



DEVELOPMENT OF SOME SUMMATION FORMULAE BASED ON HALF ARGUMENT USING GAUSS SECOND SUMMATION THEOREM

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ABSTRACT

The main object of this paper is to develop some summation formulae based on half argument by the help of Gauss second summation theorem. The results derived in this paper are of general character.

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A. INTRODUCTION:

Generalized Gaussian Hypergeometric function of one variable:

$${}_A F_B(a_1, a_2, \dots, a_A; b_1, b_2, \dots, b_B; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_A)_k z^k}{(b_1)_k (b_2)_k \dots (b_B)_k k!} \quad (1)$$

or

$${}_A F_B((a_A); (b_B); z) \equiv {}_A F_B((a_j)_{j=1}^A; (b_j)_{j=1}^B; z) = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \quad (2)$$

where the parameters  $b_1, b_2, \dots, b_B$  are neither zero nor negative integers and  $A, B$  are non negative integers.

Contiguous Relations:

[Andrews p.363 (9.16), E.D. p.51 (10), H.T.F.I. p.103(32)]

$$(a-b) {}_2F_1(a, b; c; z) = a {}_2F_1(a+1, b; c; z) - b {}_2F_1(a, b+1; c; z) \quad (3)$$

[Abramowitz p.558 (15.2.19)]

$$(a-b)(1-z) {}_2F_1(a, b; c; z) = (c-b) {}_2F_1(a, b-1; c; z) + (a-c) {}_2F_1(a-1, b; c; z) \quad (4)$$

Gauss second summation theorem is defined as [Prud., 491(7.3,7.5)]

$${}_2F_1(a, b; \frac{a+b+1}{2}; \frac{1}{2}) = 2^{b-1} \frac{\Gamma(\frac{b}{2}) \Gamma(\frac{a+b+1}{2})}{\Gamma(b) \Gamma(\frac{a+1}{2})} \quad (5)$$

Recurrence relation:

$$\Gamma(\zeta+1) = \zeta \Gamma(\zeta) \quad (6)$$

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**B. MAIN RESULT OF SUMMATION FORMULAE:**

