

A MATHEMATICAL STUDY OF SELF- ORGANIZING NERVE SYSTEMS

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ABSTRACT

We treat very simple nerve net models for the sake of tractability, such that we obtain explicitly the behaviours of the models. This enables us to know how the behaviours of the models depend on the values of the system parameters and to know the possible information processing manners of neuron nets. The study is devoted in two parts first part treats the general equation of neural learning in a unified manner. Thus, perceptrons learning, correlation learning for associative memory, automatic formation of signal or feature detectors etc, are studied in this common frame. The second part treats dynamics of neural excitation. Dynamics of neuron pools, of neural fields, and of a completion model are analysed. By combining the result of first and second parts, we can analyze self organizing neural systems: We analyze a model of formation of signal or feature detectors, and a model of topographic organization of nerve fields.

Key words: Nerve system, neuron nets, neuron pools, time sequence and membrane potential.

AMS Subject Classification (2000): 76S05, 76S99.

1. INTRODUCTION:

The brain is a self organizing system adapting to the information structures of the environment. It is highly hierarchical and enormously complex system so that it is in general difficult to analyse its model. Here we present a mathematical approach of analyzing the brain functioning. We treat very simple nerve net models for the sake of tractability, such that we obtain explicitly the behaviours of the models. This enables us to know how the behaviours of the models depend on the values of the system parameters and to know the possible information processing manners of neuron nets. Since the biological systems are structurally stable, so the results obtained by using the simplified models are expected to be applicable, at least qualitatively, to more realistic and complex model as well.

The study consist of three parts, summarizing and extending the results of Amari and Takeuchi (1978), Amari, Yoshida and Kanatani (1977), Kishimoto and Amari (1979). The first part treats the general equation of neural learning in a unified manner. Thus, perceptron learning, correlation learning for associative memory, automatic formation of signal or feature detectors, etc, are studied in this common frame. The second part treats dynamics of neural excitation. Dynamics of neuron pools, of neural fields, and of a completion model are analyzed. By combining the result of first and second parts, we can analyze self organizing neural systems. We analyze a model of formation of signal or feature detectors, and a model of topographic organization of nerve fields. A possible mechanism of formation of micro – regional structures is suggested.

2. EQUATION OF NEURAL LEARNING:

Modification of synaptic efficiency:

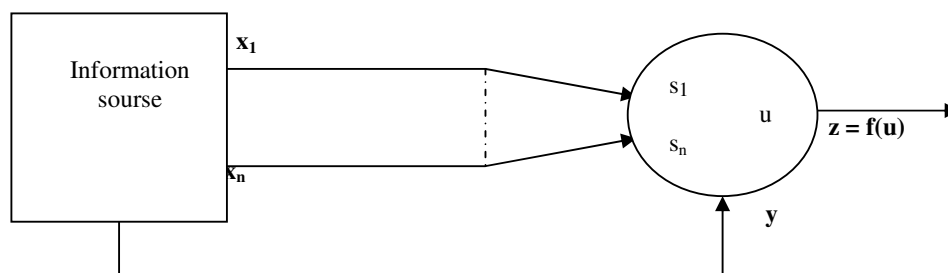


Fig. -01 Model of nerve

Let us consider a simple mathematical model of neurons (Fig.-1). The model neuron receives n inputs signals $x_1, x_2 \dots x_n$ and emits one output signals z . These signals take in general analog values between 0 and 1, representing the respective normalized pulse frequencies (they may take on two values 0 and 1). Let s_1, s_2, \dots, s_n be the synaptic efficiencies or weights of inputs x_1, x_2, \dots, x_n . Then neuron receives the weight sum $\sum s_i x_i$ of inputs, and the average membrane potential $u(t)$ at time t of the neuron changes subject to the equation

$$\tau \dot{u}(t) = -u(t) + \sum s_i x_i - h \quad (2.1)$$

where “ $\dot{\cdot}$ ” denotes the time derivative $\frac{d}{dt}$, τ is a time constant of neural excitation and h is a threshold value, $(-h)$ is the resting potential. Since τ is small, the potential quickly converges to

$$u = \sum s_i x_i - h, \quad (2.2)$$

for constant inputs. The neuron emits output pulses with frequency

$$z = f(u), \quad (2.3)$$

according to average membrane potential u , where f is a monotonically non-decreasing function. The equation (2.2) and (2.3) give the input - output relations of the model neuron.

The neuron modifies its synaptic weights s_i depending on the inputs x_i , the output z , etc. In some cases, the neuron can make use of an extra signal y which plays the role of a teacher. In this case self-organization is called learning with teacher. The teacher signal is set identically equal to 0 in case of learning without teacher.

