^{n_1+n_2+n_3-k} = 1 + (n_1-b_1+1)2^{b_1-1} + (n_2 - b_2 + 1)\, 2^{b_2-1} + (n_3 - b_3 + 1)\, 2^{b_3-1}. \tag{2}$$

Let us now examine, in cases 1 to 8, the possibilities of the existence of (n_{1b_1}, n_{2b_2}, n_{3b_3}) optimal codes for different values of n_1, n_2, n_3, b_1, b_2, b_3 and k satisfying (2).

Case: 1 When sub-blocks are of equal size and burst length is also same, i.e. $n_1 = n_2 = n_3 = N$ and $b_1 = b_2 = b_3 = b$. Then, from equation (2) we have

$$2^{3N-k} = 1+3(N-b+1)2^{b-1}. \tag{3}$$

For this, we note that $3(N-b+1)2^{b-1}$ should always be odd. This is possible only when N = 5, 21, 85, 341 ...and b=1. We show the existence of such codes in the following example.

Example 1.1: For b=1, N=5, the value of k=11. This may give rise to (15, 11) optimal code. If we consider the matrix

$$H = \begin{pmatrix} 1&0&0&0&1 & 1&0&0&1&0 & 1&1&0&1&1 \\ 0&1&0&0&0 & 1&1&0&0&1 & 1&1&1&0&1 \\ 0&0&0&1&0 & 0&1&1&1&0 & 1&1&1&1&0 \\ 0&0&1&0&1 & 0&0&1&0&1 & 1&0&1&1&1 \end{pmatrix}$$

as the parity check matrix for the desired code, then it can be verified that the code is (5_1, 5_1, 5_1) BBEC-optimal code which is, in fact, (15,11) Hamming code and is also a byte correcting code.

Case: 2 When sub-blocks are of the same size as in previous case and burst length is same only in two sub-blocks i.e. $n_1 = n_2 = n_3 = N$ and $b_1 = b_2 = b$, and $b_3 = b'$. Then, we have from equation (2),

$$2^{3N-k} = 1+2^{b}\,(N-b+1) + (N-b'+1)2^{b'-1} \tag{4}$$

For this, we note that $(N-b'+1)2^{b'-1}$ should always be odd. This is possible only when N = 7, b' = 1 and b = 2. We show the existence of such codes in the following example 2.1

Example 2.1: For N=7, b=2, b'=1, the value of k turns out to be 16. This may give rise to (21, 16) optimal code. If we consider the matrix

$$H = \begin{pmatrix} 1&0&0&0&0&1&0 & 1&0&1&0&1&1&0 & 1&0&0&0&0&1&0 \\ 0&1&0&0&0&0&1 & 0&1&1&1&0&1&0 & 0&1&0&1&0&1&1 \\ 0&0&1&0&0&0&0 & 1&0&1&0&0&1&0 & 1&0&1&1&1&0&1 \\ 0&0&0&1&0&1&0 & 1&0&1&1&0&0&0 & 0&1&0&1&1&0&1 \\ 0&0&0&0&1&0&1 & 0&1&0&1&1&0&1 & 0&0&1&0&1&1&1 \end{pmatrix}$$

as the parity check matrix, then it can be verified that the code under discussion is (7_2, 7_2, 7_1) BBEC-optimal code.

Case: 3. When the length of the burst errors is same in every sub-block and all the sub-blocks are of different size, i.e. $b_1 = b_2 = b_3 = b$; $n_1 \neq n_2 \neq n_3$, we have from equation (2)

$2^{n-k} = 1+ (n-3b+3)2^{b-1}$. (5)

Here also, we note that 2^{b-1} (n-3b+3) should always be odd so that R.H.S. in (5) becomes power of 2. This is possible only when n ($=n_1+n_2+n_3$) takes the values 15, 31, 63… and b=1. We show the existence of such codes by constructing following example 3.1.

