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RECURRENCE RELATIONS FOR MOMENTS OF K – TH LOWER RECORD VALUES FROM EXPONENTIATED LOG-LOGISTIC DISTRIBUTION AND A CHARACTERIZATION

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ABSTRACT

In this study we give some explicit expressions and recurrence relations satisfied by single and product moments of $k-$th lower record values from exponentiated Log-logistic distribution. Further, using a recurrence relation for single moments we obtain characterization of exponentiated Log-logistic distribution.

Key Words: *Record, single moments, product moments, recurrence relations, exponentiated Log-logistic distribution, characterization.*

AMS Subject Classification: *62G30, 62G99, 62E10.*

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1. INTRODUCTION:

The model of record statistics defined by Chandler (1952) as a model for successive extremes in a sequence of independent and identically distributed (iid) random variables. This model takes a certain dependence structure into consideration. That is, the life-length distribution of the components in the system may change after each failure of the components. For this type of model, we consider the lower record statistics. If various voltages of equipment are considered, only the voltages less than the previous one can be recorded. These recorded voltages are the lower record value sequence.

Let $X_1, X_2, \ldots$ be a sequence of iid random variables with distribution function (df) $F(x)$ and probability density function (pdf) $f(x)$. Suppose $Y_n = \min\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We say X_j, $j \geq 1$ is a lower record value of this sequence, if $Y_j < Y_{j-1}$ for $j > 1$. And we suppose that X_1 is a first lower record value. The indices at which the lower record values occur are given by record times $\{L(n), n \geq 1\}$, where $L(n) = \min\{j \mid j > L(n-1),\ X_j < X_{L(n-1)}\}$, $n > 1$, with $L(1) = 1$. For more details and references, see Ahsanullah (1995) and Arnold *et al*. (1998).

For a fixed $k \geq 1$ we define the sequence $\{L_n^{(k)}, n \geq 1\}$ of $k-$th lower record times of $\{X_n, n \geq 1\}$ as follows

$$L_1^{(k)} = 1,$$
$$L_{n+1}^{(k)} = \min\{j > L_n^{(k)}, X_{k:L_k(n)+k-1} > X_{k:j+k-1}\}$$

For $k = 1$ and $n = 1,2,\ldots,$ we write $L_n^{(1)} = L_n$. Then $\{L_n, n \geq 1\}$ is the sequence of record times of $\{X_n, n \geq 1\}$. The sequence $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{L_n^{(k)}}$ is called the sequence of $k-$th lower record values of $\{X_n, n \geq 1\}$. For convenience, we shall also take $Y_0^{(k)} = 0$. Note that $k = 1$ we have $Y_n^{(1)} = X_{L_n}, n \geq 1$, which are record value of $\{X_n, n \geq 1\}$. Moreover $Y_1^{(k)} = \min\{X_1, X_2, \cdots, X_k\} = X_{1:k}$. For more details and references, see Nagaraja (1988), Ahsanullah (1995) and Arnold *et al*. (1998).

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Let $\{X_n^{(k)}, n \ge 1\}$ be the sequence of $k-$th lower record values from (1.1). Then the pdf of $X_{L(n)}^{(k)}$, $n \ge 1$ is given by

$$f_{X_{L(n)}^{(k)}}(x) = \frac{k^n}{(n-1)!}[-\ln(F(x))]^{n-1}[F(x)]^{k-1} f(x). \tag{1.1}$$

and the joint pdf of $X_{L(m)}^{(k)}$ and $X_{L(n)}^{(k)}$, $1 \le m < n$, $n > 2$ is given by

$$f_{X_{L(m)}^{(k)}, X_{L(n)}^{(k)}}(x, y) = \frac{k^n}{(m-1)!(n-m-1)!}[-\ln(F(x))]^{m-1}$$

$$\times[-\ln(F(y)) + \ln(F(x))]^{n-m-1}[F(y)]^{k-1} \frac{f(x)}{F(x)} f(y), \quad x < y. \tag{1.2}$$

We shall denote

$$\mu_{L(n):k}^{(r)} = E((X_{L(n)}^{(k)})^r), \quad r, n = 1, 2, \ldots,$$

$$\mu_{L(m,n):k}^{(r,s)} = E((X_{L(m)}^{(k)})^r (X_{L(n)}^{(k)})^s), \quad 1 \le m \le n-1 \text{ and } r, s = 1, 2, \ldots,$$

$$\mu_{L(m,n):k}^{(r,0)} = E((X_{L(m)}^{(k)})^r) = \mu_{L(m):k}^{(r)}, \quad 1 \le m \le n-1 \text{ and } r = 1, 2, \ldots,$$

$$\mu_{L(m,n):k}^{(0,s)} = E((X_{L(m)}^{(k)})^s) = \mu_{L(n):k}^{(s)}, \quad 1 \le m \le n-1 \text{ and } s = 1, 2, \ldots.$$

Recurrence relations for single and product moments of record values from generalized Pareto, lomax, exponential and generalized extreme value distribution are derived by Balakrishnan and Ahsanullah (1994a, 1994b, and 1995) and Balakrishnan *et al.* (1993) respectively. Pawlas and Szynal (1998, 2000) and Saran and Singh (2008) have established recurrence relations for single and product moments of $k-$th record values from Weibull, Gumbel and linear exponential distribution.