We propose the following rule of synapse modification by generalizing Hebbian law:

$$\tau \dot{s}_i = -s_i(t) + cr(t)x_i(t), \quad i = 1, 2, 3, \dots, n, \quad (2.4)$$

where c is a constant and $r(t)$ is a function of the synaptic weights s_i inputs x_i and the teacher signal y at time t is

$$r(t) = r[s_i(t), x_i(t), y(t)]. \quad (2.5)$$

Learning takes place only when r is not 0. We call r the reinforcement signal or learning signal. These are various kinds of neurons in the brain, so that different kinds of neurons may have different types of learning signal r . The nervous system probably combines neurons with different types of learning signal r , so that it realizes a variety of information processing.

Where r is put equal to the output z

$$r = z = f(\sum s_i x_i - h), \quad (2.6)$$

we have the Hebbian law without teacher.

When,
$$r = y - z, \quad (2.7)$$

i.e. r is equal to the difference between the output and the teacher signal, we have the perception type learning rule. When $y = r$, we have the correlation learning rule used in the model of associative memory. When

$$r = y - \sum s_i x_i, \quad (2.8)$$

we have the orthogonal learning rule.

(a) Environment, information source and average learning equation

We use the vector notation such that $\underline{X} = (x_1, x_2, x_3 \dots x_n)$ and $\underline{S} = (s_1, s_2, s_3 \dots s_n)$, then the learning equation (2.4) is written as

$$\tau \dot{\underline{S}} = -\underline{S} + cr(\underline{S}, \underline{X}, y) \underline{X}. \quad (2.9)$$

The synaptic weight vector $\underline{S}(t)$ is modified depending on the time sequence $\{\underline{X}(t), y(t)\}$ which the neuron receives. The time sequence usually carries information of the environment so that the neuron can adapt to the environment. We regard the environment as an information source I which produces the time sequence $\{\underline{X}(t), y(t)\}$. We treat a very

simple ergodic information source which produces a pair (\underline{x}, y) of input \underline{x} and associate teacher signal y (which does not necessarily exist) with probability (density) distribution $p(\underline{x}, y)$. Here we assume that I produces a pair (\underline{x}, y) with probability $p(\underline{x}, y)$ and that the chosen pair lasts for a fixed time duration Δt . Then another pair is produced independently of the previous pair.

Since (2.9) is random equation, we consider the average equation

$$\dot{\underline{s}} = -\underline{s} + c \langle r(\underline{s}, \underline{x}, y) \underline{x} \rangle, \quad (2.10)$$

where $\langle \rangle$ denotes the average over (\underline{x}, y) . Since I is ergodic, it is expected that the behaviour of (2.10) gives a good approximation to (2.9). (Geman, 1979). We call (2.10) the average learning equation.

We see that, when r is of the form $r = r(\underline{s}, \underline{x}, y)$ where “ \cdot ” is the inner product, we have a potential function of learning

$$R = (\underline{s}, \underline{x}, y) = \frac{1}{2} |\underline{s}|^2 - c \int_0^{\underline{s} \cdot \underline{x}} r(u, y) du, \quad (2.11)$$

by which the learning equation is written as

$$\dot{\underline{s}} = - \frac{\partial R}{\partial \underline{s}}. \quad (2.12)$$

The average learning equation is

$$\dot{\underline{s}} = - \frac{\partial L}{\partial \underline{s}}, \quad (2.13)$$

where

$$L(\underline{s}) = \langle R(\underline{s}, \underline{x}, y) \rangle, \quad (2.14)$$

is a function of \underline{s} only. Hence we see that the synaptic weight vector \underline{s} converges to one of the minima of the function $L(\underline{s})$. This clarifies the characteristics of neural learning. It plays an important role in self- organization without teacher that $L(\underline{s})$ has a number of minima, as will be shown later.

The equilibrium state of learning is given by solving the equation, $\underline{s} = 0$ or

$$\underline{s} = c \langle r(\underline{s}, \underline{x}, y) \underline{x} \rangle, \quad (2.15)$$

which reflects the structure of I , because the average $\langle \rangle$ is taken over I .