$$\begin{aligned}
 {}_2F_1(a, b; \frac{a+b+15}{2}; \frac{1}{2}) &= 2^b \frac{\Gamma(\frac{a+b+15}{2})}{(a-b)\Gamma(b)} \left[ \frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a+1}{2})} \left\{ \frac{64a(10395-19524a+12139a^2-3480a^3+505a^4-36a^5+a^6)}{[\prod_{\alpha=1}^6(a-b-(2\alpha-1))][\prod_{\omega=1}^3(a-b-(2\omega-1))]} \right. \right. \\
 &+ \frac{64a(-4056b+58526ab-12896a^2b+8788a^3b-520a^4b+78a^5b+33111b^2-1144ab^2+25454a^2b^2-1144a^3b^2)}{[\prod_{\alpha=1}^3(a-b-(2\alpha-1))][\prod_{\omega=1}^3(a-b-(2\omega-1))]} \\
 &+ \frac{64a(715a^4b^2+3952b^3+17732ab^3+1716a^2b^3+2561b^4+572ab^4+1287a^2b^4+104b^5+286ab^5+13b^6)}{[\prod_{\alpha=1}^6(a-b-(2\alpha-1))][\prod_{\omega=1}^3(a-b-(2\omega-1))]} \\
 &+ \frac{64b(10395-4056a+33111a^2+3952a^3+2561a^4+104a^5+13a^6-19524b+58526ab-1144a^2b+17732a^3b)}{[\prod_{\alpha=1}^3(a-b-(2\alpha-1))][\prod_{\beta=1}^3(a-b-(2\beta-1))]} \\
 &+ \frac{64b(572a^4b+236a^5b+12139b^2-12896ab^2+25454a^2b^2+1287a^4b^2-3480b^3+8788ab^3-1144a^2b^3)}{[\prod_{\alpha=1}^3(a-b-(2\alpha-1))][\prod_{\beta=1}^3(a-b-(2\beta-1))]} \\
 &+ \left. \frac{64b(1716a^5b^5+505b^4-520ab^4+715a^2b^4-36b^5+78ab^5+b^6)}{[\prod_{\alpha=1}^3(a-b-(2\alpha-1))][\prod_{\beta=1}^3(a-b-(2\beta-1))]} \right\} \\
 &- \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a}{2})} \left\{ \frac{128(10395+4056a+33111a^2-3952a^3+2561a^4-104a^5+13a^6+19524b+58526ab+1144a^2b+17732a^3b)}{[\prod_{\alpha=1}^6(a-b-(2\alpha-1))][\prod_{\omega=1}^3(a-b-(2\omega-1))]} \right. \\
 &+ \frac{128(-572a^4b+286a^5b+12139b^2+12896ab^2+25454a^2b^2+1287a^4b^2+3480b^3+8788ab^3+1144a^2b^3)}{[\prod_{\alpha=1}^6(a-b-(2\alpha-1))][\prod_{\omega=1}^3(a-b-(2\omega-1))]} \\
 &+ \frac{128(1716a^5b^5+505b^4+520ab^4+715a^2b^4+36b^5+78ab^5+b^6)}{[\prod_{\alpha=1}^6(a-b-(2\alpha-1))][\prod_{\omega=1}^3(a-b-(2\omega-1))]} + \frac{128(10395+19524a+12139a^2+3480a^3)}{[\prod_{\alpha=1}^3(a-b-(2\alpha-1))][\prod_{\beta=1}^3(a-b-(2\beta-1))]} \\
 &+ \frac{128(505a^4+36a^5+a^6+4056b+58526ab+12896a^2b+8788a^3b+520a^4b+78a^5b+33111b^2+1144ab^2)}{[\prod_{\alpha=1}^3(a-b-(2\alpha-1))][\prod_{\beta=1}^3(a-b-(2\beta-1))]} \\
 &+ \frac{128(25454a^2b^2+1144a^3b^2+715a^4b^2-3952b^3+17732ab^3+1716a^5b^3+2561b^4-572ab^4+1287a^2b^4)}{[\prod_{\alpha=1}^3(a-b-(2\alpha-1))][\prod_{\beta=1}^3(a-b-(2\beta-1))]} \\
 &+ \left. \frac{128(-104b^5+286ab^5+13b^6)}{[\prod_{\alpha=1}^3(a-b-(2\alpha-1))][\prod_{\beta=1}^3(a-b-(2\beta-1))]} \right\} \quad (7)
 \end{aligned}$$