Example 3.1: For $n_1 = 4$, $n_2 = 5$, $n_3 = 6$, b = 1 we have k=11. This may give rise to (15, 11) BBEC – optimal code. If we consider the matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & & 1 & 0 & 0 & 1 & 0 & & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & & 1 & 1 & 0 & 0 & 1 & & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & & 0 & 1 & 1 & 1 & 0 & & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 & 1 & 0 & 1 & & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

as the parity check matrix, then it can be verified that the code is (4_1, 5_1, 6_1) BBEC optimal code or byte correcting code.

Case: 4 When both, the burst length as well as the sub-block size are different i.e. $b_1 \neq b_2 \neq b_3$; $n_1 \neq n_2 \neq n_3$. We have only equation (2) to deal with. This is possible only when at least one of the n_1; i=1, 2, 3 is odd and one of the b_i; i = 1, 2, 3 is 1 along with other values of the parameters as (n_1, n_2, n_3) = (1, 2, 3**;** 2, 4, 5**;** 3, 5, 11**;** 5, 6, 15**;** .....) with corresponding values of (b_1, b_2, b_3) = (1, 2, 3**;** 2, 4, 1**;** 3, 5, 1; 5, 6, 1**;** ......). We give in example 4.1, the parity check matrix for (2_2, 4_4, 5_1) code with k=7 to justify our results.

Example 4.1: Taking $n_1 = 2$, $n_2 =4$, $n_3 = 5$, $b_1 = 2$, $b_2 = 4$, $b_3 =1$ in (2), we have k=7. This may give rise to (11, 7) optimal code. If we consider the matrix

$$H = \begin{pmatrix} 0 & 0 & & 0 & 0 & 1 & 0 & & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & & 0 & 1 & 0 & 0 & & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & & 1 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & & 0 & 0 & 0 & 1 & & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

as the parity check matrix for the code under discussion, then it can be verified that it is (2_2, 4_4, 5_1) optimal.

Case: 5 When the length of the burst is different in each sub-block and two sub-blocks are of the same size i.e. $b_1 \neq b_2 \neq b_2$; $N = n_1 = n_2$ and $n_3 = n'$. Then we have from equation (2).

$2^{n-k} = 1+ (N-b_1+1)2^{b_1-1} + (N-b_2+1)2^{b_2-1} + (n'-b_3+1)2^{b_3-1}$ (6)

This is possible only when at least one of N and n' is odd and one of the b_i is 1 along with the other values of the parameters as (N, n') = (3.4**;** 3,7**;** 4,15**;** 6,7**;** .....) with corresponding values of (b_1, b_2, b_3) = (1,2,3**;** 3,2,1**;** 3,4,1**;** 6,4,1**;** .....). We show the existence of such codes by constructing following example 5.1.

Example 5.1: Taking N=3, n'=4 and b_1=1, b_2=2, b_3=3 in (5), we have k=6. This may give rise to (10, 6) BBEC – optimal code. If we consider the matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & & 0 & 0 & 1 & & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & & 0 & 0 & 0 & & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & & 1 & 0 & 0 & & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & & 0 & 1 & 0 & & 0 & 0 & 0 & 1 \end{pmatrix}$$

as the parity check matrix, then it can be verified that the code is (3_1, 3_2, 4_3) optimal.

Case: 6 When the burst length is same in two sub-blocks and different in the third and all the sub-blocks are of different size, i.e. $b_1 = b_2 = b$ and $b_3 = b'$; $n_1 \neq n_2 \neq n_3$. Then, we have from equation (2)

$2^{n-k} = 1+ (n_1-b+1)2^{b-1} + (n_2-b+1)2^{b-1} + (n_3-b'+1)2^{b'-1}$ (7)

This is possible only when at least one of n_1, n_2 and n_3 is odd and b' is 1 along with other values of the parameters as (n_1, n_2, n_3) = (3, 4, 5**;** 2, 3, 9**;** 4, 5, 11; 5, 6, 15**;** ....) with corresponding values of (b, b') = (2,1; 2,1; 3,1; 5,1**;** .....). Following example 6.1 shows the existence of such codes.