(c) Associative Memory:

Let us consider the case where I includes k pairs (\underline{x}^i, y^i) , $i = 1, 2, 3 \dots k$, of input and teacher signal with an equal probabilities $\frac{1}{k}$. By correlation learning, where $r = y$, the synaptic weight converges to

$$\underline{s} = \frac{1}{k} \sum_{i=1}^k y^i \underline{x}^i. \quad (2.16)$$

Hence, when k signal are mutually orthogonal and $x^i = 1$, we have

$$\underline{s} \cdot \underline{x}^j = \frac{1}{k} y^j, \quad j = 1, 2, 3 \dots k. \quad (2.17)$$

This provides a primitive model of association such that k signal pairs (\underline{x}^i, y^i) in I are memorized in a single \underline{s} and i recall y^j from any \underline{x}^j by taking the weighted sum $\underline{s} \cdot \underline{x}^j$ of the input \underline{x}^j .

Let us consider an information source I which include k pairs of vectors (\underline{x}^i, y^i). When there are m neurons, and the learning signal r of the j^{th} neuron is the j^{th} component y_j of \underline{y} , where m is the dimension number of \underline{y} , then the pool of these m neurons together recall vector signal \underline{y}^i from input \underline{x}^i , $I = 1, 2, 3 \dots k$, because the j^{th} neurons recall the j^{th} component. This is indeed a model of associative memory studied by many researches.

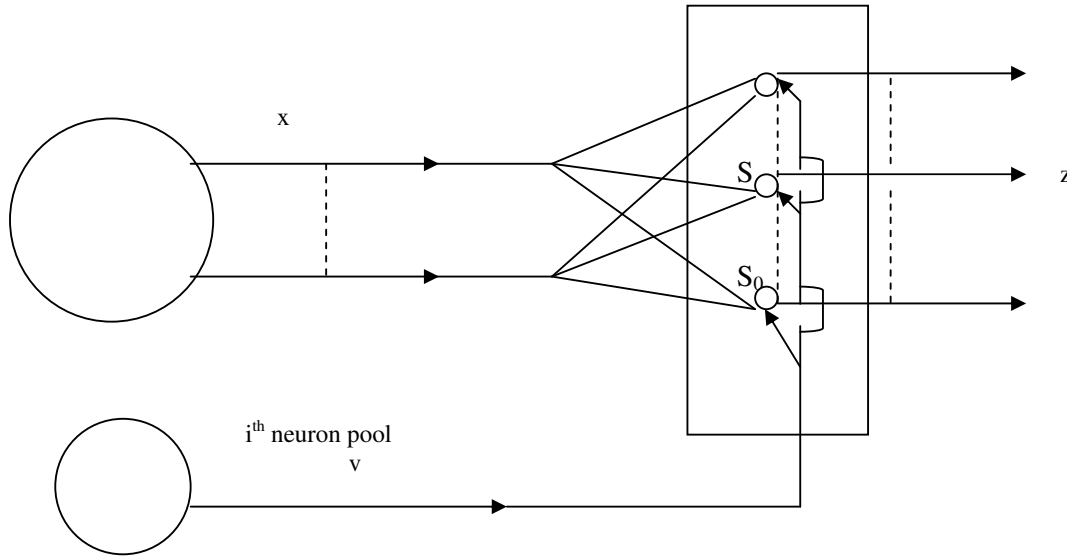


Fig.-02 Simple model of formation signal detectors

This model is well when \underline{x}^i are mutually orthogonal. The orthogonal learning rule (2.8) is effective even when \underline{x}^i are not mutually orthogonal. In this case, it has been proved (Amari, [1977]) that the synaptic weight \underline{s} of a neuron converges approximately to

$$\underline{s} = \sum_{i=1}^k y^i \underline{x}^{*i} \quad (2.18)$$

where \underline{x}^{*i} , ($i = 1, 2 \dots k$) are the dual system of \underline{x}^i , s such that \underline{x}^i are linear combinations of \underline{x}^{*i} , s satisfying

$$\begin{aligned} \underline{x}^{*i} \cdot \underline{x}^j &= 0, & (i \neq j) \\ \underline{x}^{*i} \cdot \underline{x}^i &= 1. \end{aligned}$$

In this case, we have

$$\underline{s} \cdot \underline{x}^j = y^j, \quad j = 1, 2 \dots k.$$

(d) Learning equations for neural systems:

We have so far treated learning by a single neuron. In order to analyze self-organizing neural systems in which neurons interact, we consider the dynamics of neural excitations. Let us consider a net consisting of N neuron which are mutually connected and which receive input signal \underline{x} from a common environment information source I.