$$\begin{aligned}
 {}_2F_1(a, b; \frac{a+b+16}{2}; \frac{1}{2}) &= 2^b \frac{\Gamma(\frac{a+b+16}{2})}{(a-b)\Gamma(b)} \left[ \frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})} \left\{ \frac{128(46080a-56448a^2+25984a^3-5880a^4+700a^5-42a^6)}{[\prod_{\lambda=0}^6(a-b-2\lambda)][\prod_{\mu=1}^3(a-b-2\mu)]} \right. \right. \\
 &+ \frac{128(a^7+46080b+174720a^2b-27664a^3b+14924a^4b-728a^5b+91a^6b+56148b^2+174720ab^2+56056a^3b^2)}{[\prod_{\lambda=0}^6(a-b-2\lambda)][\prod_{\mu=1}^3(a-b-2\mu)]} \\
 &+ \frac{128(-2002a^4b^2+1001a^5b^2+25984b^3+27664ab^3+56056a^2b^3+3003a^4b^3+5880b^4+14924ab^4+2002a^2b^4)}{[\prod_{\lambda=0}^6(a-b-2\lambda)][\prod_{\mu=1}^3(a-b-2\mu)]} \\
 &+ \frac{128(3003a^5b^4+700b^5+728ab^5+1001a^3b^5+42b^6+91ab^6+b^7)}{[\prod_{\lambda=0}^6(a-b-2\lambda)][\prod_{\mu=1}^3(a-b-2\mu)]} + \frac{256b(46080+14208a+4377a^2+4808a^3)}{[\prod_{\sigma=0}^6(a-b-2\sigma)][\prod_{\tau=1}^3(a-b-2\tau)]} \\
 &+ \frac{256b(1988a^4+70a^5+7a^6-14208b+113152ab+7592a^2b+17264a^3b+546a^4b+182a^5b+43776b^2-7592ab^2)}{[\prod_{\sigma=0}^6(a-b-2\sigma)][\prod_{\tau=1}^3(a-b-2\tau)]}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{256b(33176a^2b^2+572a^2b^2+1001a^4b^2-4808b^3+17264ab^3-572a^2b^3+1716a^2b^3+1988b^4-546ab^4)}{[\prod_{\sigma=0}^2(a-b-2\sigma)] [\prod_{\tau=1}^2(a-b-2\tau)]} \\
 & + \frac{256b(1001a^2b^4-70b^5+182ab^5+7b^6)}{[\prod_{\sigma=0}^2(a-b-2\sigma)] [\prod_{\tau=1}^2(a-b-2\tau)]} \cdot \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})} \left\{ \frac{256a(46080-14208a+43776a^2-4808a^2+1988a^4)}{[\prod_{\lambda=0}^2(a-b-2\lambda)] [\prod_{\mu=1}^2(a-b-2\mu)]} \right. \\
 & + \frac{256a(-70a^5+7a^6+14208b+113152ab-7592a^2b+17264a^2b-546a^4b+182a^5b+43776b^2+7592ab^2)}{[\prod_{\lambda=0}^2(a-b-2\lambda)] [\prod_{\mu=1}^2(a-b-2\mu)]} \\
 & + \frac{256a(33176a^2b^2-372a^2b^2+1001a^4b^2+4808b^3+17264ab^3+572a^2b^3+1716a^2b^3+1988b^4+546ab^4)}{[\prod_{\lambda=0}^2(a-b-2\lambda)] [\prod_{\mu=1}^2(a-b-2\mu)]} \\
 & + \frac{256a(1001a^2b^4+70b^5+182ab^5+7b^6)}{[\prod_{\lambda=0}^2(a-b-2\lambda)] [\prod_{\mu=1}^2(a-b-2\mu)]} + \frac{128(46080a+56448a^2+25984a^3+5880a^4+703a^5+42a^6+a^7)}{[\prod_{\sigma=0}^2(a-b-2\sigma)] [\prod_{\tau=1}^2(a-b-2\tau)]} \\
 & + \frac{128(46080b+174720a^2b+27664a^2b+14924a^4b+728a^2b+91a^6b-56448b^2+174720ab^2+56056a^2b^2)}{[\prod_{\sigma=0}^2(a-b-2\sigma)] [\prod_{\tau=1}^2(a-b-2\tau)]} + \\
 & + \frac{128(2002a^4b^2+1001a^2b^2+25984b^3-27664ab^3+56056a^2b^3+3003a^4b^3-5880b^4+14924ab^4)}{[\prod_{\sigma=0}^2(a-b-2\sigma)] [\prod_{\tau=1}^2(a-b-2\tau)]} \\
 & \left. + \frac{128(-2002a^2b^4+3003a^2b^4+700b^5-728ab^5+1001a^2b^5-42b^6+91ab^6+b^7)}{[\prod_{\sigma=0}^2(a-b-2\sigma)] [\prod_{\tau=1}^2(a-b-2\tau)]} \right\} \quad (8)
 \end{aligned}$$

C. DERIVATIONS OF SUMMATION FORMULAE (7) TO (8):

Derivation of (7): Replacing  $c = \frac{a+b+15}{2}$  and  $z = \frac{1}{2}$  in equation (3), we get

$$(a-b) {}_2F_1(a, b; \frac{a+b+15}{2}; \frac{1}{2}) = a {}_2F_1(a+1, b; \frac{a+b+15}{2}; \frac{1}{2}) - b {}_2F_1(a, b+1; \frac{a+b+15}{2}; \frac{1}{2})$$

