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CONVERGENCE ANALYSIS OF AN ITERATIVE METHOD FOR NONLINEAR OPERATOR EQUATIONS USING A MAJORIZING SEQUENCE

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ABSTRACT

In this paper we consider the Lavrentiev regularization method for obtaining stable approximate solution to nonlinear ill-posed operator equations $F(x)=y$, *where* $F: D(F) \subset X \to X$ *is a nonlinear monotone operator and* X *is a real Hilbert space. Under the assumption that* F *is Lipschitz continuous, the iteration* $(x_{n,\alpha}^{\delta})$ *converges to the unique solution* x_{α}^{δ} *of the equation* $F(x)=y^{\delta}+\alpha(x_0-x)$.

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1. INTRODUCTION:

In computational mathematics, an iterative method is a mathematical procedure that generates a sequence of improving approximate solutions for a class of problems. A specific implementation of an iterative method, including the termination criteria, is an algorithm of the iterative method. An iterative method is called convergent if the corresponding sequence converges for given initial approximations. A mathematically rigorous convergence analysis of an iterative method is usually performed; however, heuristic-based iterative methods are also common.

Let $F: D(F) \subset X \to X$ is a nonlinear monotone operator and X is a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and the corresponding norm $\| \ . \ \|$. Recall that F is monotone operator if it satisfies the relation

$$\langle F(x)-F(y), x-y \rangle \geq 0, \qquad \forall x, y \in D(F).$$

We are interested in obtaining a stable approximate solution for nonlinear ill-posed operator equation

$$F(x)=y, \tag{1.1}$$

when the data y is not known exactly. Further we assume that:

- Instead of an exact right-hand side y we are given only its perturbation $y^{\delta} \in X$, such that

$$\|y-y^{\delta}\| \leq \delta, \tag{1.2}$$

  where δ is the known noise level.

- The equation (1, 1) has a solution $x^{\dagger}$ (which need not be unique).

- The operator F possesses a locally uniformly bounded *Fréchet-derivative* $F'(\cdot)$ in $B_{r_0}(x_0) \cup B_{r_0}(x^{\dagger}) \subseteq D(F) \subseteq X$, $r_0 = \|x_0 - x^{\dagger}\|$ and x_0 is the initial guess for the solution.

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The numerical treatment of nonlinear ill-posed problems equation (1.1) in which the solution does not depend continuously on the data requires the application of special regularization methods. Again equation (1.1) is ill-posed, and then the problem of recovery of $x^\dagger \subseteq D(F)$ from a known noisy equation $F(x) = y^\delta$ can cause large deviation in the solution. Since F is monotone, one can use the Lavrentiev regularization method for solving (1.1). (See [9]). In this method the regularized approximation x_α^δ is obtained by solving the operator equation

$$F(x) + \alpha(x - x_0) = y^\delta, \tag{1.3}$$

It is known (cf. [9], Theorem 1.1) that the equation (1.3) has a unique solution for any $\alpha \geq 0,$ which called the regularization parameter. In practice one has to deal with some sequence $(x_{n,\alpha}^\delta)$, converging to the solution x_α^δ of (1.3), (see [2], and [6]). In [2] Bakushinsky and Smirnova considered an iteratively regularized Lavrentiev method:

$$x_{k+1}^\delta = x_k^\delta - (A_k^\delta + \alpha_\delta I)^{-1}(F(x_k^\delta) - y^\delta + \alpha_k(x_k^\delta - x_0)) \tag{1.4}$$

for $k = 0,1,2,\cdots,$ where $A_k^\delta = F'(x_k^\delta)$ and (α_k) is a sequence of positive real numbers such that $\lim_{k\to\infty} \alpha_k = 0$, as an approximate solution for (1.1). A general discrepancy principal has been considered in [2] for choosing the stopping index k_δ and showed that $x_{k_\delta}^\delta \to x^\dagger$ as $\delta \to 0$. However no error estimates for $\left\| x_{k_\delta}^\delta - x^\dagger \right\|$ has been given in [2]. Later in [7], Mahale and Nair considered the method (1.4) and obtained an error estimate for $\left\| x_k^\delta - x^\dagger \right\|$, under weaker assumptions than the assumptions in [2] (see [7]).