Kamps (1998) investigated the importance of recurrence relations of order statistics in characterization.

In the present study, we established some explicit expressions and recurrence relations satisfied by the single and product moments of $k-$th lower record values from the exponentiated Log-logistic distribution. A characterization of this distribution has also been obtained on using a recurrence relation for single moments.

A random variable X is said to have exponentiated Log-logistic distribution if its pdf is given by

$$f(x) = \frac{b\theta(x/\sigma)^{b\theta-1}}{\sigma\left(1+(x/\sigma)^b\right)^{\theta+1}}, \quad x \ge 0, \ \sigma, b, \theta > 0. \tag{1.3}$$

and the corresponding df is

$$F(x) = \left(\frac{(x/\sigma)^b}{1+(x/\sigma)^b}\right)^{\theta}, \quad x \ge 0, \ \sigma, b, \theta > 0 \tag{1.4}$$

The log-logistic distribution is considered as special cases of the exponentiated Log-logistic distribution when $\theta = 1$ and $\sigma = 1$. More details on this distribution and their applications can be found in Rosaiah *et al*. (2006, 2007).

2. RELATIONS FOR SINGLE MOMENTS:

Note that for exponentiated Log-logistic distribution defined in (1.1)

$$F(x) = \frac{1}{b\theta}\left(1+(x/\sigma)^b\right)xf(x). \tag{2.1}$$

The relation in (2.1) will be exploited in this paper to derive recurrence relations for the moments of record values from the exponentiated Log-logistic distribution.

We shall first establish the explicit expression for the single moment of $k-$th lower record values $\mu_{L(n):k}^{(r)}$. Using (1.1), we have

$$\mu_{L(n):k}^{(r)} = \frac{k^n}{(n-1)!}\int_o^\infty x^r[F(x)]^{k-1}[-\ln(F(x))]^{n-1}f(x)\,dx \tag{2.2}$$

on using the transformation $z=[F(x)]^{1/\theta}$ in (2.2), we get

$$\begin{aligned}\mu_{L(n):k}^{(r)} &= \frac{\sigma^r(\theta k)^n}{(n-1)!}\int_o^1 z^{(r/b)+\theta k-1}(1-z)^{-r/b}[-\ln z]^{n-1}dz \\ &= \frac{\sigma^r(\theta k)^n}{(n-1)!}\sum_{a=0}^{\infty}\frac{(r/b)_{(a)}}{a!}\int_o^1 z^{(r/b)+\theta k+a-1}[-\ln z]^{n-1}dz\end{aligned} \tag{2.3}$$

Again by setting $w=-\ln z$ in (2.3) and simplifying the resulting equation, we get

$$\mu_{L(n):k}^{(r)} = \sigma^r(\theta k)^n\sum_{a=0}^{\infty}\frac{(r/b)_{(a)}}{a![(r/b)+\theta k+a]^n}, \tag{2.4}$$

where

$$\alpha_{(i)} = \begin{cases}\alpha(\alpha+1)\ldots(\alpha+i-1), & i>0\\ 1, & i=0\end{cases}.$$

Remark: 2.1 Setting $k=1$ in (2.4) we deduce the explicit expression for single moments of lower record values from the exponentiated Log-logistic distribution.

Recurrence relations for single moments of $k-$th lower record values from df (1.1) can be derived in the following theorem.

Theorem: 2.1 For a positive integer $k\geq 1$ and for $n\geq 1$ and $r=1,2,\ldots,$

$$\frac{1}{\sigma^b}\mu_{L(n):k}^{(r+b)} = \frac{b\theta k}{r}\mu_{L(n-1):k}^{(r)} - \left(1+\frac{b\theta k}{r}\right)\mu_{L(n):k}^{(r)} \tag{2.5}$$

Proof From (1.1), we have

$$\mu_{L(n):k}^{(r)} = \frac{k^n}{(n-1)!}\int_0^\infty x^r[F(x)]^{k-1}[-\ln(F(x))]^{n-1}f(x)\,dx \tag{2.6}$$