Let $u_i(t)$ be the membrane potential of the i^{th} neuron, $\underline{s}_i = (s_{i1}, \dots, s_{in})$ be the synaptic weight vector for input $\underline{x} = (x_1, x_2, \dots, x_n)$,

$$Z_j = f [u_j(t)], \quad (2.19)$$

be the output of the j^{th} neuron, and w_{ij} be the synaptic weight of the output Z_j of the j^{th} neuron entering into the i^{th} neuron (fig.-3). Then for a constant input \underline{x} , the dynamics of neural excitations is described by

$$\tau \dot{u}_i(t) = -u_i - h + \underline{s}_i \cdot \underline{x} + \sum_{j=1}^n w_{ij} f[u_j(t)] \quad (2.20)$$

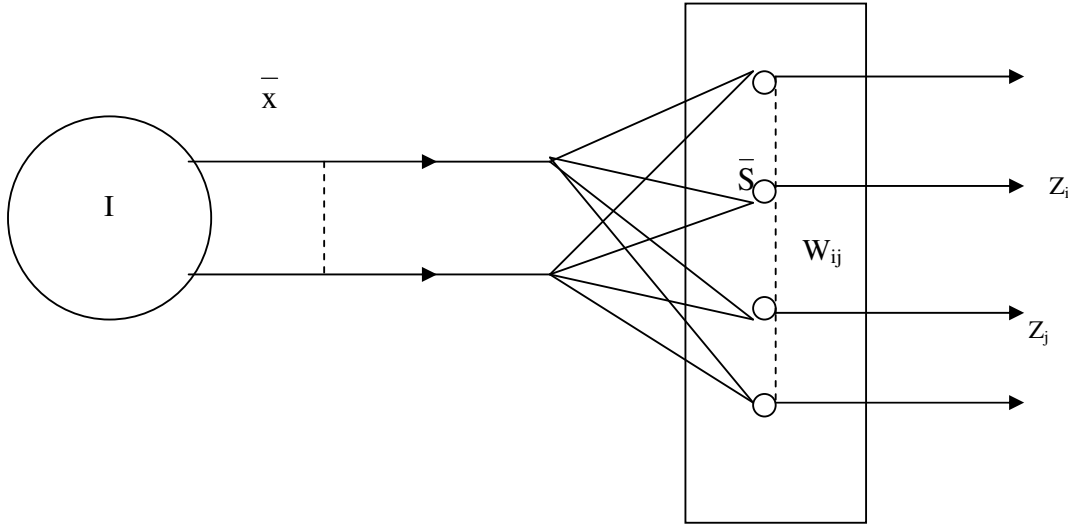


Fig.-03 Self organizing nerve net

Here we consider the case where the above dynamics is monostable, having the only equilibrium state \bar{u}_i or $\bar{\underline{u}}$ ($\bar{\underline{u}} = (\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N)$). The equilibrium $\bar{\underline{u}}$ depends on, $\underline{x}, \underline{s}_i$ and w_{ij} . Hence

$$\bar{\underline{u}} = \bar{\underline{u}}(\underline{x}, w_{ij}, \underline{s}_i), \quad (2.21)$$

which we call the equilibrium function. The equilibrium changes slowly as the synaptic weight change by self organization.

3. DYNAMICS OF NEURAL EXCITATIONS:

(a) Dynamics of neural pool:

Let us consider a very simple net consisting of N neurons. The dynamical equation of the net is described by

$$\tau \dot{u}_i(t) = -u_i + \sum w_{ij} f[u_j] + S_i, \quad (3.1)$$

where S_i is the weighted sum of input to the i^{th} neuron minus threshold h

$$S_i = \sum_i \underline{x} - h, \quad (3.2)$$

and w_{ij} is the synaptic weight connecting the output of the j^{th} neuron to the i^{th} neuron.

When the synaptic weights w_{ij} are identically and independently distributed random variables, the net is called the homogeneous neuron pool. In such a net, the behaviour of a single neuron itself is not important. Information is carried by macroscopic quantity of the net. The activity Z of the net is defined by the average of the outputs.

$$Z = \frac{1}{n} \sum Z_j, \quad (3.3)$$

and is a macroscopic quantity. The dynamical behaviour of the macroscopic quantities are studied in detail by many researches by the use of various types of neural models.

(b) Dynamics of nerve fields:

Let us consider a system consisting of m layers of nerve fields and let $u_i(x, t)$ be the average membrane potential of the neurons of the i^{th} layers at time t at spatial position x (we consider a one – dimensional fluid for the simplicity’s sake). Then the dynamics of excitations is described by the following equations

$$t \frac{\partial}{\partial t} [u_i(x, t)] = -u_i + \sum_j \int w_{ij}(x - x') f[u_j(x', t) dx'] + a_i(x, t) - h, \tag{3.4}$$

where $w_{ij}(x-x')$ is the synaptic weight from the neurons at positions x' of the j^{th} layer to the neurons at position x of the i^{th} layer (we assume the homogeneity and isotropy of the field), $a_i(x, t)$ is the intensity of the stimulation from the out side to the neurons of the i^{th} layer at position x at time t.