In [3], Elmahdy considered an iterative regularization method:

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - \beta(F(x_{n,\alpha}^\delta) - y^\delta + \alpha(x_{n,\alpha}^\delta - x_0)), \tag{1.5}$$

for solving the nonlinear equation (1,1) where $\beta > 0$ and $x_0 := x_{0,\alpha}^\delta$ is starting point of the iteration and proved that $(x_{n,\alpha}^\delta)$ converges to the unique solution x_α^δ of (1.3) under the following Assumptions.

Assumption: 1.1 There exists $r_0 > 0$ such that $B_{r_0}(x_0) \cup B_{r_0}(x^\dagger) \subseteq D(F)$ and F is *Fréchet differentiable* at all $x \in B_{r_0}(x_0) \cup B_{r_0}(x^\dagger).$

Assumption: 1.2 There exists a constant $L > 0$ such that for every $x, u \in B_{r_0}(x_0) \cup B_{r_0}(x^\dagger).$ and $v \in X$ there exists an element $\phi(x,u,v) \in X$ satisfying

$$[F'(x) - F'(u)]v = F'(u)\phi(x,u,v), \qquad \|\phi(x,u,v)\| \leq L\|v\|.$$

Assumption: 1.3 There exists a continuous, strictly monotonically increasing function $\varphi : (0,a] \to (0,\infty)$ with $a \geq \left\| F'(x^\dagger) \right\|$, satisfying $\lim_{\lambda\to 0} \varphi(\lambda) = 0$ and there exist $v \in X$ with $\|v\| \leq 1$ such that

$$x_0 - x^\dagger = \varphi(F'(x^\dagger))v \quad \text{and} \quad \sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq c_\varphi \varphi(\alpha), \qquad \forall\, \lambda \in (0, \mathrm{a}].$$

In this paper we use the following modified form of Assumption 1.2

Assumption: 1.4 There exists a constant $k_0 > 0$ such that for every $x, u \in B_{r_0}(x_0) \cup B_{r_0}(x^\dagger).$ and $v \in X$ there exists an element $\phi(x,u,v) \in X$ satisfying

$$[F'(x) - F'(u)]v = F'(u)\phi(x,u,v), \qquad \|\phi(x,u,v)\| \leq k_0\|v\|\|x - u\|.$$

Note that from Assumption 1.4 there follows Assumption 1.2 with $L = k_0\|x-u\|$. Hence Assumption 1.4 is stronger than Assumption 1.2. We will give a new stopping rule n_δ for the iteration different of the stopping rule using in [3] and provide an optimal order error estimate under a general source condition on $x_0 - x^\dagger$. Moreover we shall use the adaptive parameter selection procedure suggested by Pereverzev and Schock in [8], for choosing the regularization parameter α in $(x_{n,\alpha}^\delta)$.

The plan of this paper is as follows. In Section 2, we introduce the convergence analysis of the method and in Section 3, we give an error bounds under source conditions. Section 4 deals with starting points and algorithm. Finally, we give some concluding remarks in Section 5.

2. CONVERGENCE ANALYSIS:

Here we consider the method (1.5) for solving the nonlinear equation (1.1). The main goal of this section is to provide sufficient conditions for the convergence of method (1.5) to the unique solution x_α^δ of (1.3) by using a majorizing sequence and obtain an error estimate for $\|x_{n,\alpha}^\delta - x_\alpha^\delta\|$. Recall (see [1], Definition 1.3.11) that a nonnegative sequence (t_n) is said to be a majorizing sequence of a sequence (x_n) in X if

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n \qquad \forall n \ge 0.$$

During the convergence analysis we will be using the following Lemma on majorization, which is a reformulation of Lemma 1.3.12 in [1].

Lemma 2.1: Let (t_n) be a majorizing sequence for (x_n) in X. If $\lim_{n\to\infty} t_n = t^*$, then $x^* = \lim_{n\to\infty} x_n$ exists and

$$\|x^* - x_n\| \le t^* - t_n \qquad \forall n \ge 0. \tag{2.6}$$

Proof: Note that

$$\|x_{n+m} - x_n\| \le \sum_{j=n}^{n+m-1} \|x_{j+1} - x_j\| \le \sum_{j=n}^{n+m-1} (t_{j+1} - t_j) = t_{n+m} - t_n\,, \tag{2.7}$$

so (x_n) is a Cauchy sequence in X and hence (x_n) converges to some x^*. The error estimate in (2.6) follows from (2.7) as $m \to \infty$.

The next Lemma on majorizing sequence is used to prove the convergence of the method (1.5), proof of which is analogous to the proof of Lemma 2.2 in [3].