Integrating by parts taking $[F(x)]^{k-1}f(x)$ as the part to be integrated and the rest of the integrand for differentiation, we get

$$\mu_{L(n):k}^{(r)} = \mu_{L(n-1):k}^{(r)} - \frac{r k^n}{k(n-1)!}\int_0^\infty x^{r-1}[F(x)]^k[-\ln(F(x))]^{n-1}\,dx$$

the constant of integration vanishes since the integral considered in (2.6) is a definite integral. On using (2.1), we obtain

$$\mu_{L(n):k}^{(r)} = \mu_{L(n-1):k}^{(r)} - \frac{r k^n}{b\theta k(n-1)!}\left\{\int_0^\infty x^r[F(x)]^{k-1}[-\ln(F(x))]^{n-1} f(x)\,dx + \frac{1}{\sigma^b}\int_0^\infty x^{r+b}[F(x)]^{k-1}[-\ln(F(x))]^{n-1} f(x)\,dx\right\}$$

and hence the result.

Remark: 2.2 Setting $k=1$ in (2.5) we deduce the recurrence relation for single moments of lower record values from the exponentiated Log-logistic distribution.

3. RELATIONS FOR PRODUCT MOMENTS:

On using (1.2), the explicit expression for the product moments of $k-$th lower record values $\mu_{L(m,n):k}^{(r,s)}$ can be obtained

$$\mu_{L(m,n):k}^{(r,s)} = \frac{k^n}{(m-1)!(n-m-1)!}\int_0^\infty x^r[-\ln(F(x))]^{m-1}\frac{f(x)}{F(x)}I(x)dx \tag{3.1}$$

where,

$$I(x) = \int_0^x y^s[\ln(F(x)) - \ln(F(y))]^{n-m-1}[F(y)]^{k-1} f(y)\,dy \tag{3.2}$$

By setting $w = \ln F(x) - \ln F(y)$ in (3.2), we obtain

$$I(x) = \frac{\theta^{n-m}}{\sigma^s}\sum_{a=0}^\infty \frac{(s/b)_{(a)}[F(x)]^{((s+a)/b\theta)+k}\,\Gamma(n-m)}{a![(s/b)+a+\theta k]^{n-m}}.$$

On substituting the above expression of $I(x)$ in (3.1) and simplifying the resulting equation, we obtain

$$\mu_{L(m,n):k}^{(r,s)} = \sigma^{r+s}(\theta k)^n \sum_{a=0}^\infty\sum_{c=0}^\infty \frac{(r/b)_{(c)}\,(s/b)_{(a)}}{a!\,c!\,[(s/b)+a+\theta k]^{n-m}\,[((r+s)/b)+a+c+\theta k)]^m}\ , \tag{3.3}$$

Remark: 3.1 Setting $k=1$ in (3.3) we deduce the explicit expression for product moments of lower record values from the exponentiated Log-logistic distribution.

Making use of (1.1), we can drive recurrence relations for product moments of $k-$th lower record values from (1.4).

Theorem: 3.1 For $1 \le m \le n-2$ and $r, s = 1, 2, \cdots$

$$\frac{1}{\sigma^b}\mu_{L(m,n):k}^{(r,s+b)} = \frac{b\theta k}{s}\mu_{L(m,n-1):k}^{(r,s)} - \left(1+\frac{b\theta k}{s}\right)\mu_{L(m,n):k}^{(r,s)} \tag{3.4}$$

Proof: From equation (1.2) for $1 \le m \le n-1$, $r, s = 0, 1, 2, \cdots$

$$\mu_{L(m,n):k}^{(r,s)} = \frac{k^n}{(m-1)!(n-m-1)!}\int_0^\infty x^r[-\ln(F(x))]^{m-1}\frac{f(x)}{F(x)}I(x)\,dx\,, \tag{3.3}$$

where

$$I(x)=\int_0^x y^s\,[\ln(F(x))-\ln(F(y))]^{n-m-1}[F(y)]^{k-1}f(y)dy$$

Integrating $I(x)$ by parts treating $[F(y)]^{k-1}f(y)$ for integration and the rest of the integrand for differentiation, and substituting the resulting expression in (3.3), we get

$$\mu_{L(m,n):k}^{(r,s)}=\mu_{L(m,n-1):k}^{(r,s)}-\frac{s\,k^n}{k(m-1)!(n-m-1)!}\int_0^\infty\int_0^x x^r y^{s-1}[-\ln(F(x))]^{m-1}$$
$$\times[\ln(F(x))-\ln(F(y))]^{n-m-1}[F(y)]^k\,\frac{f(x)}{F(x)}dy\,dx$$

the constant of integration vanishes since the integral in $I(x)$ is a definite integral. On using the relation (2.1), we obtain