This kind of nerve fields have been studied by Witson and Cow in (1973), Amari (1977 a), Ellias and Grossberg (1975). When the output function f is approximately by the step function I (u) taking on two values 0 and 1, the complete categorization of the dynamical behaviour is given for the one – layes field of tateral inhibition type, i.e. when connection weight function w(x) is of tateral – inhibition type (fig-4) the field equation is in this case

$$\frac{\partial}{\partial t} [u(x, t)] = -u + \int w(x - x') I[u(x', t)] dx' + a(x, t) - h. \tag{3.5}$$

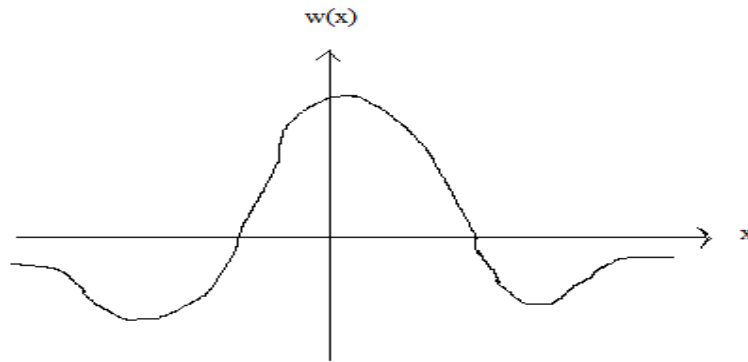


Fig.-04 Weighted function w(x)

The dynamics is in general multi-stable. Even when the input stimulus distributions $a(x,t)$ is uniform and constant.

$$a(x, t) = a, \tag{3.6}$$

the field can retain a local excitations persistently around the position where a strong stimulus came recently.

4. SELF- ORGANIZATION OF NEURAL SYSTEM:

Let us consider a revised model for formation of signal detectors, where the inhibitory neuron pool is a activated by the output of the excitatory neurons (fig.-5). The excitatory neurons interact with each other through the inhibitory neuron pool, so that we can not analyze the behaviour of each neuron independently. Instead, it is necessary before solving the average learning equation to obtain the equilibrium functions by solving the dynamical equation of neural excitations.

In the revised model, the stimulus S_i to each neuron is given by

$$S_i(\underline{x}) = \underline{s}_i \cdot \underline{x}, \tag{4.1}$$

for an input signal \underline{x} from a common information source I, where \underline{S}_i is the modifiable synaptic weight vector of the i^{th} neuron. The inhibitory synaptic weight – S_{0i} is also modifiable. We have the following equilibrium function

$$\bar{v} = \bar{N} - h \tag{4.2}$$

$$\bar{u}_i(\underline{x}, \underline{s}_i, s_{0i}) = s_{0i} g(\bar{v}) + \underline{s}_i \cdot \underline{x}, \tag{4.3}$$

where \bar{N} is determined by $\bar{T}_i > i$, where

$$T_i = \frac{s_i}{s_{0i}} + h. \tag{4.4}$$

The average learning equations are

$$\dot{\bar{s}}_i = -s_i + c \langle x I(\bar{u}_j) \rangle, \tag{4.5}$$

$$\dot{s}_{0i} = s_{0i} + c \langle g(\bar{v}) I(\bar{u}_i) \rangle. \tag{4.6}$$

Let us consider the case where I includes a finite number of signals \underline{x}_α , where

$$|\underline{x}_\alpha|^2 = a, \quad \underline{x}_\alpha \cdot \underline{x}_\beta \leq b \quad (\alpha \neq \beta),$$

are satisfied. We can then prove, by obtaining the equilibrium solution (4.5) and (4.6), that detectors of each signal \underline{x}_α are formed and its number N_α satisfies

$$\sqrt{\frac{ca}{c'}} + h > N_\alpha > \left(\frac{b}{a}\right) \sqrt{\frac{ca}{c'}} + h, \tag{4.7}$$

for all \underline{x}_α . This proves that the detectors of various signals are formed in the revised model in balanced manner in number.

5. CONCLUSION:

We have presented a mathematical method of analyzing a wide class of nerve system. A number of self – organizing nerve system are analyzed by this method. Though the mathematical analysis is carried in to effect only under some bold mathematical simplification, the behaviour of the simplified models is believed to show, at least quantitatively, the same behaviour as more realistic model have. Hence the present method is useful in analyzing nervous system as well as in building more realistic models.

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