Lemma: 2.2 Assume there exist nonnegative numbers $R, \beta, \eta,$ and $r \in (0,1)$ such that for all $n \ge 0$,

$$\sqrt{(1-\beta\alpha)^2 + \beta^2 R^2}\,\eta r^n \le r. \tag{2.8}$$

Then the sequence (t_n), $n \ge 0$, given by $t_0 = 0, t_1 = \eta$,

$$t_{n+1} = t_n + \sqrt{(1-\beta\alpha)^2 + \beta^2 R^2}\,(t_n - t_{n-1}) \tag{2.9}$$

is increasing, bounded above by $t^{**} := \frac{\eta}{1-r}$, and converges to some t^* such that $0 < t^* \le \frac{\eta}{1-r}$. Moreover, for $n \ge 0$;

$$0 \le t_{n+1} - t_n \le r(t_n - t_{n-1}) \le r^n \eta, \tag{2.10}$$

and

$$t^* - t_n \le \frac{r^n}{1-r}\eta. \tag{2.11}$$

To prove the convergence of the sequence $(x_{n,\alpha}^{\delta})$ defined in (1.5) we introduce the following notations.

- Suppose that the Lipschitz condition is satisfied for the operator F, namely there exists a constant $R>0$ such that

$$\|F(x)-F(y)\| \le R\|x-y\| \;, \qquad \forall x,y \in D(F) \tag{2.12}$$

Let

$$\omega := \|F(x_0)-y^{\delta}\| \le 1/\beta \;, \tag{2.13}$$

with $\beta < 2\alpha/(\alpha^2+R^2)$.

The following Lemma based on the Assumption 1.4 will be used in our proofs.

Lemma: 2.3 (*[5] Lemma 2.3*) For $u,v \in B_{r_0}(x_0)$

$$F(u)-F(v)-F'(u)(u-v) = F'(u)\int_0^1 \phi(v+t(u-v),u,u-v)dt.$$

Let

$$G(x) := x-\beta[F(x)-y^{\delta}+\alpha(x-x_0)]. \tag{2.14}$$

Note that with the above notation, $G(x_{n,\alpha}^{\delta}) = x_{n+1,\alpha}^{\delta}$.

Theorem 2.4: Let $t^* \le r_0$ and suppose that the equations (2.12) and (2.13) hold. Let the assumptions in Lemma 2.2 are satisfied. Then the sequence $(x_{n,\alpha}^{\delta})$ defined in (1.5) is well defined and $x_{n,\alpha}^{\delta} \in B_{t^*}(x_0)$ for all $n \ge 0$. Further $(x_{n,\alpha}^{\delta})$ is a Cauchy sequence in $B_{t^*}(x_0)$ and hence converges to $x_{\alpha}^{\delta} \in \overline{B_{t^*}(x_0)} \subset B_{t^{**}}(x_0)$ and $F(x_{\alpha}^{\delta}) = y^{\delta}+\alpha(x_0-x_{\alpha}^{\delta})$.

Moreover, the following estimate hold for all $n \ge 0$,

$$\|x_{n+1,\alpha}^{\delta}-x_{n,\alpha}^{\delta}\| \le t_{n+1}-t_n, \tag{2.15}$$

and

$$\|x_{n,\alpha}^{\delta}-x_{\alpha}^{\delta}\| \le t^*-t_n \le \frac{r^n\eta}{1-r}. \tag{2.16}$$

Proof: Let *G* be as in (2.14). Then for $u,v \in D(G)$,

$$\begin{aligned} G(u)-G(v) &= u-v-\beta[F(u)-y^{\delta}+\alpha(u-x_0)]+\beta[F(v)-y^{\delta}+\alpha(v-x_0)] \\ &= (1-\alpha\beta)(u-v)-\beta[F(u)-F(v)] \end{aligned}$$

Also we have,

$$\begin{aligned} \|G(u)-G(v)\|^2 &= \|(1-\alpha\beta)(u-v)-\beta[F(u)-F(v)]\|^2 \\ &\le \|1-\alpha\beta\|^2\|u-v\|^2+\beta^2\|F(u)-F(v)\|^2 \\ &\le [\,(1-\alpha\beta)^2+R^2\beta^2]\|u-v\|^2 \end{aligned}$$

The last equation by (2.12), so we have

$$\|G(u)-G(v)\| \le \sqrt{(1-\alpha\beta)^2+R^2\beta^2}\,\|u-v\|. \tag{2.17}$$

Take $\beta < 2\alpha/(\alpha^2+R^2)$, this the first way to prove the convergence of the sequence.