Now with the help of the derived result from Gauss second summation theorem, we get

$$\begin{aligned}
 \text{L.H.S} & = a 2^b \frac{\Gamma(\frac{a-b+15}{2})}{\Gamma(b)} \left[ \frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a+1}{2})} \left\{ \frac{64(10395-19524a+12139a^2-3480a^3+505a^4-36a^5+a^6-4056b+58526ab)}{[\prod_{\sigma=1}^2(a-b-(2\sigma-1))] [\prod_{\omega=1}^2(a-b-(2\omega-1))]} \right. \right. \\
 & + \frac{64(-12896a^2b+8788a^2b-520a^4b+78a^2b+33111b^2-1144ab^2+25454a^2b^2-1144a^2b^2+715a^4b^2+3952b^3)}{[\prod_{\sigma=1}^2(a-b-(2\sigma-1))] [\prod_{\omega=1}^2(a-b-(2\omega-1))]} \\
 & \left. + \frac{64(17732ab^3+1715a^2b^3+2561b^4+572ab^4+1287a^2b^4+134b^5+286ab^5+13b^6)}{[\prod_{\sigma=1}^2(a-b-(2\sigma-1))] [\prod_{\omega=1}^2(a-b-(2\omega-1))]} \right\} - \\
 & \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+2}{2})} \left\{ \frac{64(10395+4056a+33111a^2-3952a^3+2561a^4-104a^5+13a^6+19524b+58526ab+1144a^2b)}{[\prod_{\sigma=1}^2(a-b-(2\sigma-1))] [\prod_{\omega=1}^2(a-b-(2\omega-1))]} \right. \\
 & + \frac{64(17732a^2b-572a^4b+286a^5b+12139b^2+12896ab^2+25154a^2b^2+1287a^4b^2+3480b^3+3788ab^3)}{[\prod_{\sigma=1}^2(a-b-(2\sigma-1))] [\prod_{\omega=1}^2(a-b-(2\omega-1))]} \\
 & \left. + \frac{64(1144a^2b^3+1715a^2b^3+505b^4+520ab^4+715a^2b^4+36b^5+78ab^5+b^6)}{[\prod_{\sigma=1}^2(a-b-(2\sigma-1))] [\prod_{\omega=1}^2(a-b-(2\omega-1))]} \right\} \\
 & - b 2^{b+1} \frac{\Gamma(\frac{a+b+15}{2})}{\Gamma(b+1)} \left[ \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a}{2})} \left\{ \frac{64(10395+19524a+12139a^2+3480a^3+505a^4+36a^5+a^6+4056b+58526ab)}{[\prod_{\alpha=1}^2(a-b-(2\alpha-1))] [\prod_{\beta=1}^2(a-b-(2\beta-1))]} \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{64(12896a^2b+8788a^3b+520a^4b+78a^5b+33111b^2+1144ab^2+25454a^2b^2+1144a^3b^2+715a^4b^2-3952b^3)}{[\prod_{\alpha=1}^7(a-b-(2\alpha-1))][\prod_{\beta=1}^6(a-b-(2\beta-1))]} \\
 & + \frac{64(17732ab^3+1716a^3b^3+2561b^4-572ab^4+1287a^2b^4-104b^5+286ab^5+13b^6)}{[\prod_{\alpha=1}^7(a-b-(2\alpha-1))][\prod_{\beta=1}^6(a-b-(2\beta-1))]} \\
 & \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})} \left\{ \frac{64(10395-4056a+33111a^2+3952a^3+2561a^4+104a^5+13a^6-19524b+58526ab-1144a^2b)}{[\prod_{\alpha=1}^7(a-b-(2\alpha-1))][\prod_{\beta=1}^6(a-b-(2\beta-1))]} \right. \\
 & + \frac{64(17732a^3b+572a^4b+286a^5b+12139b^2-12896ab^2+25454a^2b^2+1287a^4b^2-3483b^3+8788ab^3)}{[\prod_{\alpha=1}^7(a-b-(2\alpha-1))][\prod_{\beta=1}^6(a-b-(2\beta-1))]} \\
 & \left. + \frac{64(-1144a^2b^3+1716a^3b^3+505b^4-520ab^4+715a^2b^4-36b^5+78ab^5+b^6)}{[\prod_{\alpha=1}^7(a-b-(2\alpha-1))][\prod_{\beta=1}^6(a-b-(2\beta-1))]} \right\}
 \end{aligned}$$

On simplification, we get the formula (7)

Similarly, we can prove the formula (8).

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