$$\mu_{L(m,n):k}^{(r,s)}=\mu_{L(m,n-1):k}^{(r,s)}-\frac{s\,k^n}{b\theta k(m-1)!(n-m-1)!}\left\{\int_0^\infty\int_0^x x^r y^s[-\ln(F(x))]^{m-1}\right.$$
$$\times[\ln(F(x))-\ln(F(y))]^{n-m-1}[F(y)]^{k-1}\,\frac{f(x)}{F(x)}dy\,dx+\frac{1}{\sigma^b}\int_0^\infty\int_0^x x^r y^{s+b}$$
$$\left.\times[-\ln(F(x))]^{m-1}[\ln(F(x))-\ln(F(y))]^{n-m-1}[F(y)]^{k-1}\,\frac{f(x)}{F(x)}dy\,dx\right\}$$

and hence the result.

Remark: 3.1 Setting $k=1$ in (3.3), we deduce the recurrence relation for product moments of lower record values from the exponentiated Log-logistic distribution.

4. CHARACTERIZATION:

Theorem 4.1 Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0)=0$ and $0<F(x)<1$ for all $x>0$,

$$\frac{1}{\sigma^b}\mu_{L(n):k}^{(r+b)}=\frac{b\theta k}{r}\mu_{L(n-1):k}^{(r)}-\left(1+\frac{b\theta k}{r}\right)\mu_{L(n):k}^{(r)}$$

if and only if

$$F(x)=\left(\frac{(x/\sigma)^b}{1+(x/\sigma)^b}\right)^\theta,\quad x\ge 0,\ \sigma,b,\theta>0. \tag{4.1}$$

Proof: The necessary part follows immediately from equation (2.5). On the other hand if the recurrence relation in equation (4.1) is satisfied, then on using equations (1.2), we have

$$\frac{k^n}{(n-1)!}\int_0^\infty x^r[F(x)]^{k-1}[-\ln(F(x))]^{n-1}f(x)\,dx$$
$$=\frac{(n-1)k^n}{k\,(n-1)!}\int_0^\infty x^{r-1}[F(x)]^{k-1}[-\ln(F(x))]^{n-2}f(x)\,dx$$
$$-\frac{r\,k^n}{b\theta k(n-1)!}\int_0^\infty x^r[F(x)]^{k-1}[-\ln(F(x))]^{n-1}f(x)\,dx$$
$$-\frac{r\,k^n}{b\theta k\sigma^b(n-1)!}\int_0^\infty x^{r+b}[F(x)]^{k-1}[-\ln(F(x))]^{n-1}f(x)\,dx \tag{4.2}$$

Integrating the first integral on the right hand side of equation (4.2), by parts, we get

$$\frac{k^n}{(n-1)!}\int_0^\infty x^r [F(x)]^{k-1}[-\ln(F(x))]^{n-1} f(x)\,dx$$

$$= \frac{r\,k^n}{k\,(n-1)!}\int_0^\infty x^{r-1}[F(x)]^{k}[-\ln(F(x))]^{n-1}\,dx$$

$$+\frac{k^n}{(n-1)!}\int_0^\infty x^r [F(x)]^{k-1}[-\ln(F(x))]^{n-1} f(x)\,dx$$

$$-\frac{r\,k^n}{b\theta k(n-1)!}\int_0^\infty x^r [F(x)]^{k-1}[-\ln(F(x))]^{n-1} f(x)\,dx$$

$$-\frac{r\,k^n}{b\theta k\sigma^b (n-1)!}\int_0^\infty x^{r+b} [F(x)]^{k-1}[-\ln(F(x))]^{n-1} f(x)\,dx$$

which reduces to

$$\frac{r\,k^n}{k\,(n-1)!}\int_0^\infty x^{r-1}[F(x)]^{k-1}[-\ln(F(x))]^{n-1}\,dx \left\{F(x)-\frac{x}{b\theta}f(x)-\frac{x^{b+1}}{b\theta\sigma^b}f(x)\right\}=0 \tag{4.3}$$

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, 1984) to equation (4.3), we get

$$\frac{f(x)}{F(x)}=\frac{b\theta}{(1+(x/\sigma)^b)\,x}$$

which prove that

$$F(x)=\left(\frac{(x/\sigma)^b}{1+(x/\sigma)^b}\right)^\theta, \quad 0\le x<\infty, \qquad \sigma,b,\theta>0$$

5. CONCLUSION:

In this study some explicit expression and recurrence relations for single and product moments of $k-$th lower record values from the exponentiated Log-logistic distribution have been established. Further, characterization of this distribution has also been obtained on using a recurrence relation for single moments.

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