On the other hand we shall prove that the sequence $(t_n),\ n \geq 0$ defined in Lemma 2.2 is a majorizing sequence of the sequence $(x_{n,\alpha}^\delta)$ and $x_{n,\alpha}^\delta \in B_{t^*}(x_0)$ for all $n \geq 0$.

Note that $\left\|x_{1,\alpha}^\delta - x_0\right\| = \left\|\beta(F(x_0) - y^\delta)\right\| \leq 1 = \eta = t_1 - t_0$, assume that

$$\left\|x_{i+1,\alpha}^\delta - x_{i,\alpha}^\delta\right\| \leq t_{i+1} - t_i, \qquad \forall\, i \leq k \tag{2.18}$$

for some *k*. Then

$$\begin{aligned}\left\|x_{k+1,\alpha}^\delta - x_0\right\| &\leq \left\|x_{k+1,\alpha}^\delta - x_{k,\alpha}^\delta\right\| + \left\|x_{k,\alpha}^\delta - x_{k-1,\alpha}^\delta\right\| + \cdots + \left\|x_{1,\alpha}^\delta - x_0\right\| \\ &\leq t_{k+1} - t_k + t_k - t_{k-1} + \cdots + t_1 - t_0 \\ &\leq t_{k+1} \leq t^*.\end{aligned}$$

So $x_{i+1,\alpha}^\delta \in B_{t^*}(x_0)$ for all $i \leq k$, and hence, by (2.17) and (2.18),

$$\left\|x_{k+2,\alpha}^\delta - x_{k+1,\alpha}^\delta\right\| \leq \sqrt{(1-\alpha\beta)^2 + R^2\beta^2}\left\|x_{k+1,\alpha}^\delta - x_{k,\alpha}^\delta\right\| \leq \sqrt{(1-\alpha\beta)^2 + R^2\beta^2}\,(t_{k+1} - t_k) = t_{k+2} - t_{k+1}\ .$$

Thus by induction $\left\|x_{n+1,\alpha}^\delta - x_{n,\alpha}^\delta\right\| \leq t_{n+1} - t_n$, for all $n \geq 0$ and hence $(t_n),\ n \geq 0$ is a majorizing sequence of the sequence $(x_{n,\alpha}^\delta)$. In particular $\left\|x_{n,\alpha}^\delta - x_0\right\| \leq t_n \leq t^*$, i.e., $x_{n,\alpha}^\delta \in B_{t^*}(x_0)$ for all $n \geq 0$. So by Lemma 2.1, the sequence $(x_{n,\alpha}^\delta)$, $n \geq 0$ is a Cauchy sequence and converges to some $x_\alpha^\delta \in \overline{B_{t^*}(x_0)} \subset B_{t^{**}}(x_0)$ and

$$\left\|x_{n,\alpha}^\delta - x_\alpha^\delta\right\| \leq t^* - t_n \leq \frac{r^n}{1-r}\eta.$$

Now by letting $n \to \infty$ in (1.5) we obtain $F(x_\alpha^\delta) = y^\delta + \alpha(x_0 - x_\alpha^\delta)$.

3. ERROR BOUNDS UNDER SOURCE CONDITIONS:

To obtain an error estimate for $\left\|x_{n,\alpha}^\delta - x^\dagger\right\|$ it is enough to obtain an error estimate for $\left\|x_\alpha^\delta - x^\dagger\right\|$. To obtain an error estimate for $\left\|x_\alpha^\delta - x^\dagger\right\|$ we use the error estimate for $\left\|x_\alpha^\delta - x_\alpha\right\|$ and $\left\|x_\alpha - x^\dagger\right\|$ where x_α is the unique solution of the equation $F(x) + \alpha(x - x_0) = y$. It is known (cf. [9] Proposition 3.1) that

$$\left\|x_\alpha^\delta - x_\alpha\right\| \leq \frac{\delta}{\alpha} \tag{3.19}$$

and (cf. [5] Theorem 3.1) that

$$\left\|x_\alpha - x^\dagger\right\| \leq (k_0 r_0 + 1)c_\varphi \varphi(\alpha). \tag{3.20}$$