Example 6.1: Taking n_1 = 3, n_2 = 4, n_3 = 5, b = 2, b'=1 in (7), we have k=8. This may give rise to the (12, 8) BBEC optimal code. If we consider the matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & & 1 & 0 & 0 & 1 & & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & & 0 & 0 & 0 & 1 & & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & & 0 & 1 & 0 & 1 & & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & & 1 & 0 & 1 & 1 & & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

as the parity check matrix, then it can be verified that code is (3_2, 4_2, 5_1) optimal.

Case: 7 When length of two bursts and size of their respective sub-blocks is same i.e. b_1 = b_2 = b and b_3 = b', n_1 = n_2 = N and n_3 = n' etc. Then we have from equation (2).

$$2^{n-k} = 1+ (N-b+1)2^{b} + 2^{b'-1} (n'-b'+1). \tag{8}$$

This is possible only when at least one of N and n' is odd and b' is 1 along with other values of the parameters as (N, n') = (2,3**;** 4,3**;** 3,7**;** 5,7**;** .....) and corresponding values of (b, b') as (2,1**;** 2,1**;** 3,1**;** .....). We show the existence of such codes by constructing the following example 7.1.

Example 7.1: For N=-4, n=3, b=2, b'=1, we have from equation (8), k=7. This may give rise to the (11, 7) optimal code. For this, if we consider the matrix

$$H = \begin{pmatrix} 0 & 0 & 1 & & 1 & 0 & 1 & 0 & & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & & 0 & 0 & 1 & 0 & & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & & 1 & 0 & 1 & 1 & & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & & 0 & 1 & 0 & 1 & & 0 & 0 & 0 & 1 \end{pmatrix}$$

as the parity check matrix, then it can be verified that code is (3_1, 4_2, 4_2) optimal.

Case: 8 When the length of bursts are same and the size of sub-blocks are equal only in two sub-blocks i.e. b_1 = b_2 = b_3 = b; n_1 = n_2 = N and n_3 = n'. We have from equation (2).

$$2^{n-k} = 1+ (N-b+1)\, 2^{b} + (n'-b+1)2^{b-1} . \tag{9}$$

For this, we note that 2^{b-1} (n'-b+1) should always be odd so that R.H.S. in (9) becomes power of 2. This is possible only when (N, n') = (2, 3**;** 4, 7**;** 6, 3**;** 12, 7**;** .....) and b=1. We show the existence of such codes by constructing following example 8.1.

Example 8.1: For n_1=2, n_2=2, n_3=3, b=1, we have from equation (9), k=4. This may give rise to the (7, 4) BBEC optimal code. If we consider the matrix

$$H = \begin{pmatrix} 1 & 0 & & 0 & 1 & & 0 & 1 & 1 \\ 0 & 1 & & 0 & 1 & & 1 & 1 & 0 \\ 0 & 0 & & 1 & 0 & & 1 & 1 & 1 \end{pmatrix},$$

as the parity check matrix then it can be verified that the code is (2_1, 2_1, 3_1) – BBEC optimal which is , infact , (7,4) Hamming code and also byte-correcting code.

OPEN PROBLEMS AND REMARKS:

We have shown, in this paper, the existence of (n_{1b_1}, n_{2b_2}, n_{3b_3})-optimal codes, with the help of examples, some of which turn out to be byte correcting. Especially in case 1, 3 and 8, the codes turns out to be byte correcting.

However, the problem needs further investigation to

(1) find byte oriented ($n_{1b_1}, n_{2b_2}, n_{3b_3}$) optimal codes;
(2) find the possibilities of the existence of (n_{1b_1}' n_{2b_2}, n_{3b_3}) optimal codes in non- binary cases, and
(3) to find the possibilities of the existence of optimal codes for *more than three sub blocks.*

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