Theorem: 3.1 Let x_α^δ be the unique solution of (1.3) and $x_{n,\alpha}^\delta$ be as in (1.5). Let the assumptions in Theorem 2.4 and (3.19), (3.20) be satisfied. Then we have the following:

$$\begin{aligned}\left\|x_{n,\alpha}^\delta - x^\dagger\right\| &\leq \left\|x_{n,\alpha}^\delta - x_\alpha^\delta\right\| + \left\|x_\alpha^\delta - x_\alpha\right\| + \left\|x_\alpha - x^\dagger\right\| \\ &\leq \frac{r^n\eta}{1-r} + \frac{\delta}{\alpha} + (k_0 r_0 + 1)c_\varphi \varphi(\alpha).\end{aligned} \tag{3.21}$$

Theorem: 3.2 Let $x_{n,\alpha}^\delta$ be as in (1.5). Let the assumptions in Theorem 2.4 and Theorem 3.1 be satisfied. Let $n_\delta := \min\{n : \frac{r^n\eta}{1-r} \leq \frac{\delta}{\alpha}\}$. Then we have:

$$\left\|x_{n,\alpha}^\delta - x^\dagger\right\| \leq \max\{2, (k_0 r_0 + 1)c_\varphi\}(\varphi(\alpha) + \frac{\delta}{\alpha}). \tag{3.22}$$

3.1. A PRIORI CHOICE OF THE PARAMETER:

The error estimate $\varphi(\alpha)+\frac{\delta}{\alpha}$ in (4.33) is of optimal order if $\alpha:=\alpha_\delta$ satisfies, $\varphi(\alpha_\delta)\alpha_\delta=\delta$. Now using the function $\psi(\lambda):=\lambda\varphi^{-1}(\lambda)$, $0<\lambda\le a$ we have $\delta=\alpha_\delta\varphi(\alpha_\delta)=\psi(\varphi(\alpha_\delta))$, so that $\alpha_\delta=\varphi^{-1}(\psi^{-1}(\delta))$.

Theorem: 3.3 Let $\psi(\lambda):=\lambda\varphi^{-1}(\lambda)$ for $0<\lambda\le a$ and assumptions in Theorem 3.2 holds. For $\delta>0$, let $\alpha_\delta=\varphi^{-1}(\psi^{-1}(\delta))$ and let n_δ $n_\delta:=\min\{n:\frac{r^n\eta}{1-r}\le\frac{\delta}{\alpha}\}$ then

$$\left\|x^\delta_{n_\delta,\alpha}-x^\dagger\right\|=O(\psi^{-1}(\delta)).$$

3.2. AN ADAPTIVE CHOICE OF THE PARAMETER:

Now, we will present a parameter choice rule based on the adaptive method studied in [8].

In practice, the regularization parameter α is often selected from some finite set

$$D_M(\alpha):=\{\alpha_i=\mu^i\alpha_0, i=0,1,\cdots,M\} \tag{3.23}$$

Where $\mu>1$, M is big enough but not too large and $\alpha_0=\sqrt{\delta}$.

Let

$$n_M:=\min\{n:\frac{r^n\eta}{1-r}\le\frac{\delta}{\alpha_M}\}. \tag{3.24}$$

Then we have

$$\left\|x^\delta_{n_M,\alpha_i}-x^\delta_{\alpha_i}\right\|\le\frac{\delta}{\alpha_i}, \qquad \forall\ i=0,\ 1,\ \cdots,\ M. \tag{3.25}$$

Let $x_i:=x^\delta_{n_M,\alpha_i}$. The parameter choice strategy that we are going to consider in this paper, we select $\alpha=\alpha_i$ from $D_M(\alpha)$ and operate only with corresponding x_i, $i=0,\ 1,\ \cdots,\ M.$

Theorem 3.4: Assume that there exists $i\in\{0,\ 1,\ \cdots,\ M\}$ such that $\varphi(\alpha_i)\le\frac{\delta}{\alpha_i}$. Let assumptions of Theorem 3.3 and equations (3.24), (3.25) hold and let

$$l:=\max\{i:\varphi(\alpha_i)\le\frac{\delta}{\alpha_i}\}<M\ ,$$

$$k:=\max\{i:\left\|x_i-x_j\right\|\le(4+2(k_0r_0+1)c_\varphi)\frac{\delta}{\alpha_j},\qquad j=0,1,\cdots,i\}. \tag{3.26}$$

Then $l\le k$ and

$$\left\|x^\dagger-x_k\right\|\le c\psi^{-1}(\delta),$$

where $c=3(2+(k_0r_0+1)c_\varphi)\mu.$

Proof: To see that $l\le k$, it is enough to show that, for each $i\in\{1,\ \cdots,\ M\}$,

$$\varphi(\alpha_i)\le\frac{\delta}{\alpha_i}\Rightarrow\left\|x_i-x_j\right\|\le(4+2(k_0r_0+1)c_\varphi)\frac{\delta}{\alpha_j},\qquad\forall j=0,1,\cdots,i.$$

For $j\le i$, by (3.24) and (3.26) we have

$$\begin{aligned}\|x_i - x_j\| &\le \|x_i - x^\dagger\| + \|x^\dagger - x_j\| \\ &\le (k_0 r_0 + 1)c_\varphi \varphi(\alpha_i) + 2\frac{\delta}{\alpha_i} + (k_0 r_0 + 1)c_\varphi \varphi(\alpha_j) + 2\frac{\delta}{\alpha_j} \\ &\le (4 + 2(k_0 r_0 + 1)c_\varphi)\frac{\delta}{\alpha_j}.\end{aligned}$$

Thus the relation $l \le k$ is proved. Next we observe that

$$\begin{aligned}\|x^\dagger - x_k\| &\le \|x^\dagger - x_l\| + \|x_l - x_k\| \\ &\le (k_0 r_0 + 1)c_\varphi \varphi(\alpha_l) + 2\frac{\delta}{\alpha_l} + (4 + 2(k_0 r_0 + 1)c_\varphi)\frac{\delta}{\alpha_l} \\ &\le 3(2 + (k_0 r_0 + 1)c_\varphi)\frac{\delta}{\alpha_l}.\end{aligned}$$

Now since $\alpha_\delta \le \alpha_{l+1} \le \mu\alpha_l$, it follows that

$$\frac{\delta}{\alpha_l} \le \mu\frac{\delta}{\alpha_\delta} = \mu\varphi(\alpha_\delta) = \mu\psi^{-1}(\delta).$$

$$\|x^\dagger - x_k\| \le 3(2 + (k_0 r_0 + 1)c_\varphi)\mu\psi^{-1}(\delta).$$

This completes the proof of the theorem.

4. IMPLEMENTATION OF ADAPTIVE CHOICE RULE:

Here we provide an algorithm for the determination of a parameter fulfilling the balancing principle (3.26) and also provide a starting point for the iteration (1.5) approximating the unique solution x_α^δ of (1.3). The choice of the starting point involves the following steps:

- Choose $0 < \alpha_0 < 1$, $0 < r < 1$ and $\mu > 1$.
- Choose $\eta > 0$ such that $\sqrt{(1-\beta\alpha_0)^2 + \beta^2 R^2}\,\eta \le r$.
- Choose $x_0 \in D(F)$ such that $\|F(x_0) - y^\delta\| \le \frac{1}{\beta}$, with $\beta < 2\alpha_0 / (\alpha_0{}^2 + R^2)$.

The choice of the stopping index n_M involves the following two steps:

- Choose the parameter $\alpha_M = \mu^M \alpha_0$ big enough with $\mu > 1$, not too large.
- Choose n_M such that $n_M := \min\{n : \frac{r^n \eta}{1-r} \le \frac{\delta}{\alpha_M}\}$.

Finally the adaptive algorithm associated with the choice of the parameter specified in Theorem 3.4 involves the following steps:

4.1. ALGORITHM:

- Set $i \leftarrow 0$
- Solve $x_i := x_{n_M,\alpha_i}^\delta$ by using the iteration (1.5).
- If $\|x_i - x_j\| \ge (4 + 2(k_0 r_0 + 1)c_\varphi)\frac{\sqrt{\delta}}{\mu^j}$, $\quad j \le i$, then take $k = i - 1$.
- Set $i = i + 1$ and return to step 2**.**

5. CONCLUDING REMARKS:

In this paper we used the adaptive method considered by Pereverzev and Schock in [8] for choosing the regularization parameter α in (1.3) and we provided a method for computing the unique solution x_α^δ of the equation (1.3). Here, an optimal error estimate has been obtained under a general source condition. We also provide a new stopping rule for the iteration index as (3.24). Note that our method always gives the optimal order $O(\psi^{-1}(\delta))$ under a general source condition; $x_0 - x^\dagger = \varphi(F'(x^\dagger))v$, $v \in X$ with $\|v\| \le 1$. Here φ is a monotonically increasing function as in Assumption 1.3 and $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$.

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